Chapter 1

Introduction to Modules and their Structure Theory

Before getting to our main subject matter, we open the text with another kind of structure, namely **module**, which plays a key role behind the scenes throughout algebra. In Chapter 1 we are introduced to modules, together with their basic structure theory, which generalize Abelian groups. Although the "correct" language in which to study modules may be that of categories, as laid out in the appendix, we stick with an elementary formulation, for ease of presentation.

Definition 1.0. Given a ring R that is not necessarily commutative, a (left) R**-module** M is an Abelian group written in additive notation $(M,+)$, together with a product $R\times M \to M$ (called **scalar multiplication**) which satisfies the following properties, for all r_i in R and a_i in M:

M1. $1a = a$;
M2. $(r_1r_2)a = r_1(r_2a)$;
M3. $(r_1 + r_2)a = r_1a + r_2a$;
M4. $r(a_1 + a_2) = ra_1 + ra_2$.

Intuitively, we have required every version of associativity and distributivity that makes sense.

Remark 1.0′. Right modules are defined analogously, with scalar multiplication on the right, i.e., $M \times R \to M$, and the axioms changed accordingly. Over a commutative ring, any module could be written as a right module, just by reversing the scalar multiplication. However, this is not necessarily the case for modules over noncommutative rings, since axiom M2 is twisted around to $a(r_1r_2) = (ar_2)r_1$, whereas the associative law should read $a(r_1r_2) = (ar_1)r_2$.

Occasionally we need M to be both a left and right module. In this case, we can define both $(r_1a)r_2$ and $r_1(ar_2)$, for $r_i \in R$ and $a \in M$, and would like these to be equal. Accordingly, we define M to be an R, R-**bimodule** if M is both a left R-module and a right R-module, satisfying the extra axiom

$$(r_1a)r_2 = r_1(ar_2), \quad \forall r_i \in R, \ \forall a \in M.$$

Of course when R is commutative, any R-module M naturally becomes a bimodule when we define ar to be ra for all a in M and r in R. The situation for noncommutative rings is more delicate, to be discussed in Volume 2.

The module axioms are the same as for vector spaces over fields. Thus, in case R is a field, its modules are precisely the vector spaces over R. Other motivating examples:

Example 1.1.

1. Any Abelian group G is a $\mathbb{Z}$-module, under the action

$$ng = \begin{cases} g + \cdots + g \quad \text{taken} \quad n \quad \text{times} & \text{for} \quad n > 0; \\ 0 & \text{for} \quad n = 0; \\ -((-n)g) & \text{for} \quad n < 0. \end{cases}$$

   (In fact, this is the *only* $\mathbb{Z}$-module multiplication possible, as seen easily by induction on $|n|$.)
2. R itself is an R-module, under the usual ring operations.
3. (Generalizing (2)) If R is a subring of T, then T is an R-module when we define ra according to the given ring multiplication in T.
4. (Generalizing (3)) If $\varphi: R \to T$ is a ring homomorphism, then any T-module M can be viewed as an R-module, by defining ra to be $\varphi(r)a$.

There is another example which comes up in applications to the theory of matrices, but we defer that until Theorem 2.64. Other important examples of modules will arise later.

Remark 1.1′. One could view modules in terms of universal algebra as described in Chapter 0, via the following framework: An R-module is an Abelian group M together with the extra structure provided by unary operations $\ell_r : M \to M$ corresponding to left multiplication by r, for each $r \in R$;

these are group homomorphisms satisfying the following extra sentences for all $r, s \in R$, all $a, b \in M$:

$\ell_1(a) = a,$

$\ell_{rs}(a) = \ell_r(\ell_s(a)),$

$\ell_{r+s}(a) = \ell_r(a) + \ell_s(a),$

$\ell_r(a + b) = \ell_r(a) + \ell_r(b).$

From this point of view, we could forego various ring properties of R while building the theory; cf. Exercise 2. Nonetheless, here we study modules in terms of the given ring R. In Volume 2, we shall need theories of modules over groups (Chapter 19) and over Lie algebras (Chapter 21).

There is some ambiguity in the notation 0, which could apply to the zero element either of R or of M, but the meaning should be clear from its context. We leave it to the reader to check basic properties such as $r0 = 0$, and

$$(-r)a = r(-a) = -(ra)$$

for all $r \in R,\ a \in M$ (so, in particular, $0a = 0$).

Our first results are proved by modifying basic theorems about Abelian groups, to include scalar multiplication. Thus we shall use results from [**Row1, Chapter 4**] at will, with additive notation instead of multiplicative notation since we are dealing here with the additive group $(M, +)$. (Also, wherever possible, we pattern module theory after the theory of vector spaces over a field, which it generalizes, but we often are hampered by the lack of multiplicative inverses.) It is convenient to start with the following notion.

Definition 1.2. A **submodule** of M is an additive subgroup N that also is closed under scalar multiplication, i.e., $ra \in N$ for all $r \in R,\ a \in N$. We write $N \leq M$ to denote that N is a submodule of M.

We write 0 for $\{0\}$; 0 and M itself are submodules of M. Any submodule of a submodule of M is itself a submodule of M. Concerning our two main examples:

1. If G is an Abelian group viewed as $\mathbb{Z}$-module, then the submodules of G are precisely its subgroups;
2. Viewing the ring R as a module over itself, the submodules of R are precisely the left ideals of R. Thus $L \leq R$ denotes that L is a left ideal of R.

Remark 1.2′. Given an R-module M, and $A \subseteq R$ and $S \subseteq M$, we write AS for

$$\left\{ \sum_{\text{finite}} a_i s_i : a_i \in A,\ s_i \in M \right\}.$$

When $A \triangleleft R$, this is a submodule of M, as seen by direct verification. When $A = R$, the submodule RS is called the submodule **generated by** S. We shall review this important concept as we come across the special cases that we need.

One major approach to module theory is to study a module M in terms of its submodules. The following facts are basic in this perspective.

LEMMA 1.3. *Suppose M is a module and $M_1, M_2 \leq M$. Then*

1. $M_1 + M_2 \leq M$;
2. $M_1 \cap M_2 \leq M$;
3. *If $\{M_i : i \in I\}$ is a chain of submodules of M, then*

$$\bigcup_{i \in I} M_i \leq M.$$

Proof. These are all additive subgroups, and are easily seen to be closed under scalar multiplication. For example, in (3), if $r \in R$ and $a \in \cup M_i$, then $ra \in M_i$ for some i, implying $ra \in \bigcup_{i \in I} M_i$. □

Remark 1.3′. Generalizing Lemma 1.3(1), given submodules $\{M_i : i \in I\}$ of M, we define $\sum M_i$ to be the set of all *finite* sums of elements from the M_i; this is the smallest submodule of M containing each M_i.

Maps of modules

Definition 1.4. A **module homomorphism** $f\colon M \to N$ (where M, N are R-modules) is a homomorphism of the additive group structure, which also "preserves" scalar multiplication in the sense that

$$f(ra) = rf(a) \tag{1.1}$$

for all a, b in M and all r in R.

Module homomorphisms are also called **maps**, and we favor this terminology, in order to avoid confusion with ring homomorphisms.

Remark 1.5. To check that $f: M \to N$ is a map, one needs show that $f(a+b) = f(a) + f(b)$ and $f(ra) = rf(a)$ for all a, b in M and all r in R.

Note. Recall that in linear algebra, the vector space $V = F^{(n)}$ is often studied in terms of the ring of linear transformations from V to itself, which is identified with the matrix ring $M_n(F)$. Likewise, any R-module M gives rise to the important (noncommutative) ring of maps from M to itself; although some aspects of this approach arise in Chapter 2, we defer its detailed study until Volume 2.

Remark 1.6. If $f: M \to N$ is a map, then $f(M) \leq N$. (Indeed, $f(M)$ is a subgroup of N, and $rf(a) = f(ra) \in f(M)$, for all $r \in R,\ a \in M$.)

As with groups, the **kernel** of a map $f: M \to N$ of modules, written $\ker f$, is defined to be $f^{-1}(0)$. It is easy to see $\ker f \leq M$. (Indeed, we know $\ker f$ is an additive subgroup; if $r \in R$ and $a \in \ker f$, then

$$f(ra) = rf(a) = r0 = 0,$$

implying $ra \in \ker f$.) We recall the key property of the kernel:

Remark 1.7 (cf. [**Row1, 4.8**])**.** A map f is 1:1 iff $\ker f = 0$.

A 1:1 map is also called **monic**.

Definition 1.8. A module **isomorphism** is a map $f: M \to N$ that is both monic and onto.

Remark 1.8′. If $f: M \to N$ is an isomorphism, then $f^{-1}: N \to M$ is also an isomorphism. (Indeed, f^{-1} is an isomorphism of Abelian groups, so one needs to show

$$f^{-1}(ra) = rf^{-1}(a) \tag{1.2}$$

for all $r \in R$ and $a \in M$. But $ra = rff^{-1}(a) = f(rf^{-1}(a))$, so we get (1.2) by applying f^{-1} to both sides.)

The following idea will be used dozens of times in the sequel.

Remark 1.9. Given a ring R and module M, we can take any $a \in M$ and, viewing R as R-module, define the **right multiplication** map

$$f_a: R \to M,$$

given by $r \mapsto ra$. Clearly $f_a(R) = Ra = \{ra : r \in R\}$.

The **annihilator** of a, denoted $\mathrm{Ann}_R\, a$, is defined as $\ker f_a$, or equivalently $\{r \in R : ra = 0\}$. Note that if $a \neq 0$, we have $1a = a \neq 0$, implying $1 \notin \mathrm{Ann}_R\, a$. This proves the annihilator of any nonzero element is a *proper* left ideal of R.

Similarly, for R commutative, the left multiplication map $\ell_r\colon M \to M$ of Remark 1.1 is a module map, for any r in R.

Continuing our analogy with group theory, we note that any subgroup of an Abelian group is a normal subgroup. Thus given a submodule $K \leq M$ we define M/K in the usual way [**Row1, 5.11**] as cosets $\{a + K : a \in M\}$. Then $(M/K, +)$ is a group, which becomes a module via multiplication

$$r(a + K) = ra + K,$$

easily seen to be well-defined. (Indeed, if $a_1 + K = a_2 + K$, then $a_1 - a_2 \in K$, so

$$ra_1 - ra_2 = r(a_1 - a_2) \in K,$$

yielding $ra_1 + K = ra_2 + K$.)

With this module structure, M/K is called the **factor module** (or **quotient module**). There is a natural onto map $f\colon M \to M/K$ given by $a \mapsto a + K$; clearly $\ker f = K$. Thus we see:

Remark 1.10. Every submodule is the kernel of a suitable onto map.

In contrast, there is a big difference in ring theory between a subring and an ideal (the kernel of a ring homomorphism).

Example 1.11. Often one studies a ring R in terms of its factor modules. For example, if $R = \mathbb{Z}$, viewed as a module over itself, any submodule has the form $m\mathbb{Z}$, so the factor modules have the form $\mathbb{Z}/m$. These are finite groups when $m \neq 0$, and classical number theory often involves studying $\mathbb{Z}$ by passing to $\mathbb{Z}/m$, for various m.

Noether's isomorphism theorems for modules.

As with groups and rings, we lead in Noether's isomorphism theorems for modules with a method to define maps from factor modules.

PROPOSITION 1.12. *Given a map $f\colon M \to N$ whose kernel contains a submodule K of M, we can define a map $\bar{f}\colon M/K \to N$ given by*

$$\bar{f}(a + K) = f(a).$$

Then $\ker \bar{f} = \ker f/K$, *and* $\bar{f}(M/K) = f(M)$.

Proof. By [**Row1, 5.14**], $\bar{f}$ is a well-defined group homomorphism, and we check

$$\bar{f}(r(a+K)) = f(ra) = rf(a) = r\bar{f}(a+K).$$

Hence $\bar{f}$ is a map, and the other assertions follow from [**Row1, 5.14 and 5.15**]. □

Now the identical proofs used in [**Row1, 5.16, 5.17, and 5.18**] yield

THEOREM 1.13 (Noether isomorphism theorems for modules). *Suppose M is an R-module.*

Noether I. If $f\colon M \to N$ is an onto map (of modules), then $N \cong M/\ker f$.

Noether II. If $K \le N \le M$, then $M/N \cong (M/K)/(N/K)$.

Noether III. If $K, N \le M$, then $N/(K \cap N) \cong (K+N)/K$.

Noether III has special significance: Note that the construction N/K only works if $K \le N$. If $K, N \le M$ but $K \not\le N$, then the image of N under the natural map $M \to M/K$ is $(K+N)/K$, which appears in Noether III.

Example 1.14. Suppose R is an integral domain, with $0 \ne r \in R$, and $A \subset B$ are ideals of R. Then $rB/rA \cong B/A$ as R-modules. Indeed, define $f\colon B \to rB/rA$ by $f(b) = rb + rA$, for $b \in B$. Then f is onto, and

$$\ker f = \{b \in B : rb \in rA\} = \{b \in B : b \in A\} = A.$$

Exact sequences.

Here is a schematic way of viewing Noether I.

Definition 1.15. A sequence of maps $\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \to \dots$ of modules is called "exact" at B if $f(A) = \ker g$. A sequence of maps is called **exact** if it is exact at each module.

Here are some instances of exact sequences:

(i) $0 \to A \xrightarrow{f} B$ is exact iff $f\colon A \to B$ is monic;

(ii) $A \xrightarrow{f} B \to 0$ is exact iff $f\colon A \to B$ is onto;

(iii) $0 \to A \xrightarrow{f} B \to 0$ is exact iff f is an isomorphism;

(iv) $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ is exact iff f is monic, g is onto, and (by Noether I) g induces an isomorphism $M/f(K) \cong N$.

Since the sequence (iv) is the shortest exact sequence which starts and ends with 0 and says something nontrivial, it is called a **short exact sequence**, and is fundamental in module theory.

Commutative diagrams.

Definition 1.16. A **commutative diagram** is a two-dimensional chart of arrows, denoting maps, such that following the route indicated by two successive arrows corresponds to taking the composite of the maps; when there are two routes from one position to another, this indicates that the corresponding composite maps are equal.

For example, the **commutative triangle**

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle h}\searrow & \downarrow {\scriptstyle g} \\ & & C \end{array}$$

just says $h = gf$.

Perhaps the most common commutative diagram is the **commutative square**

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{k} & D \end{array}$$

indicating $gf = kh$ as maps from A to D. This becomes more interesting when we have more complicated commutative diagrams, such as

$$\begin{array}{ccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} \\ B_1 & \xrightarrow{k_1} & B_2 & \xrightarrow{k_2} & B_3; \end{array}$$

here $g_3 f_2 f_1 = k_2 g_2 f_1 = k_2 k_1 g_1$, and so forth. When we are about to define a map we write a dashed arrow, e.g., $A \dashrightarrow B$.

The lattice of submodules of a module

Define $\mathcal{L}_R(M)$ to be the set of R-submodules of M. Our last result lifted directly from group theory is

THEOREM 1.17. *Suppose $K \leq M$. Every submodule of M/K has the form N/K for N a submodule of M containing K, uniquely determined. Thus there is a 1 : 1 correspondence*

$$\mathcal{L}_R(M/K) \to \{\textit{Submodules of } M \textit{ containing } K\},$$

given by $N/K \mapsto N$.

Proof. The correspondence of Abelian subgroups comes from [**Row1, 5.19′**]; one needs to check further that if $N/K \leq M/K$, then $N \leq M$. But this is clear, since for $r \in R,\ a \in N$ we have

$$ra + K = r(a + K) \in N/K,$$

implying $ra \in N$. □

In order to strengthen Theorem 1.17, we look for analogies between $\mathcal{L}_R(M)$ and the power set $\mathcal{P}(S)$ of a set S. Noting that the union (resp. intersection) of subsets is a subset, we introduce another kind of structure.

Definition 1.18. A **lattice** is a poset in which every pair $\{a, b\}$ of elements has a **supremum** (sup) $a \vee b$ and an **infinum** (inf) $a \wedge b$. In other words, $a \vee b \succeq a$ and $a \vee b \succeq b$, but $a \vee b \preceq c$ whenever $c \succeq a$ and $c \succeq b$; analogously for $\wedge$, where $\preceq$ and $\succeq$ are interchanged.

Note. Unfortunately, there is a different usage of the term "lattice," as defined in Appendix 2A, but its context is completely different.

Example 1.18′. Some examples of lattices:

1. The **power set lattice** $(\mathcal{P}(S), \subseteq)$, where $\vee = \cup$ and $\wedge = \cap$.
2. Any chain is a lattice with respect to the given (total) order, since for $a \prec b$ we have $a \vee b = b$ and $a \wedge b = a$.
3. $\mathcal{L}_R(M)$ is a lattice, where $M_1 \wedge M_2$ is $M_1 \cap M_2$ (as with the power set), but $M_1 \vee M_2 = M_1 + M_2$.
4. If L is a lattice, then the dual poset to L is also a lattice, called the **dual lattice**, where $\vee$ and $\wedge$ are interchanged.

A **lattice homomorphism** $f\colon L \to L'$ is a function which "preserves" sup and inf, i.e.,

$$f(a_1 \wedge a_2) = f(a_1) \wedge f(a_2) \quad \text{and} \quad f(a_1 \vee a_2) = f(a_1) \vee f(a_2).$$

Remark 1.19. The correspondence of Theorem 1.17 is actually a 1:1 lattice homomorphism $\mathcal{L}_R(M/K) \to \mathcal{L}_R(M)$ whose image consists of the lattice of submodules containing K. To wit:

$$(N_1 + N_2)/K = N_1/K + N_2/K; \quad (N_1 \cap N_2)/K = (N_1/K) \cap (N_2/K),$$

for any submodules N_1, N_2 of M containing K.

Although the power set lattice $\mathcal{P}(S)$ is "distributive" (i.e., satisfies De Morgan's laws), the analogous property can fail for $\mathcal{L}(M)$. However, $\mathcal{L}(M)$ does satisfy a significant (weaker) property.

PROPOSITION 1.20 (Modularity property). *If $N_1 \subseteq N_2$ and K are submodules of M, then $(N_1 + K) \cap N_2 = N_1 + (K \cap N_2)$.*

Proof. ($\supseteq$) The right-hand side is contained in both $N_1 + K$ and N_2.

($\subseteq$) Given $a_2 = a_1 + b \in N_2 \cap (N_1 + K)$, for $a_i \in N_i$ and $b \in K$, one has $b = a_2 - a_1 \in K \cap N_2$. $\square$

The modularity property is crucial in the theory, and there is a cute reformulation given in Exercise 1. Although somewhat technical, especially in its applications, the modularity property always boils down to the following fact: If $a = c + b$, then $c = a - b$.

We end the general discussion by considering a change of the base ring.

Remark 1.21 (Change of rings)**.** Suppose $I \triangleleft R$.

(i) Any R/I-module M can be viewed as an R-module, by defining the new scalar multiplication ra to be $(r + I)a$ for r in R, a in M. Note that $rM = 0$ for any r in I (since then $r + I = 0$ in R/I).

(ii) (Converse to (i).) Suppose M is an R-module satisfying $rM = 0$ for all r in I. Then M becomes an R/I-module, where we define $(r + I)a$ to be ra, for r in R, a in M. (The only tricky part of the proof is to prove that this new multiplication is well-defined. Suppose $r + I = r' + I$. Then $r - r' \in I$, implying $(r - r')a = 0$, so $ra = r'a$, as desired.)

In this case, there is a lattice isomorphism from $\mathcal{L}_R(M)$ to $\mathcal{L}_{R/I}(M)$, given by sending an R-submodule N of M to itself (viewing N as an R/I-module); the inverse correspondence comes from (i).

Appendix 1A: Categories

The theory of modules is an excellent example of a meta-theory called **category theory**, which enables one to unify many diverse branches of mathematics. Category theory is an extremely powerful theory that naturally draws one to many of the basic concepts of algebraic theories. Unfortunately, its generality requires redefining and reproving even the most basic concepts, and so we cannot afford the space for a proper treatment; we give a few basic definitions here and continue in the exercises.

The idea of category theory is to focus on the homomorphism as the basic notion; now it will be called a **morphism**. Accordingly, we have

Definition 1A.1. A **category** $\mathcal{C}$ is comprised of **objects** together with a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of morphisms (usually written $\mathrm{Hom}(A, B)$) for any objects A, B of $\mathcal{C}$, together with a **composition**

$$\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C),$$

written $(g, f) \mapsto gf$ (in which case we say the morphisms g, f are **compatible**) satisfying the following two rules:

(i) Associativity of composition: $(hg)f = h(gf)$ (whenever the relevant morphisms are compatible);

(ii) Each $\mathrm{Hom}(A, A)$ has a **unit morphism** 1_A satisfying $g1_A = 1_B g = g$ for all $g \in \mathrm{Hom}(A, B)$.

For example, **Set** denotes the category whose objects are the sets and whose morphisms $\mathrm{Hom}(A, B)$ are the functions from A to B. But recall that the class of all sets is not a set! Whereas the categories that behave like **Set** attract our main attention, in each case the class of objects is not a set. Accordingly, we say a category is **small** if its objects comprise a set. Although the small categories are of less immediate interest, here are some useful examples of small categories:

Example 1A.2. (i) Any monoid M defines a small category, which has a single object $*$, and whose set of morphisms $\mathrm{Hom}(*, *) = M$; composition is just the monoid multiplication in M. (The rules of composition are automatically satisfied, where the unit element of M takes the role of the unit morphism.)

(ii) Any poset $(\mathcal{S}, \prec)$ defines a small category, whose objects are the elements of $\mathcal{S}$, with $\mathrm{Hom}(a, b)$ either a set with a single element or the empty set, depending on whether or not $a \preceq b$. Associativity of composition follows from transitivity of the relation.

(iii) (Special case of (ii).) Any set I can be endowed with the trivial PO (that all distinct elements are incomparable), and thus constitutes a small category, where $\mathrm{Hom}(a, b)$ is $\emptyset$ whenever $a \neq b$, and $\mathrm{Hom}(a, a)$ consists of the single morphism 1_a. Thus, two distinct morphisms are never compatible, so the categorical axioms hold almost vacuously.

(iv) Any topological space X is a small category, whose objects are the open sets and whose morphisms are the inclusions. This can also be viewed as a special case of (ii), where $\mathcal{S}$ is the collection of open sets of X, ordered by inclusion. A common instance of this is when X has the discrete topology, so that every subset is open; then $\mathcal{S}$ is the power set $\mathcal{P}(X)$.

A category $\mathcal{D}$ is called a **subcategory** of $\mathcal{C}$ if all objects of $\mathcal{D}$ are objects of $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{D}}(A, B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, B)$ for all objects A, B of $\mathcal{D}$. If $\mathrm{Hom}_{\mathcal{D}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$ for all objects A, B of $\mathcal{D}$, we say $\mathcal{D}$ is a **full subcategory** of $\mathcal{C}$.

Example 1A.3 (Universal Algebra)**.** The categories arising in universal algebra are the subcategories of **Set** whose objects are structures of a given signature, and the morphisms are the homomorphisms in that signature. These categories are not small, cf. Exercise A1. For example, we have

1. **Mon** is the category whose objects are monoids, and $\mathrm{Hom}(A, B)$ is the set of monoid homomorphisms $f\colon A \to B$; i.e., $f(1) = 1$, and $f(a_1a_2) = f(a_1)f(a_2)$, $\forall a_1, a_2 \in A$.
2. **Grp** is the category whose objects are groups, and whose morphisms are group homomorphisms. **Grp** is a full subcategory of **Mon**.
3. **Ab** is the full subcategory of **Grp** whose objects are Abelian groups.
4. **Ring** is the subcategory of **Ab** whose objects are rings, and whose morphisms are ring homomorphisms. Note that **Ring** also is a subcategory of **Mon**, since any ring homomorphism is both a group homomorphism (with respect to addition) and a monoid homomorphism (with respect to multiplication).
5. R-**Mod** is the category whose objects are (left) modules over a given ring R, and whose morphisms are module maps.
6. **Mod**-R is the category whose objects are right modules over a given ring R, and whose morphisms are maps of right modules.
7. If $\mathcal{C}$ and $\mathcal{D}$ are categories, then one can define the category $\mathcal{C} \times \mathcal{D}$, whose objects are pairs (C, D) where C and D are objects of $\mathcal{C}$ and $\mathcal{D}$ respectively, and with

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((C_1, D_1), (C_2, D_2)) = \mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \times \mathrm{Hom}_{\mathcal{D}}(D_1, D_2).$$

Since the class of objects need not be a set, we are not permitted to use the laws of set theory with all the objects at once, and are drawn instead to $\mathrm{Hom}(A, B)$ and its morphisms. This is the great innovation of category theory, and from the outset we define concepts in terms of "arrows," often as they appear in commutative diagrams. We write $f\colon A \to B$ or $A \xrightarrow{f} B$ to denote the morphism f in $\mathrm{Hom}(A, B)$.

Properties of morphisms.

An **isomorphism** is just a morphism $f\colon A \to B$ for which there is a morphism $g\colon B \to A$ such that $gf = 1_A$ and $fg = 1_B$. However, it is less obvious how to generalize the notions of 1:1 and onto.

Definition 1A.4. (i) A morphism $f\colon A \to B$ is **monic** if for any morphisms $g \neq h\colon C \to A$ we have $fg \neq fh$.

(ii) A morphism $f\colon A \to B$ is **epic** if for any $g \neq h\colon B \to C$ we have $gf \neq hf$.

To some extent, monics correspond to 1:1 maps, and epics correspond to onto maps.

Remark 1A.5. If $\mathcal{C}$ is a subcategory of **Set**, then any 1:1 map is monic, and any onto map is epic. (Indeed, if $f: A \to B$ is 1:1 and $g \neq h: C \to A$, then taking $c \in C$ such that $g(c) \neq h(c)$, we have $fg(c) \neq fh(c)$, so $fg \neq fh$. Likewise, if f is onto and $g \neq h: B \to C$ then taking $g(b) \neq h(b)$ and writing $b = f(a)$ we see $gf(a) \neq hf(a)$.)

The converse also holds in **Set**, seen by reversing the above arguments, but does not always hold in general; cf. Exercise A2. Nevertheless, it does hold in R-**Mod**.

PROPOSITION 1A.6. *In R-**Mod**, monics are 1:1 and epics are onto.*

Proof. Suppose $f: M \to N$ is a module map. If f is monic, we want to prove $\ker f = 0$. So define $g, h: \ker f \to M$ by taking $g = 0$ and h to be the natural injection. Then $fg = fh = 0$, so $g = h = 0$, i.e., $\ker f = 0$.

If f is epic, we want to prove $f(M) = N$. So define $g, h: N \to N/f(M)$ by taking $g = 0$ and h to be the natural onto map. Then $gf = hf = 0$, implying $g = h = 0$, so $N/f(M) = 0$, as desired. □

Accordingly, "epic" and "onto" are used interchangeably for modules, the former often being used when one wants to emphasize the categoric nature of module theory. This simple proof gives us a glimpse of the power of category theory, and also elevates the category R-**Mod** to a special status; the most fundamental categorical ideas lead to important notions in module theory, such as 0 maps, cokernels, and coproducts. See Exercises A7ff. for some sample results. In the other direction, once we have notions in module theory, we are led to categorize them and push the category theory even further. For example, let us define the "kernel" of a map in categorical theoretic terms.

Definition 1A.7. A category $\mathcal{C}$ is called **pre-additive** if $\mathrm{Hom}(A, B)$ has the natural structure of an Abelian group for all objects A, B and the composition

$$\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$$

is bilinear. In particular each $\mathrm{Hom}(A, B)$ contains the **zero morphism** $0: A \to B$. (Strictly speaking we should write $0_{A,B}$, since each $\mathrm{Hom}(A, B)$ has its own 0 morphism.)

We work in a pre-additive category, so that the definitions become more intuitive.

Definition 1A.8. Define a partial order on monics $f_i: B_i \to A$ by saying $f_1 \leq f_2$ if $f_1 = f_2 h$ for some morphism $h: B_1 \to B_2$. We define the equivalence $f_1 \equiv f_2$ to hold when $f_1 \leq f_2$ and $f_2 \leq f_1$.

The **(categorical) kernel** (if it exists) of a morphism $f: A \to B$ is a monic $k: K \to A$ such that $fk = 0$, and which is largest in the sense of our partial order. (Thus the categorical kernel is unique up to equivalence).

One could take $K = \ker f$ (cf. Exercise A6), indicative of the tendency to move from objects to morphisms when we generalize from R-**Mod** to arbitrary pre-additive categories. Some notions become rather complicated with this new focus on morphisms — for example, a "subobject" of A is defined as an equivalence class of monics to A — but a few moments' thought shows that one also achieves more precision.

The Dual category.

Definition 1A.9. The **dual category** $\mathcal{C}^{\text{op}}$ of a category $\mathcal{C}$ has the same objects as C, but the morphisms are obtained by reversing the arrows, i.e., $\text{Hom}(A, B)$ in $\mathcal{C}^{\text{op}}$ is $\text{Hom}(B, A)$ in $\mathcal{C}$; likewise the composition is reversed. (If $f \in \text{Hom}_{\mathcal{C}^{\text{op}}}(B, C)$ and $g \in \text{Hom}_{\mathcal{C}^{\text{op}}}(A, B)$, then f, g correspond respectively to morphisms in $\text{Hom}_{\mathcal{C}}(C, B)$ and $\text{Hom}_{\mathcal{C}}(B, A)$ that are compatible in reverse order.)

Example 1A.10. We start with the opposites of the small categories in Example 1A.2.

(i) Viewing a monoid M as a small category, we define M^{op} to be the same set, but with multiplication in reverse order.

(ii) As a special case of (i), for any ring R, we can define R^{op} to have the same addition, but multiplication in reverse order.

(iii) Viewing a poset $(\mathcal{S}, \prec)$ as a small category, we see its dual category is the dual poset. In particular, the dual lattice is the categorical dual of a lattice (viewed as a small category).

The duals of non-small categories are harder to describe. On the other hand, the dual of an interesting categorical notion tends to be interesting. For example, the dual of "monic" is "epic." Often we define a dual notion formally, appending the prefix "co;" the following definition is in this spirit.

Definition 1A.11. The **cokernel** $\text{coker}\, f$ of a morphism $f: A \to B$ is an epic $h: B \to C$ with $hf = 0$, such that for every morphism $h': B \to C'$ with $h'f = 0$ there is a morphism $g: C \to C'$ such that $h' = gh$.

See Exercises A7–A11 for a treatment of the cokernel (when it exists). From this point of view, one would like to work with a class of categories whose dual lies in the same class; in this case, any theorem would instantly provide a theorem in the dual category and thus translate back to a new "dual" theorem in the original category. This is a major theme in category theory, and leads to the study of Abelian categories (cf. Exercise A19); unlike R-**Mod** , the general class of Abelian categories is closed under duality, and we can formulate many theories from module theory in this framework. In this way, working with Abelian categories enables one to progress much faster in certain subjects such as homology and cohomology theory, once the machinery is put into motion.

Many important module-theoretic constructions have roots in category theory; cf. Exercises A23–A31.

Functors.

Having shifted emphasis to categories, we would like a way to compare two categories.

Definition 1A.12. A **(covariant) functor** $F: \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a correspondence sending each object A of $\mathcal{C}$ to an object FA of $\mathcal{D}$, and each morphism $f: A \to B$ of $\mathcal{C}$ to a morphism $F(f): FA \to FB$ in $\mathcal{D}$, such that $F(1_A) = 1_{FA}$ for every object A of $\mathcal{C}$ and

$$F(fg) = F(f)F(g)$$

whenever f, g are compatible morphisms.

A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$.

Example 1A.13. (i) The **identity functor** $F = 1_{\mathcal{C}}$ satisfies $FA = A$ and $F(f) = f$, for every object A and every morphism f.

(ii) The **forgetful functor** "forgets" some of the structure in universal algebra. For example, F: **Grp** $\to$ **Mon** views a group as a monoid, forgetting that elements have inverses.

There are two very important functors in module theory, the so-called Hom and $\otimes$ functors, that lie at the foundation of homology theory; however, we shall deal with them only in passing, in Volume 2. Having functors at our disposal, we define composition of functors in the obvious way and formulate:

Definition 1A.14. Two categories $\mathcal{C}$, $\mathcal{D}$ are **isomorphic** if there are functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ such that $GF = 1_{\mathcal{C}}$ and $FG = 1_{\mathcal{D}}$.

Example 1A.15. (i) $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ is isomorphic to $\mathcal{C}$ as categories.

(ii) Any module M over a ring R is also a *right* module over R^{op}; thus we have a natural functor R-**Mod** $\to$ **Mod**-R^{op}, which is clearly an isomorphism of categories.

Two isomorphic categories have the same theories, so Example A15(ii) explains why module theory and right module theory are essentially the same (although a given noncommutative ring might have entirely different left module structure from its right module structure). However, there are many other situations in which two categories have the same theory. In order to understand why, we want to ascend one level, and consider a new category $\mathbf{Fun}(\mathcal{C})$, whose objects are functors. But presently we need the morphisms in this new category.

Definition 1A.16. Suppose $F, \tilde{F}: \mathcal{C} \to \mathcal{D}$ are functors. A **natural transformation** $\eta: F \to \tilde{F}$, also called a **morphism of functors**, assigns a morphism $FA \xrightarrow{\eta_A} \tilde{F}A$ for each object A of $\mathcal{C}$, such that for any morphism $f: A \to B$ in $\mathcal{C}$ we have the commutative square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ \tilde{F}A & \xrightarrow{\tilde{F}f} & \tilde{F}B, \end{array}$$

i.e., $(\tilde{F}f)\eta_A = \eta_B(Ff)$.

Natural transformations provide the appropriate way of comparing categories.

Definition 1A.17. With notation as in Definition 1A.16, a natural transformation η is called a **natural isomorphism** if each η_A is an isomorphism; in this case the functors F and $\tilde{F}$ are called **naturally isomorphic**, written $F \simeq \tilde{F}$. Categories $\mathcal{C}$ and $\mathcal{D}$ are **equivalent** if there are functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ such that $FG \simeq 1_{\mathcal{C}}$ and $GF \simeq 1_{\mathcal{D}}$.

Categorical equivalence enables us to transfer all categorical properties from a category $\mathcal{C}$ that we know to an equivalent category $\mathcal{D}$. We continue this approach in Chapter 25 of Volume 2, in connection with Morita theory. Meanwhile we conclude the present discussion with the definition of $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$.

Definition 1A.18. $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ denotes the category whose objects are the functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are the natural transformations between functors.

Important Note. This definition is only valid when the class of natural transformations between two functors comprise a set! Of course, this is true when the category $\mathcal{C}$ is small. This reservation must be borne in mind whenever we want to apply the definition.

Exercises — Part I

Exercises — Chapter 1

1. (Kasch-Mader) Reformulate the modularity property in terms of the exactness of the sequence

$$0 \to (K \cap N_2)/(K \cap N_1) \to N_2/N_1 \to (K + N_2)/(K + N_1) \to 0.$$

2. (Modules over arbitrary monoids) Given a monoid S, define an S-**module** to be an Abelian group $(M, +)$ together with a product $S \times M \to M$ satisfying the following properties, for all r_i in S and a_i in M: (i) $1a = a$; (ii) $(r_1r_2)a = r_1(r_2a)$; (iii) $r(a_1 + a_2) = ra_1 + ra_2$. Develop the Noether isomorphism theorems for S-monoids.
3. Any exact sequence $0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ yields an exact sequence

$$0 \to K/K_1 \xrightarrow{\bar{f}} M/K_1 \xrightarrow{\bar{g}} N \to 0,$$

for any $K_1 \leq K$.
4. For any collection $\{N_i : i \in I\}$ of submodules of M, there is a module map $M \to \prod M/N_i$, given by $a \mapsto (a + N_i)$, having kernel $\cap N_i$.
5. (The Five Lemma) If in the commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
\downarrow g_1 & & \downarrow g_2 & & \downarrow g_3 & & \downarrow g_4 & & \downarrow g_5 \\
B_1 & \xrightarrow{h_1} & B_2 & \xrightarrow{h_2} & B_3 & \xrightarrow{h_3} & B_2 & \xrightarrow{h_4} & B_3
\end{array}$$

g_1, g_2, g_4, g_5 are isomorphisms, then g_3 is also an isomorphism.

Appendix 1A: Categories

1. In universal algebra, there are structures of arbitrarily large cardinality, for any given signature. Thus, any category in universal algebra is *not* small!
2. The natural morphism $\mathbb{Z} \to \mathbb{Q}$ in **Ring** is epic and monic, but not onto.
3. Every monic $f\colon A \to B$ in **Ring** is 1:1. (Hint: Let

$$P = \{(a_1, a_2) \in A \times A : f(a_1) = f(a_2)\},$$

and let g, h be the restriction of the respective projections π_1, π_2 from $A \times A$ to A.)
4. In **Grp**, prove that every monic is 1:1, and every epic is onto. (Hint: For the latter assertion, given an epic $f\colon A \to B$ show that f is onto if $f(A) \triangleleft B$, so one may assume $[B : f(A)] \geq 2$. But consider homomorphisms g and h from B to the group of permutations of cosets of $f(A)$.)
5. The composition of epics is epic; the composition of monics is monic.

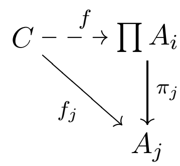
Kernels and cokernels

6. If $f\colon A \to B$ is a module map with kernel K, then the natural injection $i\colon K \to A$ is a categorical kernel of f; and, conversely, if $k\colon K \to A$ is a categorical kernel of f, then $k(K) = \ker f$.
7. Show that the definition of "cokernel" is categorically dual to "kernel."
8. If f is a kernel and $\operatorname{coker} f$ exists, then $f = \ker(\operatorname{coker} f)$.
9. Suppose $f = hg$ where g is a cokernel and h is a kernel. If f is monic, then f is equivalent to h (and thus is a kernel). More generally, $\ker f = \ker g$ and $\operatorname{coker} f = \operatorname{coker} h$.
10. Show for any morphism $f : A \to B$ in R-**mod** that $\operatorname{coker} f$ is the canonical map $B \to B/f(A)$.
11. In R-**mod** , every morhpism is the composite of a cokernel and a kernel.

Products and coproducts

12. If $\mathcal{C}$ is the set $\{1, \dots, n\}$, viewed as a small category as in Example 1A.2(iii), then $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ can be identified with $\mathcal{D}^{(n)}$.
13. A **product** (if it exists) of objects $\{A_i : i \in I\}$ in a category $\mathcal{C}$ is an object $\prod A_i$, together with morphisms $\pi_j\colon \prod A_i \to A_j$ called **projections**, such that for any object C and morphisms $f_j\colon C \to A_j$ there

is a unique morphism $f: C \to \prod A_i$ such that the diagram



commutes for each j.

14. Products can be defined in any category defined in terms of universal algebra; $(\prod_{i\in I} A_i; (\pi_j))$ is just the cartesian product $\prod_{i\in I} A_i$, with the π_i the usual projections.
15. Define the **coproduct** $\coprod A_i$ and its morphisms $\nu_i: A_i \to \coprod A_i$, dually to the product. Write this explicitly. Show in R-**mod** that the coproduct is the direct sum $\oplus A_i$, together with the canonical injections ν_i.
16. The product of a given set of objects is unique up to isomorphism, if it exists. Likewise for a coproduct.

Preadditive and Abelian categories

17. Suppose objects $A_1, \dots, A_n$ of a preadditive category have the product $(\prod A_i, (\pi_i : 1 \le i \le n))$. Find morphisms $\nu_j: A_j \to \prod A_i$ such that $\sum_{j=1}^n \nu_j \pi_j = 1_{\prod A_i}$ and $\pi_i \nu_j = \delta_{ij}$, where $\delta_{ii} = 1_{A_i}$ and $\delta_{ij}: A_j \to A_i$ is 0 for $i \ne j$. (Hint: Define $\nu_j: A_j \to \prod A_i$ such that $\pi_i \nu_j = \delta_{ij}$.)
18. Notation as in Exercise A17, $(\prod A_i, (\nu_j : 1 \le j \le n))$ is the coproduct of the A_i. Conclude that a finite set of objects in a preadditive category has a product iff it has a coproduct, in which case these are isomorphic.
19. A preadditive category $\mathcal{C}$ is **Abelian** if satisfies the extra properties:

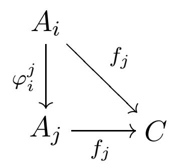
    (i) The product of any finite set of objects exists in $\mathcal{C}$; (ii) Every morphism has a kernel and a cokernel; (iii) Every morphism f can be written as hg where g is a cokernel and h is a kernel.

    If a category $\mathcal{C}$ is Abelian, then so is $\mathcal{C}^{\text{op}}$. Thus the notion of Abelian category is self-dual.
20. Show R-**mod** is an Abelian category.
21. If f is a morphism in an Abelian category, define the **image** $\operatorname{im} f = \ker(\operatorname{coker} f)$. Show that if $f = hg$, where g is a cokernel and h is a kernel, that $\operatorname{im} f = h$.
22. In an Abelian category, any monic epic is an isomorphism.

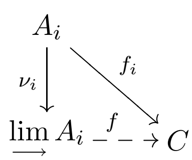
Direct and inverse limits

Here are two general constructions of algebraic objects within a given category, which reoccur repeatedly in algebraic theories. More information is given in Exercises 8.34ff.

23. Suppose I is a given poset. A **system** in a category $\mathcal{C}$ is a functor from I (viewed as a small category) to $\mathcal{C}$. This can be viewed as a set $\{A_i : i \in I\}$ of objects together with $\varphi_i^j \in \mathrm{Hom}(A_i, A_j)$ for each $i \leq j$ satisfying $\varphi_i^i = 1_{A_i}$ and $\varphi_j^k \varphi_i^j = \varphi_i^k$ for all $i \leq j \leq k$. Given an object C, we say a set of morphisms $f_i\colon A_i \to C$ is **compatible** (with the system) if the commutative diagram



holds for each $i \leq j$, i.e., if $f_i = f_j \varphi_i^j$. Define the **direct limit** (if it exists) of the system to be an object $\varinjlim A_i$ together with compatible morphisms $\nu_i\colon A_i \to \varinjlim A_i$, such that for any object C and compatible morphisms $f_i\colon A_i \to C$ there is a unique $f\colon \varinjlim A_i \to C$ completing the commutative diagram



for all $i \leq j$, i.e., $f\nu_i = f_i$.

When I has the trivial partial order, the compatibility condition becomes vacuous and the direct limit becomes the coproduct.

24. (Generalizing Exercise A15) Direct limits always exist in R-**mod**. (Hint: It is

$$(\oplus M_i) \Big/ \sum_{i \leq j} \left(\nu_j \varphi_i^j - \nu_i\right)(M_i).)$$

25. A poset I is **directed** if I has the property that each pair of elements has an upper bound. Show that whenever the index set I is directed, $\varinjlim A_i$ exists in any category $\mathcal{C}$ defined in first-order logic. (Hint: Define a relation $\sim$ on the disjoint union $\dot{\cup} A_i$ by saying $a \sim b$ (for $a \in A_i$, $b \in A_j$) if $\varphi_i^k a = \varphi_j^k b$ for some $k \geq i, j$.) Interpret this when I is a chain.

26. (Inverse, or projective, limits) Define **inverse system** dually to system, as a set $\{A_i : i \in I\}$ of objects together with $\varphi_i^j \in \mathrm{Hom}(A_j, A_i)$

for each $i \leq j$ satisfying $\varphi_i^i = 1_{A_i}$ and $\varphi_i^j \varphi_j^k = \varphi_i^k$ for all $i \leq j \leq k$. Given an object A, we say a set of morphisms $g_i\colon A \to A_i$ is **compatible** (with the system) if $g_i = \varphi_i^j g_j$ for each $i \leq j$. Define the **inverse limit** (if it exists) of the system to be an object $\varprojlim A_i$ together with compatible morphisms $\nu_i\colon \varprojlim A_i \to A_i$, such that for any object C and compatible morphisms $f_i\colon C \to A_i$ there is a unique $f\colon C \to \varprojlim A_i$ completing the commutative diagram

$$\begin{array}{ccc} C & \stackrel{f}{\dashrightarrow} & \varprojlim A_i \\ & {\scriptstyle g_i}\searrow & \downarrow{\scriptstyle \nu_i} \\ & & A_i \end{array}$$

for all $i \leq j$, i.e., $\nu_i f = g_i$.

If $\mathcal{C}$ is a category defined in first-order logic, then

$$\varprojlim A_i = \{(a_i) \in \prod A_i : a_i = \varphi_i^j a_j,\ \forall i \leq j\}$$

in the sense that it exists if the right-hand side is an object in the category. (Hint: Take ν_j to be the restrictions of the canonical projection $\pi_j\colon \prod A_i \to A_j$ to the right-hand side.)

27. The direct limit is unique up to isomorphism (if it exists). Similarly for the inverse limit.
28. An object A of a $\mathcal{C}$ is called **finite** (also called **finitely generated**) if A cannot be written as a direct limit of proper subobjects. In a category $\mathcal{C}$ defined in first-order logic, if C is the intersection of all objects that are subsets of A containing $a_1, \dots, a_n$, then C is finite.

The pullback and pushout

These exercises display the kernel and cokernel as inverse and direct limits respectively.

29. (The pullback) Given two morphisms $f_i\colon A_i \to A$, $i = 1, 2$, define their **pullback** (P, ρ_1, ρ_2), where $\rho\colon P \to \prod A_i$ is the kernel of the morphism $f_1\pi_1 - f_2\pi_2\colon \prod A_i \to A$, and $\rho_i = \pi_i \rho\colon P \to A_i$. This is an inverse limit, and thus has the property that for any object P' and morphisms $\rho'\colon P' \to A_i$ there is a unique morphism $g\colon P' \to P$ such that $\rho_i g = \rho_i'$, $i = 1, 2$.
30. Assume $\mathcal{C}$ is Abelian. Given subobjects $f_i\colon A_i \to A$ of A, define their **intersection** to be their pullback, and define their (categorical) **sum** to be the image of the natural morphism $f\colon \coprod A_i \to A$. Show that the subobjects of A form a modular lattice, where the sup and inf are respectively categorical sum and intersection.

31. Define the categorical dual of the pullback. This is called the **pushout**. Show that the pushout always exists in R-**mod**, as a suitable homomorphic image of $A_1 \oplus A_2$.

Exercises — Chapter 2

1. If M is a f.g. module, then M has maximal submodules.

Direct products and sums

2. If $M = \oplus M_i$, then $\ell(M) = \sum \ell(M_i)$.
3. If $g\colon M \to N$ and $h\colon N \to M$ satisfy $gh = 1_N$, then g is epic and h is monic.
4. (Converse to Remark 2.14.) Suppose $\{M_i : i \in I\}$ is a set of submodules of a module M, with maps $\pi_i : M \to M_i$ and $\nu_i : M_i \to M$ satisfying $\pi_i\nu_i = \mathbf{1}_{M_i}$, $\pi_i\nu_j = 0$ for all $i \neq j$, and

$$\sum_{i \in I} \nu_i\pi_i = \mathbf{1}_M,$$

whereby for any a in M, $\nu_i\pi_i(a) = 0$ for almost all $i \in I$. Then $M \cong \oplus M_i$, given by $a \mapsto (\pi_i(a))$.

Split exact sequences

5. A monic f is called a **split monic** if there is a map $j\colon M \to K$ such that $jf = 1_K$. The following conditions are equivalent, for $N \leq M$:

(i) N is a direct summand of M.

(ii) There is a map $\pi\colon M \to N$ with $N = \{a \in M : \pi(a) = a\}$.

(iii) There is a map $\pi\colon M \to M$ with $\pi^2 = \pi$ (where $N = \pi(M)$).

(iv) There is a split epic $\pi\colon M \to N$.

(v) There is an exact sequence $0 \to K \xrightarrow{\nu} M \xrightarrow{\pi} N \to 0$, where ν is a split monic and π is a split epic.

(vi) There is a split monic $\nu\colon K \to M$.

(Hint: (vi) $\Rightarrow$ (v) Take $\pi\colon M \to K$ with $\pi\nu = 1_K$.)

Free modules

6. $R^{(n)} \cong R^{(n-1)} \oplus R$, under the map

$$(r_1, \dots, r_n) \mapsto ((r_1, \dots, r_{n-1}), r_n).$$

Likewise, $R^{(n)} \cong R \oplus R^{(n-1)}$.

7. If R is commutative and some module map $\varphi\colon R^{(m)} \to R^{(n)}$ is onto, then $m \geq n$. (Hint: φ can be given by an $m \times n$ matrix B. Taking images modulo a maximal ideal P of R, assume R is a field.)