
Preface to the English Edition

The book presents basics of Riemannian geometry in its modern form as the geometry of differentiable manifolds and the most important structures on them. Let us recall that Riemannian geometry is built on the following fundamental idea: *The source of all geometric constructions is not the length but the inner product of two vectors.* This inner product can be symmetric and positive definite (classical Riemannian geometry) or symmetric indefinite (pseudo-Riemannian geometry and, in particular, the Lorentz geometry, important in physics). Moreover, this inner product can be complex Hermitian and even skew-symmetric (symplectic geometry). The modern point of view to any geometry also includes global topological aspects related to the corresponding structures. Therefore, in the book we pay attention to necessary basics of topology.

With this approach, Riemannian geometry has great influence on a number of fundamental areas of modern mathematics and its applications.

1. Geometry is a bridge between pure mathematics and natural sciences, first of all physics. Fundamental laws of nature are expressed as relations between geometric fields describing physical quantities.

2. The study of global properties of geometric objects leads to the far-reaching developments in topology, including topology and geometry of fiber bundles.

3. Geometric theory of Hamiltonian systems, which describe many physical phenomena, leads to the development of symplectic and Poisson geometry.

4. Geometry of complex and algebraic manifolds unifies Riemannian geometry with the modern complex analysis, as well as with algebra and number theory.

We tried, as much as possible, to present these ideas in our book. On the other hand, combinatorial geometry, which is based on a completely different set of ideas, remains beyond the scope of this book.

Now we briefly describe the contents of the book.

Chapters 1 and 2 are devoted to basics of various geometries in linear spaces: Euclidean, pseudo-Euclidean, and symplectic. Special attention is paid to the geometry of the Minkowski space, which is a necessary ingredient of relativity.

Chapters 3 and 4 are devoted to the geometry of two-dimensional manifolds; here we also present necessary facts of complex analysis. Some famous completely integrable systems of modern mathematical physics also appear in these chapters.

Basics of topology of smooth manifolds are presented in Chapter 5.

Some material about Lie groups is included in Chapter 6. In addition to classical theory of Lie groups and Lie algebras, here we pay attention to the theory of crystallographic groups and their modern generalizations related to the discovery of so-called quasicrystals.

Chapter 7 and 8 are devoted to classical tensor algebra and tensor calculus, whereas in Chapter 9 the theory of differential forms (antisymmetric tensors) is presented. It should be noted that already in Chapters 7 and 9 we encounter the so-called Grassmann (anticommuting) variables and the corresponding integration.

Chapter 10 contains the Riemannian theory of connections and curvature. The idea of a general connection in a bundle (Yang–Mills fields) also appears here.

Conformal geometry and complex geometry are the main subjects of Chapter 11.

In Chapter 12, we present elements of finite-dimensional Morse theory, the calculus of variations, and the theory of Hamiltonian systems. Geometric aspects of the theory of Hamiltonian systems are developed in Chapter 13, where the theory of Lagrangian and Poisson manifolds is introduced.

Chapter 14 contains elements of multidimensional calculus of variations. We should emphasize that, contrary to the majority of existing textbooks, the mathematical language of our exposition is compatible with the language of modern theoretical physics. In particular, we believe that such notions as the energy-momentum tensor, conservation laws, and the Noether theorem are an integral part of the exposition of this subject. Here we also touch

upon simplest topological aspects of the theory, such as the Morse index and harmonic forms on compact Riemannian manifolds from the point of view of Hodge theory.

In Chapter 15 we present the basic theory of the most important fields in physics such as the Einstein gravitation field, the Dirac spinor field, and the Yang–Mills fields associated to vector bundles. Here we also encounter topological phenomena such as the Chern and Pontryagin characteristic classes, and instantons.

Each chapter is supplemented by exercises that allow the reader to get a deeper understanding of the subject.

Preface

As far back as in the late 1960s one of the authors of this book started preparations to writing a series of textbooks which would enable a modern young mathematician to learn geometry and topology. By that time, quite a number of problems of training nature were collected from teaching experience. These problems (mostly topological) were included into the textbooks [DNF1]–[NF] or published as a separate collection [NMSF]. The program mentioned above was substantially extended after we had looked at textbooks in theoretical physics (especially, the outstanding series by Landau and Lifshits, considerable part of which, e.g., the books [LL1, LL2], involve geometry in its modern sense), as well as from discussions with specialists in theoretical mechanics, especially L. I. Sedov and V. P. Myasnikov, in the Mechanics Division of the Mechanics and Mathematics Department of Moscow State University, who were extremely interested in establishing courses in modern geometry needed first of all in elasticity and other branches of mechanics. Remarkably, designing a modern course in geometry began in 1971 within the Mechanics, rather than the Mathematics Division of the department because this was where this knowledge was really needed. Mathematicians conceded to it later. Teaching these courses resulted in publication of lecture notes (in duplicated form):

S. P. Novikov, *Riemannian geometry and tensor analysis*. Parts I and II, Moscow State University, 1972/73.

Subsequently these courses were developed and extended, including, in particular, elements of topology, and were published as:

S. P. Novikov and A. T. Fomenko, *Riemannian geometry and tensor analysis*. Part III, Moscow State University, 1974.

After that, S. P. Novikov wrote the program of a course in the fundamentals of modern geometry and topology. It was realized in a series of books [DNF1]–[NF], written jointly with B. A. Dubrovin and A. T. Fomenko. Afterwards the topological part was completed by the book [N1], which contained a presentation of the basic ideas of classical topology as they have formed by the late 1960s–early 1970s. The later publication [N2] also included some recent advances in topology, but quite a number of deep new areas (such as, e.g., modern symplectic and contact topology, as well as new developments in the topology of 4-dimensional manifolds) were not covered yet. We recommend the book [AN]. We can definitely say that even now there is no comprehensible textbook that would cover the main achievements in the classical topology of the 1950s–1970s, to say nothing of the later period. Part II of the book [1] and the book [2] are insufficient; other books are sometimes unduly abstract; as a rule, they are devoted to special subjects and provide no systematic presentation of the progress made during this period, very important in the history of topology. Some well-written books (e.g., [M1]–[MS]) cover only particular areas of the theory. The book [BT] is a good supplement to [DNF1, DNF2], but its coverage is still insufficient.

Nevertheless, among our books, Part II of [DNF1] is a relatively good textbook containing a wide range of basic theory of differential topology in its interaction with physics. Nowadays this book could be modernized by essentially improving the technical level of presentation, but as a whole this book fulfills its task, together with the books [DNF2] and [N1], intended for a more sophisticated reader.

As for Part I, i.e., the basics of Riemannian geometry, it has become clear during the past 20 years that this book must be substantially revised, as far as the exposition of basics and more complete presentation of modern ideas are concerned. To this end, the courses [T] given by the second author, I. A. Taimanov, at Novosibirsk University proved to be useful. We joined our efforts in writing a new course using all the material mentioned above.

We believe that the time has come when a wide community of mathematicians working in geometry, analysis, and related fields will finally turn to the deep study of the contribution to mathematics made by theoretical physics of the 20th century. This turn was anticipated already 25 years ago, but its necessity was not realized then by a broad mathematical community. The advancements in this direction made in our books such as [DNF1] had not elicited a proper response among mathematicians for a long time. In our view, the situation is different nowadays. Mathematicians understand much better the necessity of studying mathematical tools used by physicists. Moreover, it appears that the state of the art in theoretical physics itself is

such that the deep mathematical methods created by physicists in the 20th century may be preserved for future mankind only by the mathematical community; we are concerned that the remarkable fusion of the strictly rational approach to the exploration of the real world with outstanding mathematical techniques may be lost.

Anyhow, we wrote this book for a broad readership of mathematicians and theoretical physicists. As before (see the Prefaces to the books [DNF1, DNF2]), we followed the principles that the book must be as comprehensible as possible and written with the minimal possible level of abstraction. A clear grasp of the natural essence of the subject must be achieved before starting its formalization. There is no sense to justify a theory which has not been perceived yet. The formal language disconnects rather than unites mathematics, and complicates rather than facilitates its understanding.

We hope that our ideas will be met with understanding by the mathematical community.

The authors

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