Preface

To doubt all or believe all are two equally convenient solutions, in that both dispense with thinking.

Henri Poincaré, 1854-1912

Preface to the Second Edition.

Comments from different sources, experience in using the first edition as a class text, and the opportunity to teach a preliminary course in analysis to students who would subsequently use this book all contributed to the changes in this second edition. The analysis experience resulted in *Analysis: A Gateway to Understanding Mathematics* published by World Scientific (Singapore) in 2012. While maintaining the original approach, format, and list of topics, I have revised to some extent most chapters. I found it convenient to rearrange some of the material and as a result to include an additional chapter (Chapter 9). This new chapter contains material from Chapters 6 and 7 in the first edition and, additionally, a construction of Lebesgue measure using dyadic rationals and a countable product of probability spaces.

A brief paraphrasing of essentially one paragraph of the original preface is included here to help the reader navigate the second edition. Students of financial mathematics may wish to follow, as our students did, Chapters 1-5; Sections 6.1, 6.2, 6.3 and 7.4; the statements of the main results in Sections 6.3, 6.4, and 7.5; and Chapters 8 and 10-11. Students of mathematics and statistics interested in probability theory could follow Chapters 3-7, Section 8.2 and Chapters 9 and 10. Students of mathematics could follow Chapters 3-6 and 9 as an introduction to measure theory. Chapter 12 is, modulo a modest background in probability theory, a self-contained introduction to stochastic integration and the Itô integral. Finally anyone beginning their university studies in mathematics or merely interested in modern mathematics, from a philosophical or aesthetic point of view, will find Chapters 1-5 accessible, challenging and rewarding.

It is a pleasure to thank once more Michael Mackey for all his help and patience and Sergei Gelfand for his constant encouragement.

Preface to the First Edition.

Mathematics occupies a unique place in modern society and education. It cannot be ignored and almost everyone has an opinion on its place and relevance. This has led to problems and questions that will never be solved or answered in a definitive fashion. At third level we have the perennial debate on the mathematics that is suitable for non-mathematics majors and the degree of abstraction with which it should be delivered. We mathematicians are still trusted with this task and our response has varied. Some institutions offer generic mathematics courses to all and sundry, and faculties, such as engineering and business, respond by directing their students to the courses they consider appropriate. In other institutions departments design specific courses for students who are not majoring in mathematics. The response of many departments lies somewhere in between. This can lead to tension between the professional mathematicians' attitude to mathematics and the client faculties' expectations. In the first case non-mathematics majors may find themselves obliged to accept without explanation an approach that is, in their experience, excessively abstract. In the second, a recipe-driven approach often produces students with skills they have difficulty using outside a limited number of well-defined settings. Some students, however, do arrive, by sheer endurance, at an intuitive feeling for mathematics. Clearly both extremes are unsatisfactory and it is natural to ask if an alternative approach is possible.

It is, and the difficulties to be overcome are not mathematical. The understanding of mathematics that we mathematicians have grown to appreciate and accept, often slowly and unconsciously, is not always shared by non-mathematicians, be they students or colleagues, and the benefits of abstract mathematics are not always obvious to academics from other disciplines. This is not their fault. They have, for the most part, been conditioned to think differently. They accept that mathematics is useful and for this reason are willing to submit their students to our courses. We can—and it is in our own hands, since we teach the courses—show that it is possible to *combine* abstract mathematics and good technical skills. It is not easy, it is labor intensive, and the benefits are usually not apparent in the short term. It requires patience and some unconditional support that we need to earn from our students and colleagues.

Although this book is appearing as a graduate text in mathematics, it is based on a one-semester undergraduate course given to economics and finance students at University College Dublin. It is the result of an opportunity given to the author to follow an alternative approach by mixing the abstract and the practical. We feel that all students benefited, but some were not convinced that this was indeed the case.

The students had the usual mathematical background, an acquaintance with the *techniques* of one variable differential and integral calculus and linear algebra. The aim of the course was to provide a mathematical foundation for further studies in financial mathematics, a discipline that has made enormous advances in the last twenty-five years and has been the surprise catalyst in the introduction of certain high-level mathematics courses for non-mathematics majors at universities in recent years. Even though the eventual applications are concrete, the mathematics involved is quite abstract, and as a result business students, who specialize in finance, are today exposed to more demanding mathematics than their fellow students in engineering and science. The students' motivation, background, aspirations and future plans were the constraints under which we operated, and these determined the balance between the choice of topics, the degree of abstraction and pace of the presentation.

In view of its overall importance there was no difficulty in choosing the *Black-Scholes formula* for pricing a call option as our ultimate goal. This provided a focus for the students' motivation. As the students were *not* mathematics majors but the majority would have one or two further years of mathematically oriented courses, it seemed appropriate to aim for an understanding that would strengthen their overall mathematical background. This meant it was necessary to initiate the students into what has unfortunately become for many an alien and mysterious subject, *modern abstract mathematics*. For this approach to take root the security associated with recipe-driven and technique-oriented mathematics has to be replaced by a more mature and intrinsic confidence which accepts a degree of intellectual uncertainty as part of the thinking process. Even with highly motivated students, this requires a gradual introduction to mathematical abstraction, and at the same time it is necessary to remain, for reasons of motivation, in contact with the financial situation.

Probability theory, Lebesgue integration and the Itô calculus are the main ingredients in the *Black-Scholes formula*, and these rely on set theory, analysis and an axiomatic approach to mathematics. We take, on the financial side, a first principles approach and include only the minimum necessary to justify the introduction of mathematical concepts and place in context mathematical developments. We move slowly initially and provide elementary examples at an early stage. Hopefully, this makes the apparently more difficult mathematics in later chapters more intuitive and obvious. This cultural change explains why we felt it necessary on occasion to digress into non-technical, and even psychological, matters and why we attempted to present mathematics as a living culture with a history and a future. In particular, we tried to explain the importance of properly understanding questions and recognizing situations which required justification. This helped motivate, and place in perspective, the need for clear definitions and proofs. For example, in considering the concept of a convergent sequence of real numbers, on which all stochastic notions of convergence and all theories of integration rely, we begin by assuming an intuitive concept of limit in Chapter 1; in Chapter 3 we define the limit of a bounded increasing sequence of real numbers; in Chapter 4 we define the limit of a sequence of real numbers; in Chapter 6 we use upper and lower limits to characterize limits; in Chapter 9 we use Doob's upcrossing approach to limits; and in Chapter 11 we employ subsequences to obtain an equivalent definition of limit. In all cases the different ways of considering limits of sequences of real numbers are used as an introduction to similar but more advanced concepts in probability theory.

The introduction of peripheral material, the emphasis on simple examples, the repetition of basic principles, and attention to the students' motivation all take time. The real benefits only become apparent later, both to the students and their non-mathematical academic advisors, when they, the students, proceed to mix with other students in mathematically demanding courses.

The main mathematical topics covered in this book, for which we assume no background, are all essentially within probability theory. These are *measure theory, expected values, conditional expectation, martingales, stochastic processes, Wiener processes* and the *Itô integral*. We do not claim to give a fully comprehensive treatment, and we presented, even though otherwise tempted, certain results without proof. Readers who have worked their way through this book should be quite capable of following the standard proofs in the literature of The Central Limit Theorem, The Radon-Nikodým Theorem, etc., and we hope they will be motivated to do so. Our self-imposed attempt at self-sufficiency sometimes led to awkward proofs. Although probability theory was the initial focus for our studies, we found as we progressed that more and more analysis was required. Having introduced sequences and continuous functions and proved a number of their basic properties, it did not require much effort to complete the process and present with complete proofs the fundamental properties of continuous and convex functions in Sections 7.2 and 7.6 respectively.

Different groups may benefit from reading this book. Students of financial mathematics at an early, but not too early, stage in their studies could follow, as our students did, Chapters 1-5; Sections 6.1, 6.2, 6.3 and 7.3; the statements of the main results in Sections 6.3, 6.4, and 7.5; and Chapters 8-10. Students of mathematics and statistics interested in analysis and probability theory could follow Chapters 3-7 with the option of two additional topics: the combination of Section 8.2, Chapter 9 and Section 10.3 forming one topic and Chapter 11 the other. Students of mathematics could follow Chapters 3-6 as an introduction to measure theory, while Chapter 11 is, modulo a modest background in probability theory, a self-contained introduction to stochastic integration and the Itô integral. Finally anyone beginning their university studies in mathematics or merely interested in modern mathematics, from a philosophical or aesthetic point of view, will find Chapters 1-5 accessible, challenging and rewarding.

The exercises played an important role in the course, on which we based this book. Some are easy, others difficult; many are included to clarify simple points; some introduce new ideas and techniques; a few contain deep results; and there is a high probability that some of our solutions are incorrect. However, *an hour or two attempting a problem is never a waste of time*, and *to make sure that this happened* these exercises were the focus of our small-group weekly workshops. This is a secret that we mathematicians all too often keep to ourselves. Mathematics is an active discipline, progress *cannot* be achieved by passive participation, and with sustained active participation progress *will* be achieved.

It is a pleasure to see this book, written for undergraduate non-mathematics majors, appearing in a series devoted to graduate studies in mathematics. I greatly appreciate the support and encouragement that I received from the editorial staff of the *American Mathematical Society.* In particular, I would like to thank Sergei Gelfand, for being so positive and helpful, and Deborah Smith, for her suggestions and impressive proof-reading.

Dan Golden, from the Department of Banking and Finance at University College Dublin was the main inspiration behind this book. He set up the degree programme in economics and finance, and his constant advice, insight and encouragement were an enormous help over the last five years. It is a pleasure to thank Shane Whelan for numerous conversations on all matters connected with this book, Maciej Klimek for his interesting and constructive suggestions, Michael Mackey for his mathematical insight and excellent diagrams, Milne Anderson for encouragement and perspective over many years and Chris Boyd for his suggestions. Maria Meehan, Louis Murray, Cora Stack, Silvia Lassalle and David Horan helped more than they think. I would especially like to thank the students of economics and finance at University College Dublin, who were subjected to many pre-book versions of this material and who, by their questions

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