Money and Markets

There are very few things which we know, which are not capable of being reduc'd to a Mathematical Reasoning: and when they cannot, it's a sign our knowledge of them is very small and confus'd; and where a mathematical reasoning can be had, it's as great a folly to make use of any other, as to grope for a thing in the dark, when you have a candle standing by you.

John Arbuthnot, 1692,
Preface, Of the Laws of Chance

Summary

We give an extremely basic introduction to the financial markets and use some simple mathematics to examine interest rates.

1.1. Introduction

In this book we lay the mathematical foundations necessary to model certain transactions in the world of finance. Our goal is to provide a complete self-contained mathematical background to the Black-Scholes formula for pricing a call option. This involves two cultures, mathematics and finance, each having its own internal intuitions, concepts, rules and language. In finance, we confine ourselves to the minimal background necessary to achieve our purpose. This involves concepts such as interest rates, present worth or value, discounted value, hedging, risk, bonds, stocks, shares, options, expected return and arbitrage. In the first two chapters we explore these concepts and begin the process of interpreting them mathematically. To illustrate certain points we use examples, artificial from a finance perspective, but as we progress we make them more realistic.

We suppose the reader has some acquaintance with the techniques of one variable differential and integral calculus. All other mathematics required, for example, set theory, integration theory and probability theory, are developed *ab initio* as we proceed. History shows that intuition generally precedes rigor in mathematics, and, guided by this principle, we adopt an intuitive approach in the first two chapters. Afterwards we introduce the necessary rigorous mathematical definitions and provide proofs. The mathematical examples given are often elementary and are provided to improve our *understanding* of basic concepts. Complicated mathematical formulae and equations often turn out to be nothing more than clever combinations of simple well-known mathematical facts.

1.2. Money

In ancient times trade was conducted by exchanging goods, a system known as *bartering*. To simplify this process a fixed amount of a single commodity, often silver or gold, was chosen as a *unit of value* and goods were valued in units of this standard. We call this standard *money.*[1] Silver and gold are maintenance free and easily divided and thus suitable choices. Life would have been more complicated if the unit chosen had been a live chicken. Money's original role as a *medium of exchange* led to the separation of the acts of buying and selling, and it assumed a further role as a *store of value* as people realized its potential to be used *when* it suited them. Thus began the relationship between *money* and *time*.

When prices are stable, those with money feel financially secure. However, prices do change depending on *supply* and *demand*. The *rate of change over time* in the price of a commodity or a number of commodities is called *inflation.* If product A cost \$10 this time last year while today it costs \$12, then the percentage increase in price over the year is $\frac{12-10}{10} \times 100\% = 20\%$ and product A has a 20% annual rate of inflation. The inflation rate for a country is obtained by taking the weighted average of a basket of goods in the overall economy. If we call the *real*, in contrast to the *nominal*, value of money what it is capable of buying, then the presence of inflation means that the real value of money is a *function of time.*

[1] Similar to the way we have developed standard units of measurement for distance, temperature, land, etc.

Inflation is a problem for those with money. In its absence they can estimate their financial obligations and requirements. The presence of inflation reduces their financial security and forces them to confront an intrinsic problem: *how to maintain the future real value of money?* Money securely locked away is safe but may be losing value. On the other hand there are others who need money to buy houses, to set up businesses, etc. To cater to these needs, renting money became a business, and successful moneylenders prospered and became respectable bankers. Those with money and no immediate need of it rented it to the bank, and those who needed money rented it from the bank. The price of renting money is called *interest*.[2] Money deposited in a savings account grows at the prevailing rate of interest,[3] and as most deposits are insured and often guaranteed by governments, they are, for all practical purposes, a risk-free way of maintaining *some growth*. Any other way, such as investing in a business venture, involves *risk*. Interest rates and inflation rates are distinct processes, one increasing the nominal value of money, the other reducing its real value. However, it is often observed in economies that interest rates tend to be slightly higher than inflation rates. It seems savers generally demand a positive real interest rate and borrowers generally are willing to pay it. Two groups with different approaches to the management of money are *hedgers*, who wish to eliminate risk as much as possible, and *speculators*, who are willing to take risks in the expectation of higher profits.

1.3. Interest Rates

We now discuss interest rates and at the same time review some important results from one variable calculus. Interest rates, e.g. simple interest, compound interest, continuously compounded interest, etc., are calculated in different ways but since all involve the same basic concept they are comparable. We show how to compare them using the *effective rate of interest* or equivalently the *annual percentage rate* (APR) and having done so, settle on one and use it more or less exclusively afterwards. We let t denote the time variable, $t = 0$ will denote the present, while $t = 10$ will be 10 units of time, usually measured in years, into the future. Interest rates vary with time, but initially we assume they are constant.

[2]Nowadays we think of interest in this way, but essentially interest is the price of renting any object or service. Interest has been around for over *five thousand years* and for two thousand years before coins were introduced. Early Irish law, *The Brehon Law*, operated from around 200 BC to 1600 AD and relied heavily on the use of pledges to ensure that legal obligations were carried out. A *pledge* was an object of value delivered into the custody of another for a fixed period. A person who gave a pledge on behalf of another was entitled to *interest* while the pledged object was out of his possession. For example, if a lord supplied a goblet as a pledge, he was entitled to receive interest of 2 ounces of silver every three days until nine days were up and afterwards the rate of interest increased.

[3]The method of setting bank interest rates is complicated and involves central banks, governments, supply and demand, etc.

We begin with the simplest case, simple interest. Ten percent *simple interest* on a loan of $\$1,000$ for five years means that 10% of the amount borrowed, the *principal*, is charged for each year of the loan. Thus the interest charged is $\$\frac{10}{100} \times 5 \times 1,000 = \$500$. The general formula for calculating simple interest is straightforward: if an amount A is borrowed or saved for T years at a rate[4] r of simple interest, then the repayment due at time T is

$$A + ArT = A(1 + rT).$$

Simple interest is rarely used by banks, and it is easy to see why. If $\$1,000$ is deposited for 2 years at a rate of 10% simple interest, then the amount accumulated at the end of two years, the maturity date, would be $\$1,200$. If, however, at the end of year one the amount accumulated at that time, $\$1,100$, is withdrawn and immediately deposited for a further year at the same rate of simple interest, then, at maturity, the amount accumulated would be $\$1,210$, a gain of $\$10$ on the previous amount. If simple interest was the norm, people would be in and out of banks regularly withdrawing and immediately re-depositing their savings. For this reason a different method of calculating interest is normally used. This is called *compound interest*[5] and is based on applying simple interest over regular preassigned periods of the savings or loan to the amount accumulated at the beginning of each period. If a savings account offers 5% interest per annum compounded every six months, then the amount accumulated by $\$2,000$ deposited for two years is calculated as follows. The simple interest rule applied to the first six months' period shows that the amount will earn $\$50$ interest, and the amount deposited will have increased to $\$2,050$ at the end of six months. During the second six months, the $\$2,050$ will grow to $\$2,050(1 + \frac{5}{100} \times \frac{1}{2}) = \$2,101.25$, during the next period the amount will reach $\$2,153.78$, and in the final six months' period the amount will reach $\$2,207.63$.

Interest can, of course, be compounded at various other intervals of time, and the more frequent the compounding the greater the interest earned. Suppose an amount A is borrowed for T years at a rate r per annum compounded at n equally spaced intervals of time per year. Each interval of time $\frac{1}{n}$ has a simple interest rate of $\frac{r}{n}$. Thus after the first time interval the amount due has grown to $A(1 + \frac{r}{n})$, after two intervals it becomes $A(1 + \frac{r}{n})(1 + \frac{r}{n}) = A(1 + \frac{r}{n})^2$ and so on. Since there are a total of nT intervals of time, the total repayment at the end of T years will be $A(1 + \frac{r}{n})^{nT}$.

[4] That is at a $100r$ percentage rate.

[5] The word *compound* comes from the Latin words *com* (together) and *ponere* (to put) and is used because compound interest is a putting together of simple interest. The words used in mathematics are taken from our everyday language and given precise mathematical meanings. They are usually chosen because one of their common usages approximates their meaning within mathematics. By simply consulting a dictionary, one can sometimes gain helpful mathematical insights.

We compare different interest rates by finding their effective rate of interest. This is the rate of simple interest which would give the same return over *one year*. One thousand dollars borrowed for one year at a rate of 10% per annum compounded every six months would result in a repayment of $1,102.50 at the end of the year. If the same amount is borrowed for one year at 10.25% simple interest, then the amount due would also be $1,102.50. Thus we say that the rate 10% per annum compounded every six months has a 10.25% effective rate of interest. It is clear that the more frequent the compounding, the higher the effective rate of interest.

Example 1.1. By comparing effective rates of interest we find which of the following gives the highest and lowest return:

(a) 6% compounded once a year
(b) 5.8% compounded quarterly
(c) 5.9% compounded quarterly
(d) 5.8% compounded monthly
(e) 5.6% compounded daily.

In practical cases such as this it is not advisable to rush in and blindly apply a mathematical formula but to pause and examine the situation from a common sense point of view. Since (a) is compounded only once a year, its effective rate of interest is 6%. Since (b) and (c) are compounded at the same time, but (b) has a lower rate of interest, it follows that (b) will have a lower effective rate of interest. Comparing (b) and (d) we see that they have the same rate of interest but (d) is compounded more frequently and thus will have a higher effective rate of interest.

Interest rates are independent of the amount borrowed or saved, so we compare them by considering $1 borrowed for one year. For (b) the amount to be repaid is $\$1(1+\frac{.058}{4})^4 = \$1.0593$ and thus its effective rate of interest is 5.93%. For (c) we have $\$1(1+\frac{.059}{4})^4 = \$1.0603$, and its effective rate of interest is 6.03%. Similarly for (d), $\$1(1+\frac{.058}{12})^{12} = \$1.0596$, and its effective rate of interest is 5.96%; and for (e), $\$1(1+\frac{.056}{365})^{365} = \$1.0576$, and its effective rate of interest is 5.76%. Hence for the borrower (e) offers the cheapest rate, while (c) is the most expensive.

Example 1.2. A bank is offering 4% interest per annum compounded monthly to savers, and a customer wishes to save a fixed amount each month in order to accumulate a lump sum of $10,000 at the end of five years. We wish to determine how much should be saved each month.

As the customer is saving each month and also gaining interest, the amount deposited over the five years must be less than the lump sum $10,000. Since there will be a total of $12 \times 5 = 60$ deposits, the amount required each month will be less than $\$\frac{10{,}000}{60} = \$166.67$. Similar practical checks should be used whenever possible as they give some estimate of the expected answer and may alert us to patently false conclusions. This can be important in complicated situations. Let x denote the amount deposited each month. The first payment will be deposited for 60 months at a monthly interest rate of $\frac{.04}{12} = .0033$ and hence will amount to $x(1 + .0033)^{60}$. The second deposit will earn interest for 59 months and thus will amount to $x(1 + .0033)^{59}$. Proceeding in this way we see that the amount accumulated at the end of five years will be

$$\sum_{n=0}^{59} x(1.0033)^{60-n} = \sum_{n=1}^{60} x(1.0033)^n \tag{1.1}$$

and this must equal $10,000.

To calculate this sum we consider the more general problem of summing a *geometric series.*[6] If $n > m$ and

$$S = r^m + r^{m+1} + \cdots + r^n$$

then

$$rS = r^{m+1} + r^{m+2} + \cdots + r^{n+1}.$$

Hence $S(1 - r) = r^m - r^{n+1}$. If $r \neq 1$, then[7]

$$S = \frac{r^m - r^{n+1}}{1 - r} = \frac{r^{n+1} - r^m}{r - 1}. \tag{1.2}$$

By (1.1) and (1.2) we have

$$x\sum_{n=1}^{60}(1.0033)^n = x \cdot \frac{(1.0033)^{61} - 1.0033}{1.0033 - 1} = x(66.45) = 10,000.$$

Hence the monthly deposits required are $\$\frac{10000}{66.45} = \$150.49$.

Up to now we considered interest rates compounded at certain fixed finite intervals of time. If we compound over smaller and smaller intervals, we obtain in the limit *continuously compounded interest.* The above shows that A, continuously compounded at an annual rate r, amounts to $A\lim_{n\to\infty}(1 + \frac{r}{n})^{nT}$ after T years. This limit has flexible mathematical properties, due to its connection,

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[6]A Dutch military engineer from Bruges, Simon Stevin (1548-1620), published *Tables for Computing Compound Interest and Annuities* in 1582, which tabulated $(1+q)^{\pm k}$ and $\sum(1+q)^{\pm k}$. He also applied mathematics to accountancy (proposing double entry bookkeeping for the public revenues), engineering (windmills, sailing craft and hydrostatics), geography (maps) and military science.

[7]Mathematics is a unified discipline, and ideas and techniques from one area often reappear, sometimes in disguise, in other areas. The identity $a^{n+1} - b^{n+1} = (a - b)(a^n + a^{n-1}b + \cdots + b^n)$, used to factorize polynomials, is essentially (1.2) with $r = a/b$ and $m = 0$. When $n = 2$ this reduces to the well-known formula $a^2 - b^2 = (a - b)(a + b)$.

outlined in Proposition 1.3, with the exponential function, and we use it whenever possible. In particular, we assume from now on, unless otherwise stated, that *all* interest is continuously compounded. As the *exponential function* plays an essential role in many parts of our studies, we recall its basic properties.[8]

The exponential function, exp, is defined[9] for any real number x by the following power series expansion:[10]

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{1.3}$$

The exponential function maps the real numbers, $\mathbf{R}$, in a one-to-one[11] fashion onto the strictly positive real numbers. Its inverse is called the logarithm or log function. We have $\log \exp(x) = x$ for all $x \in \mathbf{R}$ and $\exp \log(x) = x$ for all $x > 0$. If $a > 0$ and b is a real number, we let $a^b := \exp(b \log a)$ and, in particular, let $a^0 = \exp(0) = 1$ for all $a > 0$. Both the exponential function and its inverse are increasing differentiable functions. Moreover,

$$\frac{d}{dx}\exp(x) = \exp(x) \quad \text{and} \quad \frac{d}{dx}\log(x) = \frac{1}{x}.$$

The following set of identities (see Section 6.5) are constantly used:

$$\begin{aligned} \exp(x+y) &= \exp(x)\exp(y) & \log(xy) &= \log x + \log y \\ \exp(0) &= 1 & \log 1 &= 0 \\ \exp(x-y) &= \frac{\exp(x)}{\exp(y)} & \log\left(\frac{x}{y}\right) &= \log x - \log y \\ x^y &:= \exp(y \log x) & \log(x^y) &= y \log x. \end{aligned}$$

Note that for every property of the exponential function there is a corresponding property of the log function. The real number $\exp(1)$ is also denoted by e and, using this notation, we have $1 = \log \exp(1) = \log(e)$ and

$$e^x = \exp(x \log e) = \exp(x).$$

The number e is irrational and approximately equal to 2.72. We always use natural logs, that is logs to the base e.[12]

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[8] See also Exercises 3.31, 3.34, and 4.22 and Section 6.5.

[9] We introduced in (1.3) the following convenient notation $A := B$, and use later the equivalent notation $B =: A$. The inclusion of " **:** " indicates, in both cases, that the equation is being used to *define* A by means of B.

[10] Convergent series, limits and continuous functions are defined rigorously in Chapters 3 and 4. In the meantime we rely on the reader's intuitive feelings.

[11] Functions which are one-to-one are said to be *injective*, onto functions are called *surjective* and the term *bijective* is used for functions which are both injective and surjective.

[12] The number e was introduced by Leonhard Euler (1707-1783). Euler, from Basel in Switzerland, was a professor in St. Petersburg during the periods 1727-1741 and 1766-1783 and spent the intervening period in Berlin. He was the most prolific mathematician of all time and made fundamental contributions to almost all areas of pure and applied mathematics, including analysis, infinite series, differential geometry, differential equations, complex analysis, number theory, the calculus of

Proposition 1.3. *For any real number r*

$$\lim_{n\longrightarrow\infty}\left(1+\frac{r}{n}\right)^n = e^r.$$

Proof. We have

$$\frac{d}{dx}\log(x) = \lim_{\Delta x\longrightarrow 0}\frac{\log(x+\Delta x)-\log x}{\Delta x} = \frac{1}{x}.$$

If we let $x = 1$ and $\Delta x = r/n$, then $\Delta x \longrightarrow 0$ as $n \longrightarrow \infty$. Since $\log 1 = 0$ this implies

$$\lim_{n\longrightarrow\infty}\frac{\log(1+\frac{r}{n})}{\frac{r}{n}} = \lim_{n\longrightarrow\infty}\frac{n}{r}\log\left(1+\frac{r}{n}\right) = \frac{1}{r}\lim_{n\longrightarrow\infty}\log\left(1+\frac{r}{n}\right)^n = 1.$$

Hence $\lim_{n\to\infty}\log(1+\frac{r}{n})^n = r$, and as exp and log are inverse functions and both are continuous[13] this implies

$$\lim_{n\longrightarrow\infty}\left(1+\frac{r}{n}\right)^n = \exp\left(\lim_{n\longrightarrow\infty}\log\left(1+\frac{r}{n}\right)^n\right) = \exp(r) = e^r.$$

This completes the proof. □

Corollary 1.4. *An amount A earning continuously compounded interest at a constant rate r per year is worth Ae^{rT} after T years.*

Corollary 1.4 illustrates mathematically a basic functional relationship between *time* and *money*.

Example 1.5. If \$10,000 is deposited today for five years at a continuously compounded rate of 4% per annum, then it will amount to

$$\$10,000\exp(.04\times 5) = \$12,214$$

at the end of five years.

This leads to an important general principle. We can *reverse* the process and say that the *present worth* or *present value* of \$12,214 in five years' time is \$10,000. In this way we can determine, for a given fixed rate of interest, the present worth of any amount at *any future time*. For example, if the interest rate is 7% per annum then the present worth of \$5,000 in six years' time, A, is given by $Ae^{(.07)6} = \$5,000$ and $A = \$5,000e^{-.42} = \$3,285.23$. The procedure of finding the present worth or value of a future amount is called *discounting* back to the present, discounting back or just discounting. The present worth

variations, etc. He revolutionized mathematics by basing his analysis on functions rather than curves. His contributions to applied mathematics, astronomy, cartography, and engineering projects, such as ship building, were also significant. Euler went blind in 1767, but with the help of his amazing memory and some assistants, he produced almost half his scientific output while blind. He wrote so much that the St. Petersburg Academy continued to publish his unpublished work for almost fifty years after his death.

[13] See Definition 4.18 and Exercises 4.22 and 12.12

of a future amount is called its *discounted value*. This allows us to introduce a way of measuring, and hence comparing, the risk-free future value of money. As discounting plays an important role in pricing stock options, we summarize the above in the form of a proposition which is a mirror image of Corollary 1.4.

Proposition 1.6. *The discounted value of an amount A at a future time T, assuming a constant continuously compounded interest rate r, is given by*

$$Ae^{-rT}.$$

Example 1.7. In this example we discount back to the present in order to evaluate a project. Suppose bank interest rates are 4% per annum continuously compounded, that an initial outlay of \$400,000 is required, and that the projected end of year returns are given in the following table.

year t	Profit/Loss	NPV $= Ae^{(-.04)t}$
0	\$ − 400,000	\$ − 400,000
1	\$60,000	\$57,647
2	\$80,000	\$73,849
3	\$140,000	\$124,169
4	\$200,000	\$170,429
Total	—	\$26,094

Thus the *Net Present Value* (NPV) of the project is \$26,094. This shows that the project, assuming all estimates are correct and interest rates remain fixed, will show a greater profit than that generated by using bank deposit accounts.

The following proposition and corollary are fundamental results from the differential calculus.

Proposition 1.8. *(Mean Value Theorem) If the function $f : [a,b] \longrightarrow \mathbf{R}$ is continuous over $[a,b]$ and differentiable over (a,b), then there exists a point c, $a < c < b$, such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Corollary 1.9. *If $f : [a,b] \longrightarrow \mathbf{R}$ is a continuous function over $[a,b]$, differentiable over (a,b) and $f'(x) = 0$ for all $x \in (a,b)$, then f is a constant function.*

Example 1.10. In this example we use Corollary 1.9 to provide another proof of Corollary 1.4. This leads, in Example 1.12, to a way of dealing with non-constant interest rates. Suppose an amount A is deposited for T years at a constant continuously compounded annual interest rate r. For $0 \leq t \leq T$ let $A(t)$ denote the amount accumulated at time t. Clearly $A(0) = A$, and we wish to find $A(T)$. During the time interval $[t, t + \Delta t]$ the amount grows by $A(t + \Delta t) - A(t)$, and since Δt is small, we suppose that the continuously compounded rate over $[t, t+\Delta t]$ is approximately the same as the simple interest rate. Hence

$$A(t + \Delta t) - A(t) \approx r\Delta t A(t)$$

where $\approx$ denotes approximately equal. This implies

$$\lim_{\Delta t \longrightarrow 0} \frac{A(t + \Delta t) - A(t)}{\Delta t} = \lim_{\Delta t \longrightarrow 0} rA(t) = rA(t),$$

and we obtain the required result by solving the *differential equation*

$$A'(t) = rA(t). \tag{1.4}$$

We have

$$\frac{A'(t)}{A(t)} = \frac{d}{dt}(\log A(t)) = r = \frac{d}{dt}(rt)$$

and hence

$$\frac{d}{dt}(\log A(t) - rt) = 0. \tag{1.5}$$

By Corollary 1.9 and (1.4) there exists a real number C such that $\log A(t) - rt = C$. Hence $\log A(t) = C + rt$, and applying the exponential function we obtain

$$A(t) = \exp \log A(t) = \exp(C + rt) = \exp(C) \cdot \exp(rt).$$

Since $\exp(0) = 1$ we have $A(0) = A = \exp(C)$ and

$$A(T) = A(0) \exp(rT) = A \exp(rT).$$

To deal with variable interest rates we need a further result from the differential calculus.

Proposition 1.11. *(Fundamental Theorem of Calculus)*[14] *If f is a continuously differentiable real valued function on the interval $[a, b]$, then*

$$\int_a^b f'(t)dt = f(b) - f(a).$$

[14] An extension of the Fundamental Theorem of Calculus, the *Radon-Nikodým Theorem*, is the key result required in Chapter 8 to prove the existence of conditional expectations.

Example 1.12. We consider again the problem of calculating the growth of an initial deposit A for a period of T years given that the interest is continuously compounded at the annual rate $r(t)$ at time t. Two different approaches are included and both will be important later.

As before, let $A(t)$ denote the amount accumulated by time t. Our previous analysis shows that

$$A(t+\Delta t) - A(t) \approx r(t)\Delta t A(t). \tag{1.6}$$

If this approximation is sufficiently accurate, then $A(t)$ is differentiable and we obtain, as in (1.4), the differential equation

$$A'(t) = r(t)A(t). \tag{1.7}$$

Hence

$$\frac{A'(t)}{A(t)} = \frac{d}{dt}(\log A(t)) = r(t),$$

and if r is continuous, the Fundamental Theorem of Calculus implies

$$\log A(T) - \log A(0) = \int_0^T \frac{d}{dt}(\log A(t))dt = \int_0^T r(t)dt.$$

Hence

$$\log \frac{A(T)}{A(0)} = \int_0^T r(t)dt.$$

Since $A(0) = A$ this implies

$$A(T) = A\exp(\int_0^T r(t)dt).$$

In place of using (1.6) to derive (1.7) we may also proceed as follows. Fix $t \in [0,T]$ and partition the interval $[0,t]$ into n subintervals of *equal length*, apply (1.6) to each of them, and add them together. If $[t_i, t_{i+1}]$ is the $(i+1)^{th}$ interval in the partition and $\Delta t_i := t_{i+1} - t_i$ for $i = 0, 1, \ldots, n-1$, then

$$A(t) - A(0) = \sum_{i=0}^{n-1} A(t_{i+1}) - A(t_i) \approx \sum_{i=0}^{n-1} r(t_i)A(t_i)\Delta t_i.$$

If we take finer and finer partitions we obtain, in the limit,

$$A(t) - A(0) = \int_0^t r(s)A(s)ds. \tag{1.8}$$

Equation (1.8) is called an *integral equation*. Integrating (1.7) we obtain (1.8) and, if we know that A is differentiable, then on differentiating (1.8) we obtain (1.7). As a general rule every differential equation gives rise to an integral equation, but the converse is *not* true[15]. This apparently rather minor point will again surface when we discuss the Itô integral in Chapters 8 and 12.

This concludes our basic introduction to money and interest. Our analysis is typical of a process known as *mathematical modeling*. We started with the basic concept of interest and examined in turn three increasingly more complex situations: simple interest, compound interest and continuously compounded interest. At each stage we developed the required mathematical model before examining the next level of complexity, and as we progressed, we used more sophisticated mathematics. We were led to *linear* growth, *geometric* growth and *exponential* growth and to three basic formulae: $A(1 + rT), A(1 + \frac{r}{n})^{nT}$ and Ae^{rT}. The final result is transparent because of our gradual development of the model and because of the continuous interacting between financial and mathematical concepts.

To expose the intrinsic nature of money and interest, we deliberately over-simplified the situation. Other *financial instruments* or *securities*, that is legally enforceable agreements that give entitlement to future contingency payments or which guarantee a risk-free return based on current interest rates, also exist. They are said to be *liquid* if they can be easily traded in a well developed market. The most common type of liquid securities are *bonds*. These are issued by many different groups, governments,[16] local authorities, banks, corporations, etc., for different purposes and with different conditions attached. We confine ourselves to a few brief comments. A typical 5 year bond might have a face value of $\$1,000$, which represents the payment made on maturity. If the interest rate on the bond[17] is 7.5% per annum the holder of the bond receives at the end of each year[18] until maturity $\$1,000 \times .075 = \$75$. The purchaser of the bond makes a commitment to a certain level of interest. If interest rates increase, the value of the bond decreases and conversely. Our interest in bonds stems from their use in constructing hedging portfolios.

We now introduce, and use bank interest rates to illustrate, an extremely important concept, *arbitrage*. Suppose we have two banks, A and B, operating side by side. Bank A offers customers a 10% interest rate per annum on savings, while Bank B offers loans to customers at an 8% rate of interest per annum. It is not difficult to see how to take advantage of this situation. Go to Bank B, borrow as much as possible and immediately place it on deposit in Bank A. If, for example, one obtains a loan for $\$1,000,000$ for one year, then at the end

[15] Every differentiable function is continuous, and hence integrable, but not every continuous function is differentiable.

[16] War bonds were issued by the Roman Senate during the second Punic war, 218-210 BC, between Rome and Carthage.

[17] In the case of bonds it is traditional to use the effective rate of interest.

[18] Or perhaps half that amount every six months until maturity. The periodic payments are called *coupons*.

of the year the principal in Bank A amounts to $\$1,000,000e^{.1} = \$1,105,171$, while the loan repayment to Bank B amounts to $\$1,000,000e^{.08} = \$1,083,287$. This gives a *risk-free guaranteed profit* of $\$21,884$ at the end of the year.

The word *arbitrage* is used to describe any situation, opportunity or price which allows a *guaranteed profit without risk*. The market recognizes very rapidly when arbitrage opportunities exist and takes advantage of them, thereby closing them down. In our example the *demand* on at least one of the banks would increase rapidly, and as a result, interest rates would quickly be adjusted until *equilibrium* was established.

In pricing derivatives we always aim to determine an arbitrage-free price, that is one in which neither buyer nor seller can realize risk-free profits. Prices determined in this fashion are said to be based on the *no arbitrage principle*.

1.4. The Market

So far we have considered the riskless growth of money where the return is guaranteed but modest. This does not suit everyone, and some are willing to take risks to increase the value of their money at a faster rate. We consider one such situation.

Business A is family owned, and during its fifty years in existence it has grown substantially and now has over 800 employees. The board of directors, all family members, feel that the time is right for a large-scale expansion and has identified an opportunity to take over a rival company of the same size as itself. To do so it needs *capital*, that is money, and at the same time the family wishes to maintain control of the company. A large bank loan is a possibility, but this could lead to difficulties if either interest rates rose sharply, business slowed down or the takeover turned out to be less successful than anticipated. Selling between 40% and 60% to a large number of individuals with diverse interests, who would not organize themselves into a control-seeking group, would be preferable. To achieve this the family puts a total value on the company and divides it into a large number of identical parts, say 250 million, each of which is called a *share*, and offers between 100 million and 150 million shares for sale to the public. The shares would be offered for sale on the *stock market*, a process known as a *flotation*. The non-family shareholders would collectively be entitled to $40\% - 60\%$ of the profits of the company. These are usually paid out semi-annually in the form of so much per share and are called *dividends*. As shares are auctioned daily the price is constantly changing, and shareholders may also profit by buying and selling shares. In buying and selling shares the *stockbroker*,[19] who acts as intermediary in these transactions, is paid a *commission*, which is often a percentage value of the total transaction, subject to a minimum charge. The difference in price between buying and selling shares is called *capital gains* or *losses*. During periods of relative stock price stability, dividends become the principal component of the return and are similar, in some ways, to the interest paid by banks. When share prices are volatile, that is subject to large swings, investors are usually more concerned with capital gains.

Who are the potential shareholders? The general public would have formed an opinion of the company and its future prospects, and financial experts would provide an informed opinion and give more detailed analysis. For a modest investment individuals could buy a small part of the company and share in its future prosperity and profits. Another important group of investors are fund managers, for example pension fund managers. In most companies, employees contribute weekly or monthly to a pension scheme which funds their retirement. The amounts contributed, especially in large companies, accumulate to substantial amounts, and it is crucial that the pension payments are available when required. Fund managers are appointed to see that this happens. Because of the conservative nature of their mission, fund managers usually spread their investments over all sectors of the economy. A well-managed established family business with reasonable profits and prospects would appeal to fund managers.

The flotation, if successful, would provide the company with the capital necessary for expansion. A reasonable mix of small shareholders, fund managers and a few large individual shareholders would not threaten the family's overall control. However, whereas previously they could make decisions behind closed doors, they would now have annual public general meetings and their affairs would be subject to more regulations and media attention.

This concludes our introduction to the stock market. We have discussed just one, not atypical, situation. For shareholders we have observed two important points: share prices go up and down, and these fluctuations occur continuously; thus the share price is a *function of time*. The price changes occur for many different reasons: economic, political and even psychological. Apart from buying and selling shares there are other commercial transactions involving shares, for instance, *contracts* or *options* to buy or sell shares at a given price at a given future date. These financial instruments are called *derivatives*, since their values are derived from underlying *assets*, in this case shares. To derive the Black-Scholes formula, which gives an arbitrage-free price for call

[19] Professional traders can be classified roughly as either *hedgers*, who try eliminating risks to maintain the real value of their assets; *speculators*, who take risks in the hope of large profits; and *arbitrageurs*, who move in when they see an opportunity to make riskless profits.

options, we need *probability theory* and some insight into how the gambler and bookmaker approach their trade. We start with the latter in the next chapter.

1.5. Exercises

(1.1) Show that for every strictly positive real number a there exists a unique real number b such that $e^{a+b} = e^a + e^b$. What happens if $a \leq 0$? Sketch for a, an arbitrary real number, the graph of

$$f(x) = e^{a+x} - e^a - e^x.$$

(1.2) If n is a positive integer, show that $\lim_{|x|\to\infty} |x|^n e^{-x^2/2} = 0$.

(1.3) If $f : \mathbf{R} \longrightarrow \mathbf{R}$ is continuous, let $\int_{-\infty}^{+\infty} f(x)dx = \lim_{n,m\to\infty} \int_{-m}^{n} f(x)dx$ whenever this limit exists. If $f, g : \mathbf{R} \longrightarrow \mathbf{R}$ are continuous, $|f(x)| \leq g(x)$ and $\int_{-\infty}^{+\infty} g(x)dx$ is finite show that $\int_{-\infty}^{+\infty} f(x)dx$ is finite. Using this result and the previous exercise show that $\int_{-\infty}^{+\infty} x^n e^{-x^2/2}dx$ is finite for all non-negative integers n. Evaluate $\int_{-\infty}^{+\infty} x^n e^{-x^2/2}dx$, assuming that $\int_{-\infty}^{+\infty} e^{-x^2/2}dx = \sqrt{2\pi}$.

(1.4) Adapt the method used to prove equation (1.2) to find $\sum_{j=1}^{n} jr^j$. Verify your answer by differentiating (1.2). By applying L'Hôpital's Rule[20] at $r = 1$ find $\sum_{j=1}^{n} j$. Find $\sum_{j=1}^{\infty} jr^j$ when $|r| < 1$. Use the same methods to find $\sum_{j=1}^{\infty} j^2 r^j$ when $|r| < 1$ and $\sum_{j=1}^{n} j^2$.

(1.5) A mortgage of \$250,000 is to be repaid over 20 years in equal monthly installments. Find a lower bound for the repayments. Suppose the interest rate is 5.2% per annum continuously compounded. If interest is added at the beginning of each year, find the total monthly repayments. Find the total amount repaid.

(1.6) Five-year government bonds have a face value of \$2,000 and annual coupons worth \$130. If interest rates (a) increased by 1%, (b) decreased by 2% immediately after the bonds were issued, find the change in value of the bonds.

(1.7) By differentiation verify for $x > 0$ that

$$e^{-x^2/2}\frac{1}{x} = \int_x^{\infty} e^{-y^2/2}\big(1 + \frac{1}{y^2}\big)dy.$$

[20] *L'Hôpital's Rule.* If f and g are defined and k-times continuously differentiable on an open interval containing the point a and if $f(a) = g(a) = f^{(1)}(a) = g^{(1)}(a) = \cdots = f^{(k)}(a) = g^{(k)}(a) = 0$, then

$$\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f^{(k)}(x)/g^{(k)}(x)$$

whenever the limit on the right-hand side exists.

Obtain a similar formula with $1 + (1/y^2)$ replaced by $1 - (3/y^4)$. Hence show for $x > 0$ that

$$\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\left(\frac{1}{x} - \frac{1}{x^3}\right) \leq \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-y^2/2}dy \leq \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\frac{1}{x}.$$

Use the same approach and higher powers of x to improved this estimate.

(1.8) At what constant rate should money be continuously deposited into a savings account in order to accumulate \$10,000 at the end of 5 years given that interest rates are 6%?

(1.9) If $\mu \in \mathbf{R}$, $\sigma \in \mathbf{R}^+$ and $f(x) = (\sqrt{2\pi}\sigma)^{-1}\exp\{(x-\mu)^2/2\sigma^2\}$ show, by using the result quoted in Exercise 1.3 and completing squares, that $\int_{-\infty}^\infty f(x)dx = 1$, $\int_{-\infty}^\infty xf(x)dx = \mu$, and $\int_{-\infty}^\infty x^2 f(x)dx = \sigma^2 + \mu^2$.

(1.10) Let $f : \mathbf{R} \longrightarrow \mathbf{R}, f(x) = 0$ when $x \leq 0$ and $f(x) = \exp(-1/x)$ when $x > 0$. Show that f has derivatives of all orders at all points. Let $b, c \in \mathbf{R}, b < c$ and let $g(x) = f(c-x)\cdot f(x-b)$ for all $x \in \mathbf{R}$. Show that $g = 0$ when $x < b$ and $x > c$. Let $\alpha^{-1} = \int_b^c g(x)dx$ and $h(x) = \alpha\int_x^c g(t)dt$ for all $x \in \mathbf{R}$. Show that h has derivatives of all orders at all points, that $h(x) = 1$ for $x \leq b$, $0 \leq h(x) \leq 1$ for $b \leq x \leq c$, and $h(x) = 0$ for $x \geq c$.