Chapter 6

Global Extrinsic Geometry

6.1. Introduction and historical notes

We have already seen in Chapter 3 convincing reasons to believe that the second fundamental form and the corresponding curvatures associated with the Gauss map are efficient tools for controling the shape of a surface. In addition, in Chapter 5 we have developed a theory of integration on compact surfaces with the usual properties of Lebesgue integration. These are the two main bases for carrying out a deeper study of the geometry of surfaces in $\mathbb{R}^3$.

From the point of view of modern differential geometry, which has been greatly influenced by topology, *deeper* means being able, even in a small measure, to skip from local to global properties. For example, the Brouwer-Samelson theorem, the Jordan-Brouwer separation theorem, and the Brouwer fixed point theorem, proved in Chapters 4 and 5, are of course results of a global nature, but they are theorems that perhaps belong more to differential topology. In this chapter, we deal with a class of results and proofs that we think of as the core of differential geometry.

Whenever one thinks about global problems in geometry, it is usual to impose some *global* condition to avoid the case when *pieces* of larger surfaces may appear as possible solutions, that is, surfaces that are proper open subsets of another larger surface. One of these conditions is clearly *compactness*. Another global condition that is weaker than compactness, which appears in many texts, is *geodesic completeness*. Since our point of view is mainly concerned with extrinsic geometry, we will restrict ourselves to surfaces that are closed subsets of $\mathbb{R}^3$. Thus, we will restrict ourselves in this chapter to studying closed (as subsets of Euclidean space) surfaces and mainly to compact surfaces. In this way, we will be able to use, as a fundamental analytical tool, the theory of integration of Chapter 5.

The first problem that we will consider, which will repeat some questions that we have already posed for curves, is to determine compact surfaces with one or more of their curvatures having some specific behavior. For example, can one characterize compact surfaces with constant Gauss or mean curvature? We first consider the Gauss curvature. The local geometry of surfaces with constant Gauss curvature was studied by F. Minding in 1838, with respect to their relation to non-Euclidean geometries. In the compact case, we know from Chapter 3 that the only example is the sphere, as the Hilbert-Liebmann theorem asserted. This result was first proved by H. Liebmann in 1899 and was rediscovered by D. Hilbert in 1901—our proof is a simplification of Hilbert's. It is then natural to generalize the question and consider compact surfaces whose Gauss curvature does not change sign. From what we have seen until now, we deduce that this sign must be positive. We take up the study of compact surfaces with positive Gauss curvature, the so-called *ovaloids*, and, more generally, positively curved closed surfaces. Study of this class of surfaces was initiated at the beginning of the 20th century by H. Minkowski and W. Blaschke in relation to the theory of convexity. We will prove Hadamard's theorem (1857), which states that ovaloids are topologically spheres, and Stoker's theorem (1936), which states that closed non-compact surfaces with positive Gauss curvature are diffeomorphic to $\mathbb{R}^2$. Furthermore, both are characterized as boundaries of convex domains in Euclidean space. Just this information about ovaloids will enable us to again prove the Hilbert-Liebmann theorem using as a tool some integral expressions known as the Minkowski formulas. We will also obtain a new version of J. H. Jellett's result (1853) which, in this context, states that a compact surface with constant mean curvature, which is star-shaped with respect to a given point, must be a sphere centred at this point.

The last results about ovaloids lead to the second question that we posed above: which compact surfaces have constant mean curvature? We know, as mentioned in the introduction to Chapter 3, that the Gauss and mean curvatures were introduced in the 18th century and, since then, they have competed with each other. In spite of the fact that the initial competition was won, in some sense, by the Gauss curvature, once Gauss proved its intrinsic character, the global problems on the mean curvature have proved to be more complicated and, in some sense, more interesting. In fact, it was only in the years between 1956 and 1962 that A. D. Alexandrov succeeded in proving his celebrated theorem that, in our setting, asserts that the only compact surfaces with constant mean curvature are, again, spheres. This theorem was quite a surprising result, not because of its contents, but because of the method that Alexandrov invented to prove it, one that was completely different from those of Liebmann and Jellett. It was based on an ingenious use of properties of certain partial differential equations. Including this proof is not feasible for a book of our level, and that is also why this theorem, though it is basic to the global theory of surfaces, does not even appear in the most prominent books on the subject. In 1978, R. Reilly found a new proof that, although less geometrical, was easier to understand and, especially, to explain. Unfortunately, it is also difficult to include in a first course in differential geometry. During the last ten years, the authors have studied some problems related to the mean curvature which have allowed them to outline the proof that we will present here, based on a paper by H. Heintze and H. Karcher.

Finally, the determination of compact surfaces with constant mean curvature will lead us to answer one of the oldest questions in geometry, posed by the Greeks: the isoperimetric problem, which can be stated as follows. Among all the compact surfaces in $\mathbb{R}^3$ whose inner domains have a given volume, which has the least area? The ancient Greeks themselves knew that the answer was spheres, as was first shown by H. A. Schwarz according to earlier ideas of Steiner and Minkowski, who had looked for the solution among convex domains. Our proof will consist, following [**5**] and [**6**], of obtaining the Brunn-Minkowski inequality, which is nothing more than another property of Lebesgue measure, and showing that surfaces, for which equality is attained, must have constant mean curvature.

Although we think that this chapter should furnish a more than adequate introduction to global geometry of surfaces in Euclidean space, the compactness hypothesis required by the theory of integration we use and our desire to not complicate this text excessively will keep us from studying negatively curved surfaces or surfaces with vanishing Gauss curvature. Thus, a suitable follow-up for the reader would be the search for readable proofs of the Hilbert and Efimov theorems on the non-existence of closed negatively curved surfaces in $\mathbb{R}^3$ or for the Hartman-Nirenberg and Massey theorems, asserting that the only surfaces with null curvature are cylinders having any plane curve as a directrix. (See Appendix 7.7.)

6.2. Positively curved surfaces

Before determining the compact surfaces with constant Gauss or mean curvature, we will consider surfaces whose Gauss curvature does not change sign. Recall—Exercise 3.42—that, on any compact surface, there is always an elliptic point. Therefore, if the sign of the Gauss curvature does not

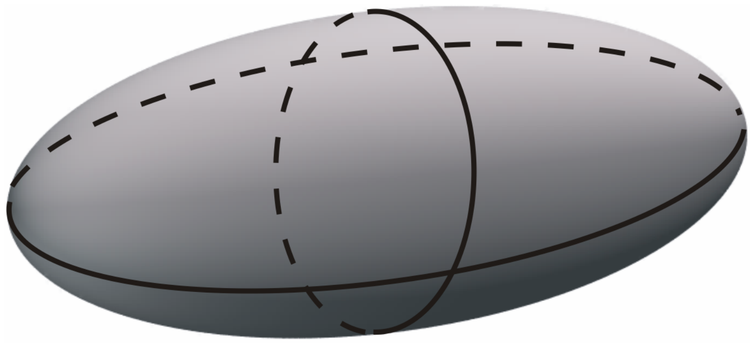

Figure 6.1. *Ovaloid*

change, then it must be positive everywhere. We will use the term *ovaloid* for any compact connected surface with positive Gauss curvature everywhere. This is equivalent, according to definition (3.5), to the fact that the second fundamental form is definite at each point and, taking Exercise (4) of Chapter 4 into account, to the statement that the second fundamental form relative to the inner normal is positive definite at each point. For example, according to Exercise 3.24, all ellipsoids are ovaloids.

More generally, in this section, we will give a complete description of the shape that *positively curved closed surfaces* can have. In the following, S will denote a connected surface that is closed as a subset of the Euclidean space $\mathbb{R}^3$ and has strictly positive Gauss curvature everywhere. By Exercise (6) of Chapter 4, which generalizes the Jordan-Brouwer and Brouwer-Samelson theorems to closed surfaces, we know that the surface S separates $\mathbb{R}^3$ into two connected components and that it is orientable. Since the second fundamental form of S, relative to any choice of normal field, is definite, we can choose as the orientation of S the one corresponding to the Gauss map $N : S \to \mathbb{S}^2$ for which the second fundamental form is *positive* definite at each point. Then, all the half-lines starting at a point $p \in S$ with direction $N(p)$ lie locally on the same side of the surface, since any two points of S belong to a relatively compact open subset of S, which possesses a tubular neighbourhood. This privileged connected component Ω of $\mathbb{R}^3 - S$ toward all the half-lines in the direction of the so-chosen Gauss map point will be called the *inner domain* of the surface S. It is clear that, if S is compact, this notion of inner domain coincides with the one that was already defined in Remark 4.20. Examples of closed connected surfaces with positive Gauss curvature are, according to Example 3.45, the elliptic paraboloids.

A concept related, in a natural way, to the Gauss curvature is that of the convexity of surfaces. Usually, a surface is said to be *convex* when it lies on one of the closed half-spaces determined by the affine tangent plane at

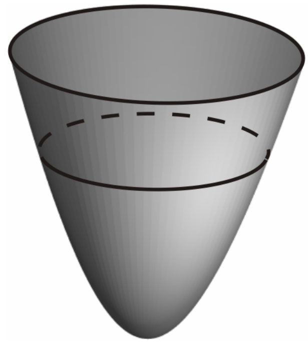

Figure 6.2. *Inner domain*

each of its points. We will say that it is *strictly convex* when it is convex and touches each tangent plane only at one point. It is worth pointing out that, at first, this notion of convexity does not have much to do with the notion of affine convexity. However, we have the following result relating them. The one that refers to compact surfaces was first shown by Hadamard in 1857 and the analogous statements for closed surfaces are consequences of a result due to Stoker in 1936.

Theorem 6.1 (First part of the Hadamard-Stoker theorem)**.** *Let S be a connected surface closed as a subset of $\mathbb{R}^3$ with strictly positive Gauss curvature, and let Ω be its inner domain. Then the following are true.*

- **A:** *For each $x, y \in \overline{\Omega}$, $]x, y[\subset \Omega$. In particular, Ω is a convex subset of $\mathbb{R}^3$.*
- **B:** *The intersection of S with the affine tangent plane of S at any of its points is only the point itself. Moreover, $\overline{\Omega}$ is contained in all the closed half-spaces of $\mathbb{R}^3$ determined by the affine tangent planes and the inner normals of the surface.*

Proof. Part **A**. We first show that Ω is convex. Consider the subset $A = \{(x, y) \in \Omega \times \Omega \,|\, [x, y] \subset \Omega\}$ of the product $\Omega \times \Omega$. Since Ω is connected, if it were not convex, then the boundary of A in $\Omega \times \Omega$ would be non-empty. Take a pair (x, y) in Bdry A. By definition of the topological boundary, there are sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{x'_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$, and $\{y'_n\}_{n\in\mathbb{N}}$ of points of Ω such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x'_n = x$, $\lim_{n\to\infty} y_n = \lim_{n\to\infty} y'_n = y$, and $[x_n, y_n] \subset \Omega$, $[x'_n, y'_n] \not\subset \Omega$, for each $n \in \mathbb{N}$. This implies that one could find a point $z \in]x, y[\cap S$ of first contact between the segment and the surface. From Lemma 4.15, we deduce that $]x, y[\subset \overline{\Omega}$ would lie in the affine tangent plane of S at z and that the height function relative to this plane, when restricted to the normal section at z in the direction of the tangent segment $]x, y[$, would attain a maximum at z. On the other hand, we said

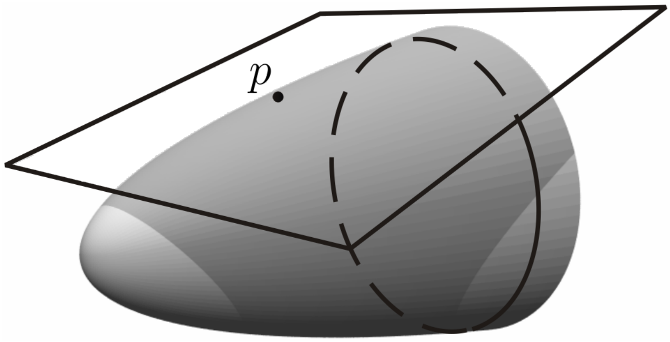

Figure 6.3. *Hadamard-Stoker's theorem*

that the positivity of the Gauss curvature forces the second fundamental form relative to the orientation chosen on S to be positive definite at each point. Then, Remark 3.28—see also Proposition 3.37—says that this same height function attains a local strict maximum at z. This is a contradiction and, thus, Ω is a convex subset of Euclidean space. Now, an elementary fact about convex sets is that the closure of a convex set is convex (the reader should try to prove this as an exercise). Therefore, if part **A** were not true, then there would exist two points $x, y \in S$ such that $]x, y[\cap S \neq \emptyset$ and this is impossible from the above. In fact, arguing in the same way, we would have a first contact point at each $z \in]x, y[\cap S$ and this is again a contradiction.

Part **B**. Let $p \in S$ be a point of the surface. Let Π_p denote the affine tangent plane of S at p, and let Π_p^+ denote the closed half-space of $\mathbb{R}^3$ determined by this plane Π_p and the inner normal vector $N(p)$. In other words,

$$\Pi_p^+ = h_p^{-1}[0, +\infty) \quad \text{where} \quad h_p(q) = \langle q - p, N(p) \rangle,$$

for each $q \in \mathbb{R}^3$, is the height function relative to Π_p.

Suppose now that $q \in \Pi_p \cap S$ for some $p \in S$. Then p and q belong to $\overline{\Omega}$ and, from part **A**, we would have $]p, q[\subset \Omega \cap \Pi_p$. But, since p is an elliptic point, Proposition 3.37 and Remark 3.28 tell us that all the points in a neighbourhood of p in Π_p, except p itself, lie in the outer domain of S, and this leads us to a contradiction, unless $p = q$ and so $]p, q[= \emptyset$. Therefore, $\Pi_p \cap S = \{p\}$, for each $p \in S$, which is the first assertion of part **B**.

To see that the second assertion is also true, observe that, for $p \in S$, the set $S - \{p\}$ is connected and does not intersect Π_p. But, since there are points of S near p that lie in Π_p^+, since p is an elliptic point, we have $S - \{p\} \subset \Pi_p^+$ and, since $p \in \Pi_p$, $S \subset \Pi_p^+$. Hence $\mathbb{R}^3 - \Pi_p^+ \subset \mathbb{R}^3 - S$. Thus, the connected open set $\mathbb{R}^3 - \Pi_p^+$ belongs to one of the two connected components of the complement of S and has points in the outer domain of S, again since p is elliptic. Consequently, $\mathbb{R}^3 - \Pi_p^+ \subset \mathbb{R}^3 - \overline{\Omega}$. Taking complements, we conclude that $\overline{\Omega} \subset \Pi_p^+$, and this holds for each $p \in S$. □

Remark 6.2. If $p \in S$, assertion **B** of Theorem 6.1, $\overline{\Omega} \subset \Pi_p^+$, is equivalent to

$$h_p(q) = \langle q - p, N(p) \rangle \geq 0, \qquad \forall q \in \overline{\Omega},$$

and the equality implies that $q = p$. Hence

$$\langle p - q, N(p) \rangle < 0, \qquad \forall p \in S \quad \forall q \in \Omega;$$

that is, the support function of S corresponding to the inner normal based on any point of Ω is negative everywhere. In particular—see Exercise (4) of Chapter 3—we have that S is star-shaped with respect to any point of its inner domain.

Remark 6.3 (Intersection of a straight line and an ovaloid)**.** Assume that the surface S is an ovaloid and that Ω is the inner domain determined by it. Let R be a straight line of $\mathbb{R}^3$ cutting the inner domain, that is, with $R \cap \Omega \neq \emptyset$. As a consequence of the Hadamard-Stoker Theorem 6.1, $R \cap \Omega$ is a non-empty bounded convex set, which is open as a subset of the line R. Thus, it has to be an open interval. This, along with the fact that R cannot be tangent at any point of S, by part **B** of Theorem 6.1, implies that $R \cap S$ consists of exactly two points; that is, *a straight line touching the inner domain of an ovaloid cuts it at exactly two points*. See Exercise (6) at the end of this chapter.

Theorem 6.1 above asserts that the inner domain of a closed connected surface with positive Gauss curvature is convex and that, furthermore, the surface itself is strictly convex. To proceed with this type of result, we will need a lemma, in which we will use the following notation. If $x \in \mathbb{R}^3$ is a point of Euclidean space and $v \in \mathbb{R}^3 - \{0\}$ is a non-null vector, we denote the straight line passing through x in the direction of v by $R(x, v)$ and by $R^+(x, v)$ the set

$$R^+(x, v) = \{x + tv \mid t \geq 0\},$$

that is, the half-line with origin at x and direction v.

Lemma 6.4. *Let S be a closed connected surface with positive Gauss curvature and let Ω be its inner domain. Then the following are true.*

- **a:** *If $R^+(x, v) \subset \overline{\Omega}$ for some $x \in \overline{\Omega}$ and $v \in \mathbb{R}^3 - \{0\}$, then $R^+(y, v) \subset \overline{\Omega}$ and, moreover, $R^+(y, v) - \{y\} \subset \Omega$, for each $y \in \overline{\Omega}$.*
- **b:** *There are no straight lines contained in $\overline{\Omega}$.*

Proof. Part **a**. If we suppose that, for some $x \in \overline{\Omega}$ and a certain non-null vector v, $R^+(x, v) \subset \overline{\Omega}$, then $x + tv \in \overline{\Omega}$ for each $t \geq 0$. If $y \in \overline{\Omega}$, by part

A of the Hadamard-Stoker Theorem 6.1 above, we have $]y, x+tv[\subset \Omega$, for $t \geq 0$. Taking the limit as $t \to \infty$, we obtain that $R^+(y,v) \subset \overline{\Omega}$. If this half-line had some point z of S different from its origin, we would have a contradiction to part **A** of Theorem 6.1.

Part **b**. Suppose that there is a straight line R in $\overline{\Omega}$. We pick a point $x \in R$ and let $v \in \mathbb{S}^2$ be a direction vector of R. Then the two half-lines $R^+(x,v)$ and $R^+(x,-v)$ are in $\overline{\Omega}$. Thus, if $p \in S \subset \overline{\Omega}$, by part **a**, the half-lines $R^+(p,v)$ and $R^+(p,-v)$ are contained in $\overline{\Omega}$. Therefore,

$$p+v \in R^+(p,v) \subset \overline{\Omega} \quad \text{and} \quad p-v \in R^+(p,-v) \subset \overline{\Omega}.$$

Using again part **A** of the Hadamard-Stoker Theorem, we would get

$$p \in]p-v, p+v[\subset \Omega,$$

which is impossible. □

Theorem 6.5 (Second part of the Hadamard-Stoker Theorem)**.** *Let S be a connected surface, closed as a subset of $\mathbb{R}^3$ and with strictly positive Gauss curvature.*

- **C:** *If S is an ovaloid, that is, if S is compact, then the Gauss map $N : S \to \mathbb{S}^2$ is a diffeomorphism. Consequently, S is diffeomorphic to a sphere.*
- **D:** *If S is not compact, then the Gauss map $N : S \to \mathbb{S}^2$ is injective and S is a graph on an open convex subset of a plane.*

Proof. We are assuming that the surface S has positive Gauss curvature everywhere. Then, equation (3.4) implies that the Gauss map $N : S \to \mathbb{S}^2$ is a local diffeomorphism, by the inverse function theorem (Theorem 2.75). We now show that, under our hypotheses, N must be injective. Indeed, if $p, q \in S$ are two points of the surface with $N(p) = N(q)$, then the two corresponding affine tangent planes Π_p and Π_q are parallel. By part **B** of Theorem 6.1, we have $S \subset \overline{\Omega} \subset \Pi_p^+ \cap \Pi_q^+$. This intersection coincides with one of the two half-spaces, since the boundary planes are parallel, say, with Π_p^+. Hence $q \in \Pi_q^+ \subset \Pi_p^-$ and thus $q \in \Pi_p^+ \cap \Pi_p^- = \Pi_p$ and $\Pi_p = \Pi_q$. Assertion **B** also gives us that

$$\{p\} = S \cap \Pi_p = S \cap \Pi_q = \{q\}.$$

Then $p = q$ and N is injective, whether or not S is compact.

Part **C**. If we suppose now that S is compact, then $N(S) \subset \mathbb{S}^2$ is compact, and thus, it is closed as a subset of the unit sphere. Since N is a local diffeomorphism, this image is also an open subset of $\mathbb{S}^2$ and, by connectedness, $N(S) = \mathbb{S}^2$; that is, N is surjective. In conclusion, the Gauss map N is a diffeomorphism between S and the sphere $\mathbb{S}^2$.

Part **D**. Now consider the case where S is not compact and, hence, it is not bounded. Then, we may select a sequence $\{q_n\}_{n\in\mathbb{N}}$ of points $q_n \in S$ such that $\lim_{n\to\infty} |q_n| = +\infty$. Thus, if we fix every $q \in S$, since $q, q_n \in S \subset \overline{\Omega}$, by part **A** of Theorem 6.1, we have $[q, q_n] \subset \overline{\Omega}$ for each $n \in \mathbb{N}$. Extracting a subsequence, if necessary, we can find $a \in \mathbb{S}^2$ such that $R^+(q, a) \subset \overline{\Omega}$. We apply part **a** of Lemma 6.4 above to deduce that

$$R^+(p, a) - \{p\} \subset \Omega, \qquad \forall p \in S.$$

From this, if $p \in S$ and if there were another point $p' \in S$ in $R(p, a)$, either p or p' would be on the open half-line starting from the other point in the direction of a and, so, it would belong to Ω. Therefore

$$R(p, a) \cap S = \{p\}, \qquad \forall p \in S.$$

Furthermore, none of these lines can be tangent to S. The reason for this is that, if $R(p, a)$ were tangent to S, of course at p, then there would be a neighbourhood of p in the line that would be contained, except for p itself, in the outer domain of S—as we have already argued several times above—which would contradict the fact that $R^+(p, a) - \{p\} \subset \Omega$. Consequently, the orthogonal projection P of $\mathbb{R}^3$ to the plane

$$\Pi = \{x \in \mathbb{R}^3 \mid \langle x, a\rangle = 0\}$$

passing through the origin and perpendicular to a is injective when restricted to S. Moreover, if $p \in S$ and $v \in T_pS$ is in the kernel of $(dP)_p$, we have

$$0 = (dP)_p(v) = P(v) = v - \langle v, a\rangle a.$$

Since $a \notin T_pS$, we infer that $v = 0$. That is, the projection P is also a local diffeomorphism, by the inverse function theorem. The final conclusion is that P is a diffeomorphism between S and its image in Π, which is an open convex subset of this plane. In other words, S is a graph over this open set of a function which is necessarily convex, since Ω is convex. □

Remark 6.6. If S is an ovaloid, assertion **C** of the second part of the Hadamard-Stoker Theorem 6.5 says that $N : S \to \mathbb{S}^2$ is a diffeomorphism. We can apply the change of variables formula (Theorem 5.14) to the constant function 1 on the unit sphere and obtain

$$4\pi = A(\mathbb{S}^2) = \int_{\mathbb{S}^2} 1 = \int_S |\mathrm{Jac}\, N|.$$

But—see Exercise (2) at the end of Chapter 5—if $p \in S$,

$$|(\mathrm{Jac}\, N)(p)| = |(dN)_p(e_1) \wedge (dN)_p(e_2)|,$$

where $\{e_1, e_2\}$ is any orthonormal basis of T_pS. For example, taking e_1 and e_2 as principal directions of S at the point p, we have

$$|(\mathrm{Jac}\, N)(p)| = |k_1(p)k_2(p)||e_1 \wedge e_2| = K(p),$$

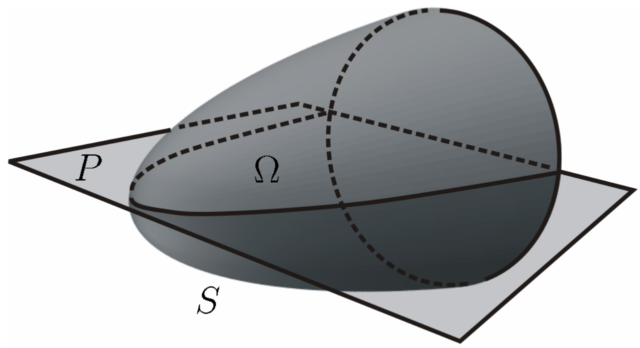

Figure 6.4. *Lemma* 6.7

since the Gauss curvature is positive. It follows that

$$\int_S K(p)\, dp = 4\pi$$

on any ovaloid S.

These two Hadamard-Stoker theorems are the strongest results that one can expect for positively curved closed surfaces. However, we can obtain more information about this type of surface if we assume that at least one of the curvatures of the surface, Gauss or mean curvature, is bounded away from zero. To arrive at this kind of result, we will need to show that some pieces of the surfaces that we are dealing with can be expressed as graphs.

Lemma 6.7. *Let P be a plane cutting the inner domain Ω of a closed connected surface S with positive Gauss curvature. Then there exists an open subset V of S that is a graph over $P \cap \Omega$.*

Proof. Denote the non-empty open subset $P \cap \Omega$ of the plane P by U and suppose that, given $v \in \mathbb{S}^2$, normal to P, there is a point $y \in U$ such that the half-line $R^+(y, v)$ does not intersect the surface S. Then $R^+(y, v)$, which is connected, must lie in one of the two connected components of $\mathbb{R}^3 - S$ and, since $y \in R^+(y, v) \cap \Omega$, we will have $R^+(y, v) \subset \Omega$. Thus, by part **a** of Lemma 6.4, we will obtain that $R^+(x, v) \subset \Omega$ for each $x \in U$. But part **b** of the same lemma asserts that Ω contains no straight lines. Consequently, any half-line $R^+(x, -v)$ cannot be entirely included in Ω, whatever $x \in U$ is. However, each of them has the point $x \in R^+(x, -v)$ in Ω. Hence, we have as a consequence that $R^+(x, -v) \cap S \neq \emptyset$ for each $x \in U$.

We may assume from the above that there exists a vector $v \in \mathbb{S}^2$, normal to P, such that all the half-lines $R^+(x, v)$ starting at the points x of U in the direction of v meet the surface S. Thus, we have, by Theorem 6.1, that each intersection $R^+(x, v) \cap \Omega$ is a non-void bounded convex open subset of the half-line $R^+(x,v)$, which, moreover, contains its origin x, for every $x \in U$. Therefore, each $R^+(x,v) \cap S$ has a unique point for each $x \in U$ and also the straight line $R(x,v)$ is not tangent to S at this point, since otherwise, by part **B** of Theorem 6.1, $R(x,v)$ could not contain any points of the inner domain Ω. This means that, if $f : S \to P$ is the restriction to S of the orthogonal projection onto the plane P, we have that f is a diffeomorphism from the open subset $V = f^{-1}(U)$ of S to U, that is, V is a graph over the open set U. □

We will now show that, on a closed connected positively curved surface that is not compact, the Gauss curvature must approach zero at infinity and the fact that the mean curvature is bounded away from zero on this class of surfaces supplies us with some bounding properties.

Theorem 6.8 (Bonnet's theorem). *Let S be a connected surface that is closed as a subset of $\mathbb{R}^3$. The following assertions are true:*

- **A:** *If $\inf_{p\in S} K(p) > 0$, then S is compact, that is, S is an ovaloid.*
- **B:** *If S is non-compact, has positive Gauss curvature, and its mean curvature satisfies $\inf_{p\in S} H(p) = h > 0$, then S is a graph over a convex open subset of a plane whose diameter is less than or equal to $2/h$.*

Proof. Part **A**. Assume that S is not compact and denote by $k > 0$ the infimum of the Gauss curvature on S. By part **D** of the Hadamard-Stoker Theorem 6.5, we know that S is a graph over a convex open subset of a plane P and that the Gauss map is a diffeomorphism of S over an open subset of $\mathbb{S}^2$. Thus, if $\mathcal{R}$ is any region of the surface S, one has

$$k\,A(\mathcal{R}) = \int_{\mathcal{R}} k \leq \int_{\mathcal{R}} K(p)\,dp = \int_{\mathcal{R}} |(\operatorname{Jac} N)(p)|\,dp.$$

Using the change of variables formula, we see that

$$k\,A(\mathcal{R}) \leq \int_{N(\mathcal{R})} 1 = A(N(\mathcal{R})) \leq 4\pi,$$

and this bound serves for each region $\mathcal{R}$ of the surface S. On the other hand, if $]a,b[$ is an open segment of the inner domain Ω of S, we will have $R^+(x,v) \subset \Omega$ for some $v \in \mathbb{S}^2$ normal to the plane P and each $x \in]a,b[$. But, if $n \in \mathbb{N}$, the set

$$U_n = \{x + tv \mid x \in]a,b[,\ 0 < t < n\}$$

is an open subset of the plane Q determined by $]a,b[$ and the vector v. Applying Lemma 6.7 to this plane Q, we deduce that, for each $n \in \mathbb{N}$, there exists an open subset $\mathcal{R}_n$ of the surface that is a graph over the open subset U_n of the plane Q. Therefore, by Exercise (13) of Chapter 5, we obtain that

$$\frac{4\pi}{k} \geq A(\mathcal{R}_n) \geq A(U_n) = n|a-b|$$

for all natural numbers n. This is absurd and, hence, S must be a compact surface.

Part **B**. As in the previous case **A**, our hypotheses and part **D** of the Hadamard-Stoker Theorem 6.5 imply that S is a graph over a convex open subset U of some plane P. Let q_1 and q_2 be any two points of U, and let p_1 and p_2 be the two points of S projecting onto them. One of the two unit vectors normal to the plane P, which we will represent by v, satisfies $R^+(p_1, v) \subset \overline{\Omega}$ and, from Lemma 6.4, $R^+(x, v) \subset \Omega$ for each $x \in]p_1, p_2[$. Then, the set

$$V = \{x + tv \,|\, x \in]p_1, p_2[, \; t > 0\}$$

is an open subset of the plane Q determined by the segment $]p_1, p_2[$ and the vector v. Applying Lemma 6.7 to this plane Q, we infer the existence of an open subset S' of the surface S that is a graph over this open subset V of Q. Note now that this V belongs to a plane strip whose width is $|q_1 - q_2|$. We complete the proof using Exercise (18) at the end of Chapter 3. □

Remark 6.9. The above result implies that for positively curved closed surfaces, there are two or three independent directions, depending on whether we do or do not allow the mean curvature or the Gauss curvature to come near zero, with respect to which the surface is bounded. Assertion **A** concerning the Gauss curvature is a weak version of a result, due to Bonnet, which has a very well-known generalization to Riemannian geometry: the Bonnet-Myers theorem.

6.3. Minkowski formulas and ovaloids

Let S be a compact surface. We will work with differentiable vector fields V on this surface, as introduced in Definition 3.1. That is, $V : S \to \mathbb{R}^3$ is a differentiable map. If $N_\varepsilon(S) \subset \mathbb{R}^3$, with $\varepsilon > 0$, is a tubular neighbourhood of the surface, we define a vector field X on it—see Definition 5.30—by extending V as a constant along the normal segments of $N_\varepsilon(S)$. That is,

$$(6.1) \qquad (X \circ F)(p,t) = X(p + tN(p)) = V(p), \qquad \forall (p,t) \in S \times (-\varepsilon, \varepsilon),$$

where the map F was introduced in (4.1). As a consequence of this definition, X is differentiable as well. We now apply to this vector field X the divergence theorem (Theorem 5.31) on the regular domain $V_a(S)$, $a \in (0, \varepsilon)$—see Remark 5.33—and obtain

$$\int_{V_a(S)} \operatorname{div} X = -\int_S \langle X, N \rangle - \int_{S_a} \langle X, N_a \rangle,$$

where N and N_a are, respectively, the inner and the outer normal fields of the surface S and of the parallel S_a at distance $a \in (0, \varepsilon)$. We use the change of variables formula in the second integral on the right-hand side for the diffeomorphism $F_a : S \to S_a$ that we already studied in Remark 4.29. Moreover, if we take into account that $N_a \circ F_a = -N$ and that $X \circ F_a = V$, we have—see also Example 5.17—

$$\int_{S_a} \langle X, N_a \rangle = -\int_S \langle V, N \rangle (1 - 2aH + a^2 K).$$

Therefore

$$\int_{V_a(S)} \operatorname{div} X = -2a \int_S \langle V, N \rangle H + a^2 \int_S \langle V, N \rangle K, \tag{6.2}$$

for all a with $0 < a < \varepsilon$. On the other hand, we will now compute the divergence of the vector field X that we have built on the tubular neighbourhood $N_\varepsilon(S)$ from the vector-field V given on the surface S. For each $p \in S$ and $t \in (-\varepsilon, \varepsilon)$, we have, from Definition (6.1),

$$(dX)_{p+tN(p)}(N(p)) = 0 \quad \text{and} \quad (dX)_{p+tN(p)}(e_i) = \frac{(dV)_p(e_i)}{1 - tk_i(p)}, \quad i = 1, 2,$$

where $\{e_1, e_2, N(p)\}$ is the basis of $\mathbb{R}^3$ formed by the unit normal of S at p and the principal directions of the surface at this point. Thus

$$(\operatorname{div} X)(p + tN(p)) = \operatorname{trace}(dX)_{p+tN(p)} = \frac{\langle (dV)_p(e_1), e_1 \rangle}{1 - tk_1(p)} + \frac{\langle (dV)_p(e_2), e_2 \rangle}{1 - tk_2(p)},$$

for all $p \in S$ and $t \in (-\varepsilon, \varepsilon)$. The left-hand side of (6.2), after using (5.1) and Fubini's Theorem, becomes

$$\int_{V_a(S)} \operatorname{div} X = \int_0^a \int_S [(\operatorname{div} X) \circ F](p, t) |\operatorname{Jac} F|(p, t) \, dp \, dt.$$

If we substitute the expression $|\operatorname{Jac} F|(p, t) = (1 - tk_1(p))(1 - tk_2(p))$ and the value of the divergence of X that we have just computed, we obtain

$$[(\operatorname{div} X) \circ F](p, t) |\operatorname{Jac} F|(p, t) = \langle (dV)_p(e_1), e_1 \rangle + \langle (dV)_p(e_2), e_2 \rangle$$

$$-t[k_2(p) \langle (dV)_p(e_1), e_1 \rangle + k_1(p) \langle (dV)_p(e_2), e_2 \rangle].$$

Incidently, this calculation shows that the functions

$$p \in S \longmapsto \langle (dV)_p(e_1), e_1 \rangle + \langle (dV)_p(e_2), e_2 \rangle,$$

$$p \in S \longmapsto k_2(p) \langle (dV)_p(e_1), e_1 \rangle + k_1(p) \langle (dV)_p(e_2), e_2 \rangle,$$

defined on the surface S from the vector field V are differentiable, where $\{e_1, e_2\}$ is an orthonormal basis of the principal directions of T_pS. The first function will be denoted by $\operatorname{div} V$ and will be called the divergence of the field V. The reason is obvious if we recall Definition 5.30 of the divergence of a vector field defined on Euclidean space. Putting all this into (6.2), it follows that

$$a\int_S \operatorname{div} V - \frac{1}{2}a^2\int_S [k_2(p)\langle (dV)_p(e_1), e_1\rangle + k_1(p)\langle (dV)_p(e_2), e_2\rangle]\,dp$$

$$= -2a\int_S \langle V, N\rangle H + a^2\int_S \langle V, N\rangle K,$$

which is valid for each $a \in (0, \varepsilon)$. Comparing coefficients in this polynomial equality, we have established the following theorem.

Theorem 6.10 (Divergence theorem for surfaces)**.** *Let S be a compact surface and let $V : S \to \mathbb{R}^3$ be a differentiable vector field defined on S. Then the following hold.*

A: $\displaystyle\int_S \operatorname{div} V = -2\int_S \langle V, N\rangle H.$

B: $\displaystyle\int_S [k_2(p)\langle (dV)_p(e_1), e_1\rangle + k_1(p)\langle (dV)_p(e_2), e_2\rangle]\,dp = -2\int_S \langle V, N\rangle K,$ *where $\{e_1, e_2\}$ is an orthonormal basis of principal directions at the point $p \in S$.*

As usual, H and k_1, k_2 stand for the mean and the principal curvature functions of the surface.

As a direct consequence we can obtain the Minkowski integral formulas; the reader should try to deduce them also from Exercise (9) of Chapter 5 and Example 5.19.

Theorem 6.11 (Minkowski formulas)**.** *Let S be any compact surface, N its inner Gauss map, and H and K its mean and Gauss curvatures. Then, we have the following two integral equalities:*

A: $\displaystyle\int_S (1 + \langle p, N(p)\rangle H(p))\,dp = 0,$

B: $\displaystyle\int_S (H(p) + \langle p, N(p)\rangle K(p))\,dp = 0.$

Proof. It suffices to apply Theorem 6.10 to the vector field $V : S \to \mathbb{R}^3$ given by $V(p) = p$ for all $p \in S$. □

From these two Minkowski formulas we can deduce, in an alternative way, the two global theorems that we established in Section 3.6: the Jellett-Liebmann and Hilbert-Liebmann results in Corollaries 3.48 and 3.49.

Corollary 6.12 (The Hilbert-Liebmann theorem)**.** *A compact connected surface with constant Gauss curvature must be a sphere.*

Proof. Let S be a surface satisfying our hypotheses. We know, by Exercise 3.42, that K is a positive constant. Then S is an ovaloid to which we can apply the Hadamard-Stoker Theorem 6.5. After doing this, we are assured that the support function based on any inner point is negative. Now, we may also suppose that the origin of $\mathbb{R}^3$ is in Ω. If not, a suitable translation would take it to this situation without changing either the hypotheses or the conclusions of the theorem. The Minkowski formulas—the second one divided by $\sqrt{K}$—give us

$$\int_S (1 + \langle p, N(p)\rangle H(p))\, dp = 0 \quad \text{and} \quad \int_S \left(\frac{H(p)}{\sqrt{K}} + \langle p, N(p)\rangle \sqrt{K}\right) dp = 0.$$

Subtracting these two equalities, we have

$$\int_S \left((1 - \frac{H(p)}{\sqrt{K}}) + \langle p, N(p)\rangle (H(p) - \sqrt{K})\right) dp = 0.$$

But, since $H^2 \geq K$, we obtain that either $H(p) \geq \sqrt{K}$ or $H(p) \leq -\sqrt{K}$, for all $p \in S$. The correct alternative is the first one, as Exercise (4) of Chapter 4 shows. Therefore

$$1 - \frac{H(p)}{\sqrt{K}} \leq 0 \quad \text{and} \quad (H(p) - \sqrt{K})\langle p, N(p)\rangle \leq 0$$

at each point p of S. Then, the integrand above is non-positive and has vanishing integral. Using Proposition 5.6, we infer that the integrand is zero. This implies that $H^2 = K$ and then S is totally umbilical. Thus, to complete the proof, apply Corollary 3.31. □

Corollary 6.13 (Jellett's theorem)**.** *The only compact connected surfaces that are star-shaped with respect to a given point and have constant mean curvature are the spheres centred at this point.*

Proof. If S is a compact connected surface and it is star-shaped with respect to, say, the origin of $\mathbb{R}^3$, then $0 \in \Omega$, where Ω is the inner domain of S. Otherwise, from Exercise (9) of Chapter 6, there would exist some line tangent to S passing through the origin. Since the mean curvature H of S is constant, we write the Minkowski formulas, the first one multiplied by the constant H, and we have

$$\int_S (H + H^2\langle p, N(p)\rangle)\, dp = 0 \quad \text{and} \quad \int_S (H + K(p)\langle p, N(p)\rangle)\, dp = 0.$$

Subtracting the two equalities, we get

$$\int_S (H^2 - K(p))\langle p, N(p)\rangle\, dp = 0.$$

But the integrand is non-positive, from inequality (3.6) and Exercise (4) of Chapter 3. We conclude the proof as we did in the previous result. □

Exercise 6.14. ↑ Use the Minkowski formulas to show that, if the Gauss and mean curvatures coincide everywhere on an ovaloid, then it is necessarily a unit sphere.

Exercise 6.15. ↑ Manipulate the Minkowski formulas to prove that an ovaloid such that $HK = 1$ has to be a unit sphere.

6.4. The Alexandrov theorem

The study and characterization of compact surfaces with constant mean curvature are typically more involved than the case of constant Gauss curvature. The Alexandrov theorem solves this problem completely. As we did in the case of the Gauss curvature, we will examine the case in which this curvature does not change its sign before considering the problem of constant mean curvature. It is worth noting that the fact that this sign is either positive or negative is not significant, but it depends on the orientation that we have chosen on the surface. In any case, if the mean curvature does not vanish anywhere when we take the inner normal field on a compact surface, then it must be positive, as Exercise (4) of Chapter 4 shows.

Theorem 6.16 (The Heintze-Karcher inequality). *Let S be a compact surface whose (inner) mean curvature H is positive everywhere. Then*

$$\operatorname{vol}\Omega \leq \frac{1}{3}\int_S \frac{1}{H(p)}\,dp,$$

where Ω is the inner domain determined by S. Moreover, equality holds if and only if S is a sphere.

Proof. Define a subset A of $S \times \mathbb{R}$ by

$$A = \{(p,t) \in S \times \mathbb{R} \,|\, 0 \leq t \leq \frac{1}{k_2(p)}\},$$

where k_2 is the largest principal curvature of S corresponding to the inner normal which, since $k_1 \leq H \leq k_2$, is positive everywhere. If we denote by $a > 0$ a real number larger than the maximum on S of the continuous function $1/k_2$, then A is a compact set contained in $S \times [0,a)$. Exercise (8) of Chapter 4 says that the inner domain Ω determined by S satisfies

$$\Omega \subset F(A),$$

where $F : S \times \mathbb{R} \to \mathbb{R}^3$ is the map defined in (4.1) by $F(p,t) = p + tN(p)$ for all $p \in S$ and $t \in \mathbb{R}$. Now, apply the area formula of Theorem 5.27 on $S \times (0,a)$ to the map F and the characteristic function χ_A of the set A to obtain the integral equality

$$\int_{\mathbb{R}^3} n(F,\chi_A) = \int_{S\times(0,a)} \chi_A\,|\operatorname{Jac} F|.$$

Clearly $n(F, \chi_A) \geq 1$ on $F(A) \supset \Omega$ and, then, also using Fubini's theorem, we have

$$\operatorname{vol}\Omega \leq \int_S \left(\int_0^a \chi_A(p,t) \, |\operatorname{Jac} F|(p,t) \, dt \right) dp.$$

We now substitute the absolute value of the Jacobian of F and take into account the definition of the set A. Thus we have

$$\operatorname{vol}\Omega \leq \int_S \left(\int_0^{1/k_2(p)} |(1 - tk_1(p))(1 - tk_2(p))| \, dt \right) dp.$$

But when $0 \leq t \leq 1/k_2(p)$, the previous integrand is non-negative. Moreover, using inequality (3.6), we see that

$$|(1 - tk_1(p))(1 - tk_2(p))| = 1 - 2tH(p) + t^2 K(p) \leq (1 - tH(p))^2$$

and equality is attained only at the umbilical points. Hence, since $1/k_2(p) \leq 1/H(p)$ for each $p \in S$ and since the function $(1 - tH(p))^2$ is non-negative,

$$\operatorname{vol}\Omega \leq \int_S \left(\int_0^{1/H(p)} (1 - tH(p))^2 \, dt \right) dp,$$

with equality only if the surface S is totally umbilical. The proof is completed by computing the integral on the right-hand side and recalling Corollary 3.31, which classified closed totally umbilical surfaces. □

The Heintze-Karcher inequality that we have just proved, applied to the case of surfaces of constant mean curvature, and the first Minkowski formula supply complementary information. They can be combined to yield one of the most beautiful theorems of classical differential geometry.

Theorem 6.17 (Alexandrov's theorem)**.** *If a compact connected surface has constant mean curvature, then it is a sphere.*

Proof. If S is a surface satisfying the conditions of the theorem, then Theorem 6.16 implies that $\operatorname{vol}\Omega \leq A(S)/3H$, where Ω is the inner domain of S and $H > 0$ is the value taken by the mean curvature relative to the inner normal. Moreover, if equality occurs, then S is a sphere. But, indeed, equality does occur, since in our case the first Minkowski formula of Theorem 6.11 gives

$$A(S) + H \int_S \langle p, N(p) \rangle \, dp = 0.$$

It is enough to note that

$$\int_S \langle p, N(p) \rangle \, dp = -3 \operatorname{vol}\Omega,$$

according to Remark 5.32. □

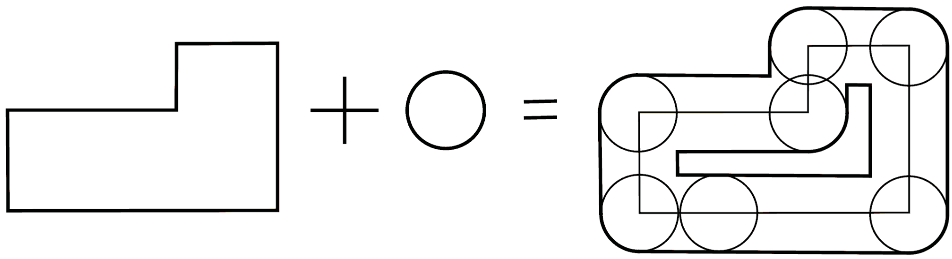

Figure 6.5. *Sum of sets*

6.5. The isoperimetric inequality

Given a compact connected surface S in Euclidean space, it makes sense, after Chapter 4 where we proved the Jordan-Brouwer theorem, to talk about the volume enclosed by S and, after Chapter 5, about the area of the surface S. Thus, it is natural in this context to pose the isoperimetric question, which can also be considered in the case of plane curves (see Chapter 9): Among all the compact surfaces enclosing a given volume, which has the least area? To give an answer, we will first establish a useful geometrical property of the Lebesgue integral on $\mathbb{R}^3$: the Brunn-Minkowski inequality.

If A and B are two arbitrary subsets of Euclidean space $\mathbb{R}^n$, we define its *sum* $A+B$ as the set

$$A+B=\{a+b\,|\,a\in A,\ b\in B\}.$$

For example, if A is the open interval (a_1,a_2) of $\mathbb{R}$ and B is the open interval (b_1,b_2) of $\mathbb{R}$, the sum is

$$A+B=(a_1,a_2)+(b_1,b_2)=(a_1+a_2,b_1+b_2).$$

The following exercises, which point out some of the properties of this sum of sets, will be used in what follows.

Exercise 6.18. ↑ Let $A,B\subset\mathbb{R}^n$. If at least one of these subsets is open, prove that the sum $A+B$ is open as well.

Exercise 6.19. Suppose that $A,B\subset\mathbb{R}^n$ are two bounded subsets of Euclidean space. Show that $A+B$ is bounded.

Exercise 6.20. ↑ Let I_1,I_2,I_3 and J_1,J_2,J_3 be six bounded open intervals of the real line. The two subsets of $\mathbb{R}^3$, given by

$$A=I_1\times I_2\times I_3\quad\text{and}\quad B=J_1\times J_2\times J_3,$$

are called *parallelepipeds with edges parallel to the coordinate axes.* Prove that

$$A+B=(I_1+J_1)\times(I_2+J_2)\times(I_3+J_3)$$

and that, if A and B are disjoint, there exists a plane parallel to one of the coordinate planes separating A and B.

Exercise 6.21. If A and B are two arc-wise connected subsets of $\mathbb{R}^3$, prove that the sum $A+B$ is also an arc-wise connected subset.

Theorem 6.22 (The Brunn-Minkowski inequality)**.** *Let A and B be two bounded open subsets of Euclidean space $\mathbb{R}^3$. Then*

$$(\operatorname{vol} A)^{1/3} + (\operatorname{vol} B)^{1/3} \leq (\operatorname{vol}(A+B))^{1/3}.$$

Proof. Suppose first that A and B are the parallelepipeds $I_1 \times I_2 \times I_3$ and $J_1 \times J_2 \times J_3$, respectively, where the I_i and the J_i, $i = 1, 2, 3$, are bounded open intervals of $\mathbb{R}$ with lengths a_i and b_i. Then

$$\frac{(\operatorname{vol} A)^{1/3} + (\operatorname{vol} B)^{1/3}}{(\operatorname{vol}(A+B))^{1/3}} = \frac{\left(\prod_{i=1}^3 a_i\right)^{1/3} + \left(\prod_{i=1}^3 b_i\right)^{1/3}}{\left(\prod_{i=1}^3 (a_i + b_i)\right)^{1/3}},$$

and thus

$$\frac{(\operatorname{vol} A)^{1/3} + (\operatorname{vol} B)^{1/3}}{(\operatorname{vol}(A+B))^{1/3}} = \left(\prod_{i=1}^3 \frac{a_i}{a_i + b_i}\right)^{1/3} + \left(\prod_{i=1}^3 \frac{b_i}{a_i + b_i}\right)^{1/3}.$$

But the geometrical mean of three numbers is always less than or equal to their arithmetical mean. Hence

$$\frac{(\operatorname{vol} A)^{1/3} + (\operatorname{vol} B)^{1/3}}{(\operatorname{vol}(A+B))^{1/3}} \leq \frac{1}{3}\sum_{i=1}^3 \frac{a_i}{a_i + b_i} + \frac{1}{3}\sum_{i=1}^3 \frac{b_i}{a_i + b_i} = 1.$$

So, in this particular case, the inequality claimed in the theorem is verified.

As a second case, suppose now that the bounded open sets A and B are disjoint finite unions of bounded open parallelepipeds whose edges are parallel to the coordinate axes. That is,

$$A = \bigcup_{i=1}^n A_i \quad \text{and} \quad B = \bigcup_{j=1}^m B_j,$$

where A_i and B_j are as in the first case above. We are going to prove the Brunn-Minkowski inequality for this kind of set using induction on the total number $n+m$ of parallelepipeds involved. It is clear that the result is true if $n+m = 2$, since this is the particular case above. Assume that $n+m \geq 3$ and that the desired inequality is true for disjoint finite unions of bounded open parallelepipeds with edges parallel to the coordinate axes and total number less than $n+m$. Since $n+m > 2$, we have either $n > 1$ or $m > 1$. Suppose the first case. From Exercise 6.20, there exists a plane P parallel to the coordinate axes separating A_1 and A_2. Let P^+ and P^- be the two open half-spaces in which P divides $\mathbb{R}^3$ and let $A^+ = A \cap P^+$ and $A^- = A \cap P^-$.

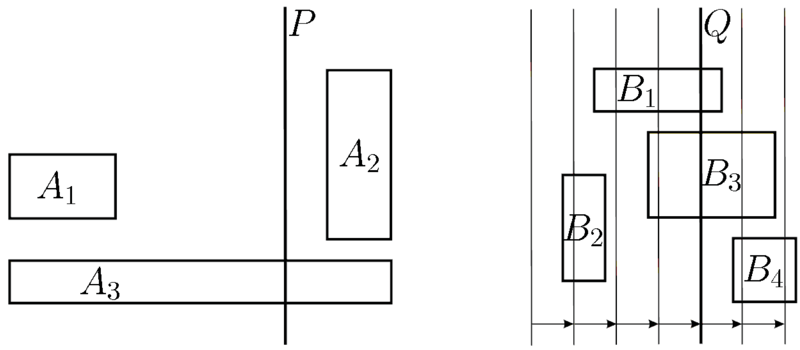

Figure 6.6. *Choice of P and Q*

Then A^+ and A^- are also finite unions of parallelepipeds whose faces are parallel to the coordinate planes, namely

$$A^+ = \bigcup_{i=1}^{n^+} A_i^+ \quad \text{and} \quad A^- = \bigcup_{i=1}^{n^-} A_i^-,$$

where $n^+ < n$ and $n^- < n$, since P separates at least A_1 and A_2. On the other hand, we can find another plane Q parallel to P such that

$$\frac{\operatorname{vol} A^+}{\operatorname{vol} A} = \frac{\operatorname{vol} B^+}{\operatorname{vol} B}, \tag{6.3}$$

where we have used the same notation as before. The reason is that, as soon as P is chosen, the fraction on the left-hand side is a number between 0 and 1 and, if one takes Q on the left of all the blocks of B and displaces it to the right until it surpasses all of them, then the fraction on the right side is a continuous function of the position of Q taking values from 0 to 1; see Figure 6.6.

In this way, since $\operatorname{vol} A^+ + \operatorname{vol} A^- = \operatorname{vol} A$ and $\operatorname{vol} B^+ + \operatorname{vol} B^- = \operatorname{vol} B$, we obtain

$$\frac{\operatorname{vol} A^-}{\operatorname{vol} A} = \frac{\operatorname{vol} B^-}{\operatorname{vol} B}. \tag{6.4}$$

Moreover, B^+ and B^- are also disjoint finite unions of open parallelepipeds with faces parallel to the coordinate planes

$$B^+ = \bigcup_{j=1}^{m^+} B_j^+ \quad \text{and} \quad B^- = \bigcup_{j=1}^{m^-} B_j^-,$$

where $m^+ \leq m$ and $m^- \leq m$, because we cannot suppose that Q separates two blocks of B. Hence we can apply the induction hypothesis to the pairs

A^+, B^+ and A^-, B^- since the total numbers of blocks satisfy $n^+ + m^+ < n + m$ and $n^- + m^- < n + m$, respectively. As a consequence,

$$\operatorname{vol}(A^+ + B^+) \geq [(\operatorname{vol} A^+)^{1/3} + (\operatorname{vol} B^+)^{1/3}]^3,$$
$$\operatorname{vol}(A^- + B^-) \geq [(\operatorname{vol} A^-)^{1/3} + (\operatorname{vol} B^-)^{1/3}]^3. \tag{6.5}$$

On the other hand, since $A^+ \subset P^+$ and $B^+ \subset Q^+$, then $A^+ + B^+ \subset P^+ + Q^+ = (P + Q)^+$ and, analogously, $A^- + B^- \subset (P + Q)^-$. Keeping in mind that $P + Q$ is another plane of $\mathbb{R}^3$, we conclude that $A^+ + B^+$ and $A^- + B^-$ are disjoint. Thus, taking (6.5) into account,

$$\begin{aligned}\operatorname{vol}(A + B) &\geq \operatorname{vol}(A^+ + B^+) + \operatorname{vol}(A^- + B^-)\\ &\geq [(\operatorname{vol} A^+)^{1/3} + (\operatorname{vol} B^+)^{1/3}]^3 + [(\operatorname{vol} A^-)^{1/3} + (\operatorname{vol} B^-)^{1/3}]^3.\end{aligned}$$

But, from (6.3) and (6.4),

$$\begin{aligned}\operatorname{vol}(A + B) &\geq \operatorname{vol} A^+ \left[1 + \left(\frac{\operatorname{vol} B}{\operatorname{vol} A}\right)^{1/3}\right]^3 + \operatorname{vol} A^- \left[1 + \left(\frac{\operatorname{vol} B}{\operatorname{vol} A}\right)^{1/3}\right]^3\\ &\geq \operatorname{vol} A \left[1 + \left(\frac{\operatorname{vol} B}{\operatorname{vol} A}\right)^{1/3}\right]^3 = [(\operatorname{vol} A)^{1/3} + (\operatorname{vol} B)^{1/3}]^3,\end{aligned}$$

and, thus, the induction is complete and the result is also true for this second particular case.

Now consider the general case. For this, denote by A and B any two bounded open sets in $\mathbb{R}^3$. Using the elementary theory of Lebesgue integration, there are two sequences A_n and B_n, $n \in \mathbb{N}$, of open sets in $\mathbb{R}^3$ of the type that we have considered in the second particular case, such that $A_n \subset A$ and $B_n \subset B$ and

$$\lim_{n\to\infty} \operatorname{vol} A_n = \operatorname{vol} A \quad \text{and} \quad \lim_{n\to\infty} \operatorname{vol} B_n = \operatorname{vol} B.$$

Hence $A_n + B_n \subset A + B$, for every $n \in \mathbb{N}$, and so

$$\operatorname{vol}(A + B)^{1/3} \geq \operatorname{vol}(A_n + B_n)^{1/3} \geq (\operatorname{vol} A_n)^{1/3} + (\operatorname{vol} B_n)^{1/3},$$

for every $n \in \mathbb{N}$. Taking limits as n tends to infinity, we obtain the Brunn-Minkowski inequality, which is what we wanted to prove. □

Now, given a compact connected surface, we look for a natural situation where we can apply the Brunn-Minkowski inequality. In fact, take $\varepsilon > 0$ sufficiently small so that $N_\varepsilon(S)$ is a tubular neighbourhood. For every $t \in (0, \varepsilon)$, the parallel surface S_t at oriented distance t is in the inner domain Ω determined by S. Hence, we also have $\Omega_t \subset \Omega$, where Ω_t is the inner domain determined by S_t; see Remark 5.33. Denote by B_t^3 the open ball of radius t centred at the origin of $\mathbb{R}^3$. Then, if $p \in (\Omega_t + B_t^3) \cap S$, we have $p = q + v$ with $q \in \Omega_t$ and $v \in B_t^3$. Thus $|p - q| = |v| < t$, and hence $\operatorname{dist}(q, S) < t$. But this is impossible since the minimum of the square of the distance function from the points of S to q has to be attained at some point $p_0 \in S$ such that q is on the normal line of S at p_0 and, since $q \in \Omega_t$, we have $|p_0 - q| \geq t$. Thus, the sum set $\Omega_t + B_t^3$, which is connected by Exercise 6.21 above, is included in one of the two connected components of $\mathbb{R}^3 - S$. Since $\Omega_t \subset \Omega$, we finally have that

$$\Omega_t + B_t^3 \subset \Omega, \qquad \forall t \in (0, \varepsilon).$$

Now applying the Brunn-Minkowski inequality that we have just proved, it can be seen that

$$\operatorname{vol}\Omega \geq \operatorname{vol}(\Omega_t + B_t^3) \geq [(\operatorname{vol}\Omega_t)^{1/3} + (\operatorname{vol} B_t^3)^{1/3}]^3,$$

for each $t \in (0, \varepsilon)$. Developing the cube on the right-hand side and removing the terms with exponent greater than 1/3 for the volume of the ball B_t^3, we conclude that

$$\operatorname{vol}\Omega \geq \operatorname{vol}\Omega_t + 3(\operatorname{vol} B_t^3)^{1/3}(\operatorname{vol}\Omega_t)^{2/3}, \qquad \forall t \in (0, \varepsilon).$$

Taking into account that $\operatorname{vol} B_t^3 = (4\pi/3)t^3$, as shown in Remark 5.32, we get

$$\frac{\operatorname{vol}\Omega - \operatorname{vol}\Omega_t}{t} \geq 3\left(\frac{4\pi}{3}\right)^{1/3} (\operatorname{vol}\Omega_t)^{2/3}, \qquad \forall t \in (0, \varepsilon).$$

In Example 5.19 we computed the volume $\operatorname{vol}\Omega_t$ enclosed by the parallel surface S_t. Then, taking the limit in the previous inequality as t goes to zero, we have

$$A(S) \geq 3\left(\frac{4\pi}{3}\right)^{1/3} (\operatorname{vol}\Omega)^{2/3}.$$

This inequality involving the area of a compact connected surface and the volume of its inner domain is the *isoperimetric inequality*, which can also be written as

$$\frac{A(S)^3}{(\operatorname{vol}\Omega)^2} \geq 36\pi = \frac{A(\mathbb{S}^2)^3}{(\operatorname{vol} B^3)^2}. \tag{6.6}$$

That is, the function $S \mapsto A(S)^3/(\operatorname{vol}\Omega)^2$, defined for each compact connected surface S, attains its minimum at any sphere. But the proof that we have just given does not permit us to show that these are indeed the only minima. However, we can do this in a small roundabout way.

For each differentiable function $f : S \to \mathbb{R}$ we can find a number $\delta > 0$ such that, if $|t| < \delta$, then $tf(S) \subset (-\varepsilon, \varepsilon)$, where $\varepsilon > 0$ is chosen so that $N_\varepsilon(S)$ is a tubular neighbourhood. In this way, it makes sense to consider the set

$$S_t(f) = \{x \in N_\varepsilon(S) \mid x = p + tf(p)N(p),\ p \in S,\ |t| < \delta\},$$

which, according to Exercise (3) of Chapter 4, is a compact surface such that $S_0(f) = S$, for any f. The family of surfaces $S_t(f)$ with $|t| < \delta$ will be called the *variation* of S corresponding to the function f. Note that each $S_t(f)$ is diffeomorphic to S under the map $\phi_t : S \to S_t(f)$, given by

$$\phi_t(p) = p + tf(p)N(p) = F(p, tf(p)), \qquad \forall p \in S,$$

where $F : S \times \mathbb{R} \to \mathbb{R}^3$ is, as usual, the map defined in (4.1), which is injective and which has as an inverse the projection π of the tubular neighbourhood $N_\varepsilon(S)$ restricted to $S_t(f)$; see Exercise (1) at the end of Chapter 4. The following two examples, which are studied by using the change of variables formula, the divergence theorem, and the differentiable dependence of parameters, will help us see what happens in the case of equality in the isoperimetric inequality.

Example 6.23 (First variation of area)**.** Let $S_t(f)$, $|t| < \delta$, for some $\delta > 0$, be the variation of the surface S corresponding to the function $f : S \to \mathbb{R}$. We will show that the function

$$t \in (-\delta, \delta) \longmapsto A(t) = A(S_t(f))$$

is differentiable and that

$$\left.\frac{d}{dt}\right|_{t=0} A(S_t(f)) = -2\int_S f(p)H(p)\,dp.$$

In fact, if e_1, e_2 are the principal directions of S at the point $p \in S$, we have

$$(d\phi_t)_p(e_i) = (1 - tf(p)k_i(p))e_i + t(df)_p(e_i)N(p), \qquad i = 1, 2.$$

Then, we see that

$$(6.7)\quad (d\phi_t)_p(e_1) \wedge (d\phi_t)_p(e_2) = [1 - 2tf(p)H(p)]N(p) - t(\nabla f)_p + t^2 G(p,t),$$

where $p \in S$, $|t| < \delta$, ∇f is the gradient field of the function f—see Exercise (17) at the end of this chapter—and G is a certain differentiable function defined on $S \times (-\delta, \delta)$ taking values in $\mathbb{R}^3$. Thus, since from the change of variables formula

$$A(t) = A(S_t(f)) = \int_{S_t(f)} 1 = \int_S |\mathrm{Jac}\,\phi_t|,$$

equation (6.7) says, keeping in mind the differentiable dependence of parameters of Remark 5.22, that $A(t)$ is differentiable at $t = 0$ and

$$A'(0) = \int_S \left.\frac{d}{dt}\right|_{t=0} |\mathrm{Jac}\,\phi_t| = \int_S \left.\frac{d}{dt}\right|_{t=0} |(d\phi_t)_p(e_1) \wedge (d\phi_t)_p(e_2)|\,dp.$$

Therefore, using (6.7), we have

$$\left.\frac{d}{dt}\right|_{t=0} |(d\phi_t)_p(e_1) \wedge (d\phi_t)_p(e_2)| = -2f(p)H(p).$$

From this we can easily obtain the required formula.

Example 6.24 (First variation of volume)**.** Let $\Omega_t(f)$ denote the inner domain determined by the compact surface $S_t(f)$. We will prove that the function

$$t \in (-\delta, \delta) \longmapsto V(t) = \operatorname{vol} \Omega_t(f)$$

is differentiable and that

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{vol} \Omega_t(f) = -\int_S f(p)\, dp.$$

For this, recall that

$$V(t) = -\frac{1}{3}\int_{S_t} \langle N_t(p), p\rangle\, dp,$$

by Remark 5.32, where N_t is the inner normal of $S_t(f)$. Again using the change of variables formula for the diffeomorphism ϕ_t, we obtain

$$V(t) = -\frac{1}{3}\int_S \langle N_t \circ \phi_t, \phi_t\rangle |\operatorname{Jac} \phi_t|.$$

Since the diffeomorphism ϕ_t between S and $S_t(f)$ is the identity map when $t = 0$, we have

$$(N_t \circ \phi_t)(p) = \frac{(d\phi_t)_p(e_1) \wedge (d\phi_t)_p(e_2)}{|(d\phi_t)_p(e_1) \wedge (d\phi_t)_p(e_2)|} = \frac{(d\phi_t)_p(e_1) \wedge (d\phi_t)_p(e_2)}{|\operatorname{Jac} \phi_t|(p)}$$

and thus, from (6.7), we have

$$V(t) = -\frac{1}{3}\int_S \langle (d\phi_t)_p(e_1) \wedge (d\phi_t)_p(e_2), \phi_t(p)\rangle\, dp$$

$$= -\frac{1}{3}\int_S tf(p)\, dp - -\frac{1}{3}\int_S [(1 - 2tf(p)H(p))\langle N(p), p\rangle - t\langle (\nabla f)_p, p\rangle]\, dp$$

$$-\frac{t^2}{3}\int_S D(p,t)\, dp,$$

where D is a differentiable function defined on $S \times (-\delta, \delta)$ and ∇f again denotes the gradient of f; see Exercise (17) at the end of this chapter. Hence V is differentiable at $t = 0$ and

$$V'(0) = \frac{1}{3}\int_S [-f(p) + 2f(p)H(p)\langle N(p), p\rangle + \langle (\nabla f)_p, p\rangle]\, dp.$$

Now, assuming we have solved Exercise (20) at the end of this chapter, we have that

$$2\int_S f(p)\langle p, N(p)\rangle H(p)\, dp = -\int_S [2f(p) + \langle (\nabla f)_p, p\rangle]\, dp.$$

Substituting this into the previous equality, we get the integral formula that we called the first variation of volume formula.

Theorem 6.25 (Space isoperimetric inequality)**.** *Let S be a compact connected surface. Then $A(S)^3 \geq 36\,\pi\,(\mathrm{vol}\,\Omega)^2$, where Ω is its inner domain. Moreover, equality occurs if and only if S is a sphere.*

Proof. The inequality was already proved in (6.6). Furthermore, we know that equality is satisfied for spheres of arbitrary radius. If S is a surface that attains equality and $f : S \to \mathbb{R}$ is any differentiable function, we consider the variation $S_t(f)$ of S relative to f, which is defined for $|t| < \delta$ with $\delta > 0$. Then, the function $h : (-\delta, \delta) \to \mathbb{R}$, given by

$$h(t) = A(S_t(f))^3 - 36\,\pi\,(\mathrm{vol}\,\Omega_t(f))^2,$$

is differentiable from the above examples and has a minimum at $t = 0$ by our hypothesis. Consequently,

$$0 = h'(0) = 3A(S)^2 \left.\frac{d}{dt}\right|_{t=0} A(S_t(f)) - 72\,\pi\,\mathrm{vol}\,\Omega \left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}\,\Omega_t(f).$$

That is, if S attains equality in the isoperimetric inequality, one has, using the first two variation formulas computed earlier, that

$$\int_S f(p)[12\,\pi\,\mathrm{vol}\,\Omega - A(S)^2 H(p)]\,dp = 0$$

for any differentiable function f defined on S. For instance, choosing f as

$$f(p) = 12\,\pi\,\mathrm{vol}\,\Omega - A(S)^2 H(p), \qquad \forall p \in S,$$

we deduce that f itself must be identically zero. In conclusion, the surface S has constant mean curvature and we complete the proof by applying Alexandrov's Theorem 6.17. □

Exercises

(1) Let S be an ovaloid in $\mathbb{R}^3$. Show that

$$\int_S H^2 \geq 4\pi$$

and that equality occurs if and only if S is a sphere.

(2) ↑ Suppose that there exists some $c > 0$ such that

$$cA(S) \leq \left(\int_S H(p)\,dp\right)^2$$

for all ovaloids S and that there exists an ovaloid that attains equality. Using the parallel surfaces of S, prove that $c = 4\pi$.

(3) ↑ Let S be an ovaloid and $N : S \to \mathbb{S}^2$ its Gauss map. Show that there is a unique diffeomorphism $\phi : S \to S$ such that $N \circ \phi = -N$. If $|\phi(p) - p|$ is the constant function $c \in \mathbb{R}$ on S, we say that the ovaloid has constant *width* c. Prove that, in this case, $k_i \circ \phi = k_i/(1 - ck_i)$, $i = 1, 2$.

(4) ↑ Let S be a compact connected surface and let N be a Gauss map for S. Let s and k_1, k_2 be the corresponding support function based on the origin and the principal curvatures. If there is a real number $c \in \mathbb{R} - \{0\}$ such that neither s takes the value c nor k_1, k_2 the value $-1/c$, prove—applying the *monodromy theorem*, which asserts that a local diffeomorphism from a compact surface to a sphere has to be a diffeomorphism, to the map $p \in S \mapsto (p - cN(p))/|p - cN(p)| \in \mathbb{S}^2$—that S is diffeomorphic to the sphere.

(5) ↑ Let S be a compact connected surface contained in an open ball of radius $r > 0$. If its principal curvatures satisfy $k_1, k_2 > -1/r$, then S is diffeomorphic to the sphere. In particular, prove that a compact connected surface whose principal curvatures are non-negative everywhere must be diffeomorphic to a sphere.

(6) ↑ Let S be an ovaloid and let R be a line in $\mathbb{R}^3$. Prove that $S \cap R = \emptyset$, or $S \cap R = \{p\}$ and R is tangent to S at p, or $S \cap R$ has exactly two points.

(7) Show that the following integral formulas are always valid:

$$\int_S \langle N, a\rangle H = 0 \quad \text{and} \quad \int_S \langle N, a\rangle K = 0,$$

where S is a compact surface, N its Gauss map, H and K its mean and Gauss curvatures, and $a \in \mathbb{R}^3$ an arbitrary vector.

(8) ↑ Let S be a compact surface contained in a closed ball of radius $r > 0$ and such that its mean curvature satisfies $|H| \leq 1/r$. Prove that S is a sphere of radius r.

(9) ↑ Let S be an ovaloid and let $F_t : S \to \mathbb{R}^3$ be the map given by $F_t(p) = p + tN(p)$, where N is the outer Gauss map of S. Show that, for each $t > 0$, the image $S_t = F_t(S)$ is an ovaloid and, moreover, $F_t : S \to S_t$ is a diffeomorphism. (Use the monodromy theorem—see Exercise (4)—for the map $\phi = F_t/|F_t|$.)

(10) ↑ Use the isoperimetric inequality to prove that on each ovaloid S we have

$$\left(\int_S H(p)\, dp\right)^2 \geq 4\pi A(S).$$

(11) ↑ Let S be a compact connected surface with $K \geq 1$. If one can find an open unit ball inside the inner domain of S, then S is a unit sphere.

(12) $\uparrow$ An ovaloid whose (inner) mean and Gauss curvatures satisfy $H+K=2$ must be a unit sphere.

(13) An ovaloid whose mean and Gauss curvatures satisfy $H^2+K^2=2$ must also be a unit sphere.

(14) $\uparrow$ Let S be an ovaloid with $H \geq 1/r$ and $K \leq H/r$, where $r>0$. Show that S is a sphere of radius r.

(15) Let S be a compact surface. Show that S has constant mean curvature if and only if, for each differentiable function f defined on S, such that $\operatorname{vol}\Omega_t(f) = \operatorname{vol}\Omega$, $|t|<\delta$, one has that

$$\left.\frac{d}{dt}\right|_{t=0} A(S_t(f)) = 0.$$

(16) Let S be a compact surface and $V : S \to \mathbb{R}^3$ a tangent vector field. Prove that

$$\int_S (\operatorname{div} V)(p)\, dp = 0,$$

$$\int_S \{k_1(p)\langle (dV)_p(e_1), e_1\rangle + k_2(p)\langle (dV)_p(e_2), e_2\rangle\}\, dp = 0,$$

where $\{e_1, e_2\}$ is a basis of principal directions at T_pS for each $p \in S$.

(17) Let $f : S \to \mathbb{R}$ be a differentiable function defined on a surface S. We use the term *gradient* of f for the vector field of tangent vectors denoted by $\nabla f : S \to \mathbb{R}^3$ and given by

$$\begin{cases} \langle (\nabla f)(p), v\rangle = (df)_p(v), \quad v \in T_pS, \\ \\ \langle (\nabla f)(p), N(p)\rangle = 0, \end{cases}$$

where $N(p)$ is a unit normal to S at p. Prove that ∇f is a differentiable vector field and that, if it is identically zero, f is constant on each connected component of S.

(18) If $f : S \to \mathbb{R}$ is a differentiable function on a surface S, we define the *Laplacian* of f by $\Delta f = \operatorname{div} \nabla f$. Prove that

$$\Delta(fg) = f\,\Delta g + g\,\Delta f + 2\,\langle \nabla f, \nabla g\rangle,$$

for any two functions f and g defined on S.

(19) $\uparrow$ Let S be a compact surface and let $f : S \to \mathbb{R}$ be a differentiable function. Prove that, if $\Delta f \geq 0$, then f is constant on each connected component of S.

(20) If $f : S \to \mathbb{R}$ is a differentiable function on a compact surface, apply the divergence theorem to the vector field V defined on S by $V(p) = f(p)p$

for each $p \in S$ and show that

$$\int_S \langle (\nabla f)_p, p\rangle \, dp + 2\int_S f(p)\, dp = -2\int_S f(p)H(p)\langle p, N(p)\rangle \, dp.$$

Hints for solving the exercises

Exercise 6.14: If $H = K$, by subtracting the two Minkowski formulas, we have

$$\int_S 1 = \int_S H.$$

On the other hand, inequality (3.6) gives us $H = K \leq H^2$ and, thus, $H \geq 1$, since $H = K > 0$. We complete the exercise by using Proposition 5.6, which concerns properties of integration.

Exercise 6.15: Subtract the two Minkowski formulas again. If $HK = 1$, we see that

$$\int_S \left(1 - H(p) + \langle N(p), p\rangle (H(p) - \frac{1}{H(p)})\right) dp = 0.$$

Again inequality (3.6) and the hypothesis $HK = 1$ imply that $H \geq 1$. To complete the exercise, use the fact that the support function based on the origin—assumed to be in Ω—is strictly negative.

Exercise 6.18: Suppose that $A \subset \mathbb{R}^n$ is open. By definition,

$$A + B = \bigcup_{b\in B} (A + b).$$

But each $A + b$, with $b \in B$, is open because translations of $\mathbb{R}^n$ are homeomorphisms. Then $A + B$ is open.

Exercise 6.20: The required equality follows directly from the definition of the sum of sets. Suppose that $A \cap B = \emptyset$. Then, at least one of the intersections $I_1 \cap J_1$, $I_2 \cap J_2$, $I_3 \cap J_3$ has to be empty. Suppose that it is the first one. Thus, there exists $a \in \mathbb{R}$ such that the intervals I_1 and J_1 are on different sides of a. Hence, the plane of the equation $x = a$ separates the two sets A and B.

Exercise (2): Let S be an ovaloid that attains equality and let S_t be its parallel surface at distance t, with t small. The function

$$t \longmapsto \left(\int_{S_t} H_t(p)\,dp\right)^2 - cA(S_t)$$

has a minimum at $t = 0$. By using Exercise (16) of Chapter 3, Theorem 5.14 on change of variables, and Example 5.17, we can explicitly compute this function, namely

$$t \longmapsto \left(\int_S H - t\int_S K\right)^2 - c\left(A(S) - 2t\int_S H + t^2\int_S K\right).$$

This function is differentiable and its derivative at $t = 0$ vanishes. Hence

$$\left(\int_S H\right)\left(c - \int_S K\right) = 0.$$

Since S is an ovaloid, $H^2 \geq K > 0$. Therefore

$$c = \int_S K = 4\pi,$$

where the last equality follows from the Hadamard-Stoker Theorem 6.5.

Exercise (3): It is clear that the required involution $\phi : S \to S$ exists since the Gauss map $N : S \to \mathbb{S}^2$ is a diffeomorphism and, thus, it is enough to put $\phi = N^{-1} \circ A \circ N$, where A is the antipodal map of the sphere. Applying the chain rule to the equality $N \circ \phi = -N$, we have

$$(*) \qquad (dN)_{\phi(p)} \circ (d\phi)_p = -(dN)_p, \qquad \forall p \in S.$$

On the other hand, if $|\phi(p) - p| = c$ for each $p \in S$ and some $c \in \mathbb{R}$, we have that $c > 0$ and

$$(**) \qquad \langle \phi(p) - p, (d\phi)_p(v) - v\rangle = 0, \qquad \forall p \in S \quad \forall v \in T_pS.$$

We now show that $(d\phi)_p - I_{T_pS}$ is trivial. In fact, if $(d\phi)_p(v) = v$ for some $v \in T_pS$, from $(*)$ we have $(dN)_{\phi(p)}(v) = -(dN)_p(v)$. After scalar multiplication by v, we see that

$$\sigma_{\phi(p)}(v, v) = -\sigma_p(v, v).$$

Since S is an ovaloid, σ is definite and, thus, $v = 0$. Then $(d\phi)_p - I_{T_pS}$ is injective and, as a consequence, its image is the whole of the plane T_pS. Equality $(**)$ can then be rewritten in another way:

$$\phi(p) - p \perp T_pS, \qquad \forall p \in S.$$

Therefore $\phi(p) - p = cN(p)$, where N is the inner normal (for the choice of sign, consult Remark 6.2). Therefore, if $e_1, e_2 \in T_pS$ are principal directions, we obtain

$$(d\phi)_p(e_i) - e_i = -ck_i(p)e_i, \qquad i = 1, 2.$$

Hence $e_1, e_2 \in T_pS = T_{\phi(p)}S$ are eigenvectors for $(dN)_{\phi(p)}$ and we are done.

Exercise (4): The map $\phi : S \to \mathbb{S}^2$, given by

$$\phi(p) = \frac{p - cN(p)}{|p - cN(p)|}, \qquad \forall p \in S,$$

is well-defined and differentiable, since $p - cN(p)$ cannot vanish anywhere because the support function does not take the value c. From the definition we obtain

$$(d\phi)_p(v) = \frac{v - c(dN)_p(v)}{|p - cN(p)|} - \frac{\langle p - cN(p), v - c(dN)_p(v)\rangle}{|p - cN(p)|^3}(p - cN(p)),$$

for each $p \in S$ and $v \in T_pS$. Hence, if $e_1, e_2 \in T_pS$ are principal directions at $p \in S$,

$$|\mathrm{Jac}\,\phi|(p) = \frac{1}{|p - cN(p)|^3}|1 + ck_1(p)||1 + ck_2(p)||\langle N(p), p\rangle - c|.$$

By our hypotheses, this Jacobian never vanishes. Thus ϕ is a local diffeomorphism by the inverse function theorem. We conclude the exercise by using the monodromy theorem cited above.

Exercise (5): If S is contained in an open ball of radius $r > 0$, we can and do assume the origin is the centre of this ball. Thus, if $p \in S$,

$$|\langle p, N(p)\rangle| \leq |p| < r.$$

Then, the support function based on the origin never takes the value r. Now apply Exercise (4) above.

Exercise (6): By Remark 6.3, if $R \cap \Omega \neq \emptyset$, then $R \cap S$ consists of exactly two points. Suppose, then, that $R \cap \Omega = \emptyset$ and that $R \cap S \neq \emptyset$. Hence $R \cap S = R \cap \overline{\Omega}$ must, by Theorem 6.1, be a non-empty closed convex subset of R, that is, a non-empty closed interval. If it had more than one point, by Exercise 3.33, S would have non-positive Gauss curvature at the interior points of this interval. Since S is an ovaloid, $R \cap S$ has only a point. Finally, apply Exercise (7) of Chapter 4 to see that R is tangent to S at this point.

Exercise (8): If the centre of the ball where S is included is the origin, we use the first Minkowski formula and obtain

$$A(S) = -\int_S \langle p, N(p)\rangle H(p)\, dp \leq \int_S |p||H(p)|\, dp \leq A(S).$$

Then we complete the proof, for example, by using Alexandrov's Theorem 6.17 or Exercise (4) of Chapter 2.

Exercise (9): The same calculation as in Exercise (4) gives us

$$\left|\operatorname{Jac}\frac{F_t}{|F_t|}\right|(p) = \frac{1}{|F_t(p)|^3}|1 - tk_1(p)||1 - tk_2(p)||\langle p, N(p)\rangle + t|,$$

which does not vanish, since $k_1, k_2 \leq 0$, and $\langle p, N(p)\rangle > 0$, $p \in S$, since N is the outer normal. The monodromy theorem implies that $\phi = F_t/|F_t|$ is a diffeomorphism for each $t \geq 0$. As a consequence, ϕ and, thus, each F_t are injective. From Exercise (9) of Chapter 2, $S_t = F_t(S)$ is a compact surface and $F_t : S \to S_t$ is a diffeomorphism. Finally, Exercise (16) of Chapter 3 asserts that its Gauss curvature is always positive.

Exercise (10): We know, after solving Exercise (9), that it makes sense to consider outer parallel surfaces S_t at arbitrary distance $t \geq 0$. Since each S_t is compact and connected, we can apply the isoperimetric inequality (6.6) to each of them and obtain

$$A(S_t)^3 \geq 36\pi(\operatorname{vol}\Omega_t)^2, \qquad \forall t \geq 0.$$

Taking into account Example 5.17 and the analogue of Example 5.19 for outer parallel surfaces, we obtain the polynomial inequality

$$\left(A(S) + 2t\int_S H + 4\pi t^2\right)^3 \geq 36\pi\left(\operatorname{vol}\Omega + tA(S) + t^2\int_S H + \frac{4\pi}{3}t^3\right)^2,$$

which is valid for each $t > 0$. Dividing by t^6 and denoting by s the inverse $1/t$, we have that the function

$$f(s) = \left(4\pi + 2s\int_S H + s^2A(S)\right)^3 - 36\pi\left(\frac{4\pi}{3} + s\int_S H + s^2A(S) + s^3\operatorname{vol}\Omega\right)^2$$

is non-negative for $s > 0$. We easily see that $f(0) = 0$ and $f'(0) = 0$. Therefore, we have

$$0 \leq f''(0) = 24\pi\left(\int_S H\right)^2 - 96\pi^2 A(S).$$

Exercise (11): By integrating the inequality $K \geq 1$, we obtain

$$A(S) \leq 4\pi = A(\mathbb{S}^2),$$

since S is an ovaloid. On the other hand, if an open unit ball B is in the inner domain Ω of S,

$$\operatorname{vol}\Omega \geq \operatorname{vol}\mathrm{B}.$$

Consequently,

$$A(S)^3 \leq A(\mathbb{S}^2)^3 = 36\pi(\operatorname{vol}\mathrm{B})^2 \leq 36\pi(\operatorname{vol}\Omega)^2,$$

which, along with the isoperimetric inequality, completes the exercise.

Exercise (12): Substituting $K = 2 - H$ into inequality (3.6), we have that $H^2 + H - 2 \geq 0$ and, thus, that $H \geq 1$ (with respect to the inner normal), with equality only at the umbilical points. Going back to the original equality, we see that $K \leq 1$, and then that $K \leq 1 \leq H$. Recalling that the support function is strictly negative, the two Minkowski formulas give

$$A(S) = -\int_S H(p)\langle p, N(p)\rangle\, dp \geq -\int_S K(p)\langle p, N(p)\rangle\, dp = \int_S H \geq A(S).$$

Therefore, S is totally umbilical.

Exercise (14): Since the support function is negative, keeping in mind the hypotheses, we have

$$\frac{1}{r} \leq H \qquad \text{and} \qquad -K(p)\langle p, N(p)\rangle \leq -\frac{1}{r}H(p)\langle p, N(p)\rangle,$$

for each $p \in S$. Integrating and using the Minkowski formulas, we have

$$\frac{1}{r}A(S) \leq \int_S H = -\int_S K(p)\langle p, N(p)\rangle\, dp \leq -\frac{1}{r}\int_S H(p)\langle p, N(p)\rangle\, dp = \frac{1}{r}A(S).$$

Thus S has constant mean curvature $1/r$ and constant Gauss curvature $1/r^2$. We can conclude the proof by referring either to Corollary 3.49 of Hilbert and Liebmann or Alexandrov's Theorem 6.17.

Exercise (19): The definition of the Laplacian and Exercise (16) above imply directly that the integral on S of Δf vanishes. Therefore, since Δf is continuous, we have $\Delta f = 0$. From Exercise (18), we have

$$\Delta f^2 = 2f\Delta f + 2|\nabla f|^2 = 2|\nabla f|^2.$$

Integrating this equality and taking into account the fact that the integral of Δf^2 is zero, we arrive at

$$\int_S |\nabla f|^2 = 0.$$

Thus $\nabla f = 0$ and f is constant on each connected component of S.