the formula (4.30) provides the analytic continuation of the reciprocal of (3.20) from the right half-plane to the entire complex plane. Since zeros of entire functions are isolated, this implies that the possible singularities of $\Gamma(z)$ are at worse isolated poles of finite order. Hint: Prove that the integral is differentiable with respect to $z$ by setting up the complex difference quotient and using the Lebesque Dominated Convergence Theorem.

Part (c). Using (4.30), prove that $\Gamma(z)$ has a pole at each nonnegative integer $z = -n$, $n = 0, 1, 2, \ldots$. It turns out that these are the only poles, and they are all simple.

Part (d). Apply the method of steepest descents to the identity (4.30) to give a proof of Stirling's formula valid as $z \to \infty$ with $\arg(z) = \kappa$ where the angle $\kappa$ satisfies $-\pi < \kappa < \pi$. In other words, show that

$$\lim_{\substack{z\to\infty \\ \arg(z)=\kappa\in(-\pi,\pi)}} \frac{\Gamma(z)}{\sqrt{2\pi}z^{z-1/2}e^{-z}} = 1\,.$$

Hint: To start, set $z = \lambda e^{i\kappa}$ where $\lambda > 0$. How does your analysis break down if you consider instead the limit $z \to \infty$ with $z < 0$? ▶

## 4.8. The Effect of Branch Points

Recall the general strategy in dealing with contour integrals of the form (4.1) when the positive parameter $\lambda$ is large: one tries to deform the integration contour $C$ into a new contour $C'$ coinciding exactly with a level curve of the imaginary part of $h(z)$ by carrying out the deformation in the downhill direction for the real part of $h(z)$. For closed contours or for contours with endpoints at infinity, saddle points (critical points of $h(z)$) usually play an important role. The whole procecdure depends essentially on the analyticity of $h(z)$ throughout the region between the initial contour $C$ and the desired target contour $C'$. We assume this crucial property of $h(z)$ in the current discussion.

The contour selection principles briefly reviewed above have nothing to do with the function $g(z)$ in the integrand of $F(\lambda)$ defined by the integral formula (4.1). On the other hand, one certainly needs $g(z)$ to be analytic on the contour $C$ in order to carry out any deformation at all, and then isolated singularities of $g(z)$ in the region of the complex $z$-plane between the contour $C$ and the target contour $C'$ will obstruct the deformation process.

If the only singularity of $g(z)$ lying between $C$ and $C'$ is a pole of order $N$ at a point $z = z_0$, then $g(z)$ is a single-valued function for $z \neq z_0$, and it may be represented in the form

$$g(z) = (z - z_0)^{-N}\tilde{g}(z) \tag{4.31}$$

where $\tilde{g}(z)$ is a function that is analytic at $z = z_0$ and $\tilde{g}(z_0) \neq 0$. From the Residue Theorem, we obtain

$$F(\lambda) = \int_C e^{\lambda h(z)} g(z)\, dz = \int_{C'} e^{\lambda h(z)} g(z)\, dz + 2\pi i s \operatorname*{Res}_{z=z_0} e^{\lambda h(z)} g(z)\,. \quad (4.32)$$

Here, $s = -1$ if $C$ lies to the left of $C'$ (as in Figure 4.11) and $s = +1$

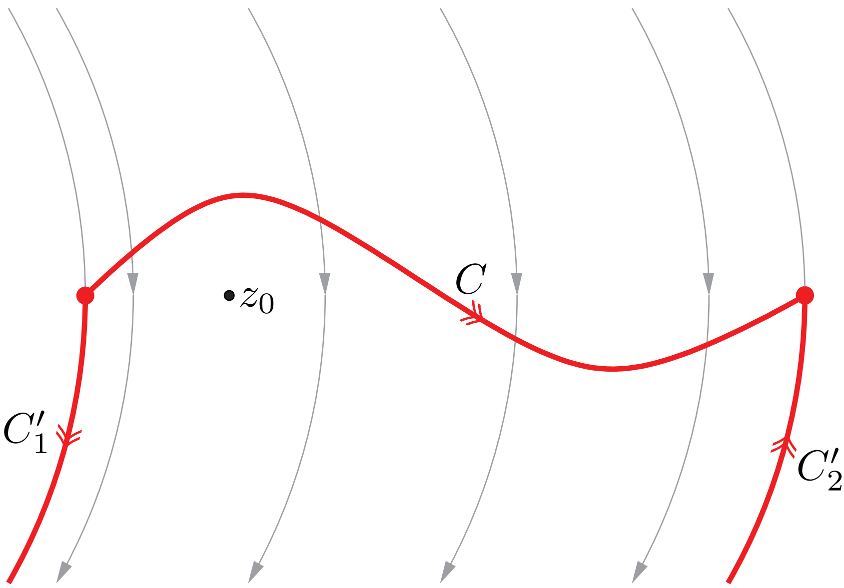


**Figure 4.11.** *A pole of $g(z)$ of order $N$ at the point $z = z_0$ between $C$ and $C'$. The gray curves indicate the direction of steepest descent for $\Re(h(z))$, and $C$ and $C' = C'_1 \cup C'_2$ are shown in red.*

if $C$ lies to the right of $C'$. The residue is the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion of the integrand at the point $z_0$. Therefore, if the analytic function $e^{\lambda h(z)}$ has the Taylor expansion

$$e^{\lambda h(z)} = \sum_{p=0}^{\infty} A_p(\lambda)(z - z_0)^p \quad (4.33)$$

for some coefficients $A_p(\lambda)$, then

$$\operatorname*{Res}_{z=z_0} e^{\lambda h(z)} g(z) = \sum_{p=0}^{N-1} \frac{\tilde{g}^{(N-1-p)}(z_0)}{(N-1-p)!} A_p(\lambda)\,. \quad (4.34)$$

This defines the discrepancy between the original contour integral and the deformed contour integral over the target path $C'$. The result of the following exercise implies that the residue contribution in (4.32) is a polynomial in $\lambda$ of degree $N - 1$ times the exponential factor $e^{\lambda h(z_0)}$.

◀ *Exercise 4.10.* Show that the coefficient $A_p(\lambda)$ in the Taylor expansion (4.33) is a product of $e^{\lambda h(z_0)}$ with a polynomial in $\lambda$ of degree $p$. Then use (4.31) and (4.33) to prove (4.34). ▶

A more general type of singularity that still is quite typical is an algebraic branch point of $g(z)$. Roughly speaking, this means that $g(z)$ may be represented near $z = z_0$ in the form

$$g(z) = (z - z_0)^\sigma \tilde{g}(z)\,, \tag{4.35}$$

where $\sigma \in \mathbb{C}$ is assumed to not be an integer and where $\tilde{g}(z)$ is an analytic function of $z$ near $z = z_0$ for which $\tilde{g}(z_0) \neq 0$. This representation fails to be precise because the factor $(z - z_0)^\sigma$ is a multivalued function in any neighborhood of $z = z_0$ when $\sigma$ is not an integer, and the multivaluedness must be resolved with a concrete choice of the branch (that is, which of the possible values of $(z - z_0)^\sigma$ is meant at some fixed point $z \neq z_0$) and of the branch cut curve emanating from $z = z_0$. Due to the relation between power functions and logarithms, the possible values of $(z - z_0)^\sigma$ at a given $z$ are proportional by $z$-independent factors of the form $e^{2\pi i\sigma n}$, where $n$ is an integer. In other words, upon making a round-trip about the point $z = z_0$ in the counterclockwise direction, the argument of $z - z_0$ increases continuously by $2\pi$, and consequently the function $(z - z_0)^\sigma$ smoothly evolves from an initial value of, say, $w$ to a final value of $we^{2\pi i\sigma}$. The introduction of a branch cut emanating from $z = z_0$ is a way to circumvent the multivaluedness; once we position the branch cut, we simply do not allow any analytic continuation to cross the cut. In particular, with a branch cut present, it is no longer possible to make a complete round-trip about the point $z = z_0$ in the process of analytic continuation.

These choices might appear to introduce some ambiguity into the calculations, but this is in fact not so. The important thing to keep in mind is that the values of the function $g(z)$ are specified unambiguously along the original contour $C$ as part of the definition of the integral $F(\lambda)$ in (4.1). This amounts to the selection of the branch of any multivalued factors of the form $(z - z_0)^\sigma$ that may be present in $g(z)$. As the given contour of integration $C$ is deformed toward the target contour $C'$ specified by the properties of the function $h(z)$, the value of $g(z)$ is determined at each point by analytic continuation from the original contour $C$.

Now if the branch point $z_0$ is the only singularity of $g(z)$ lying between $C$ and the target contour $C'$, we would like to have some way of deforming $C$ toward $C'$ that takes into account a "price" to be paid for passing the singularity. Unfortunately, there is no precise analogue of the Residue Theorem that allows such a deformation process. However, a partial deformation that is favorable for the exponent function $h(z)$ may be carried out by choosing $C'$ to avoid the branch point at $z = z_0$. It will be useful to take the branch cut emanating from the branch point to descend in the steepest possible direction for $h(z)$ (this direction will be well-defined unless $h'(z_0) = 0$, which may be regarded as unusual for unrelated functions $g(z)$ and $h(z)$). A deformation of this type is illustrated in Figure 4.12. In this

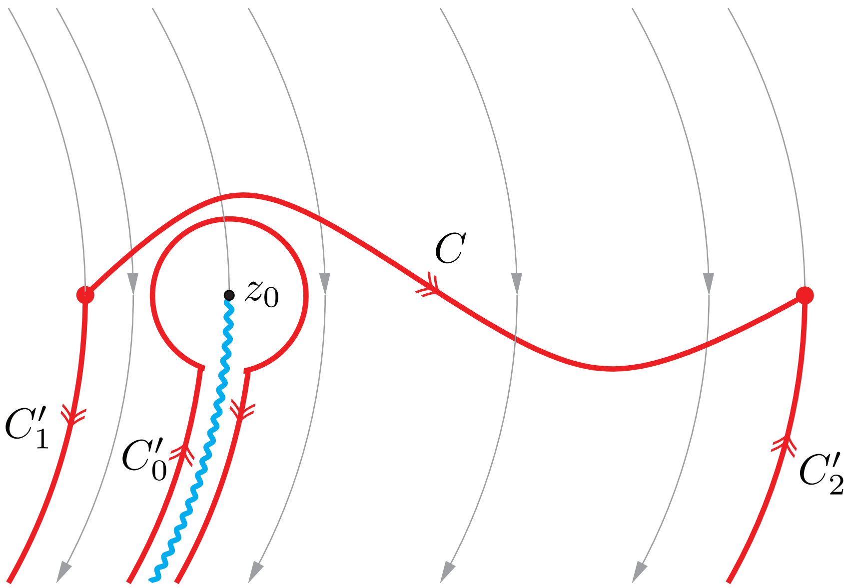


**Figure 4.12.** *The deformation of $C$ around a branch point of $g(z)$. The gray curves indicate the steepest descent directions for $\Re(h(z))$, and $C$ and $C' = C'_1 \cup C'_0 \cup C'_2$ are shown in red. A branch cut for $g(z)$, illustrated in blue, is chosen to emanate from $z = z_0$ along a contour of steepest descent.*

figure, the deformed contour $C'$ partly follows the branch cut on either side, and near the branch point it makes a round-trip about the branch point along a circle of some small radius centered at $z_0$. The reason for the small circle is that with $g(z)$ of the form (4.35), the integrand $e^{\lambda h(z)} g(z)$ is not integrable at $z = z_0$ unless $\Re(\sigma) > -1$.

The small circle in the portion $C'_0$ of the deformed contour $C'$ surrounding the branch point can be contracted to zero radius if $\Re(\sigma) > -1$ because the integrand is integrable at $z_0$ in this case. If $\Re(\sigma) \leq -1$, then for some integer $N$ it will be true that $\Re(\sigma + N) > -1$, and we can try to change the given exponent $\sigma$ into $\sigma + N$ by integration by parts. Indeed, after the deformation from $C$ to $C'$ as shown in Figure 4.12, we have

$$F_0(\lambda) := \int_{C_0'} e^{\lambda h(z)} \tilde{g}(z)(z-z_0)^\sigma \, dz$$
$$= \frac{1}{(\sigma+N)(\sigma+N-1)\cdots(\sigma+1)} \int_{C_0'} e^{\lambda h(z)} \tilde{g}(z) \frac{d^N}{dz^N}(z-z_0)^{\sigma+N} \, dz \,. \tag{4.36}$$

Clearly, this formula is only valid if $\sigma$ is not a negative integer, but in such a case the singularity at $z = z_0$ would be a pole and not a branch point. Using the identity $\Gamma(z+1) = z\Gamma(z)$, the leading factor can be rewritten in compact form

$$\frac{1}{(\sigma+N)(\sigma+N-1)\cdots(\sigma+1)} = \frac{\Gamma(\sigma+N)}{\Gamma(\sigma+N+1)} \frac{\Gamma(\sigma+N-1)}{\Gamma(\sigma+N)} \cdots \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+2)} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+N+1)} \,,$$

because the product "telescopes". Assuming that the boundary terms vanish after each of $N$ successive integrations by parts in (4.36), we then find that

$$F_0(\lambda) = (-1)^N \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+N+1)} \int_{C_0'} \frac{d^N}{dz^N} \left( e^{\lambda h(z)} \tilde{g}(z) \right) (z-z_0)^{\sigma+N} \, dz \,.$$

The following exercise shows that each differentiation contributes terms with an additional factor of $\lambda$.

◀ *Exercise 4.11.* Assuming $h(z)$ and $\tilde{g}(z)$ are analytic functions in a neighborhood of the branch point $z_0$, show that

$$\frac{d^N}{dz^N} \left( e^{\lambda h(z)} \tilde{g}(z) \right) = e^{\lambda h(z)} \sum_{p=0}^{N} \lambda^p G_p(z)$$

where $G_p(z)$ are some analytic functions in the same neighborhood that are independent of $\lambda$. In particular, show that

$$G_0(z) = \tilde{g}^{(N)}(z) \quad \text{and} \quad G_1(z) = \sum_{k=1}^{N} \binom{N}{k} h^{(k)}(z) \tilde{g}^{(N-k)}(z)$$

both hold true. ▶

In this way, $F_0(\lambda)$ is written as a linear combination of integrals involving branching exponents $\sigma + N$:

$$F_0(\lambda) = \sum_{p=0}^{N} \lambda^p \int_{C_0'} e^{\lambda h(z)} G_p(z)(z-z_0)^{\sigma+N} \, dz \,.$$

This calculation assumes that the boundary terms did not contribute to the integration by parts; if the contour $C'$ has finite endpoints or if the exponent function $h(z)$ provides relatively weak decay in the infinite parts of $C'$, these neglected boundary terms may play an important role.

To continue our discussion, we now assume without any loss of generality that $\Re(\sigma) > -1$ and therefore that the small circle in the deformed contour $C_0'$ has been contracted to zero radius. The deformed contour $C_0'$ now contains two arcs that meet at the branch point $z = z_0$. These two arcs correspond to exactly the same values of $z$, namely those lying along the contour of steepest descent for $\Re(h(z))$ from the branch point. Although the $z$-values correspond for these two arcs, the integrands differ because the arcs lie on opposite sides of the branch cut for the factor $(z - z_0)^\sigma$. The contribution to $F(\lambda)$ coming from these two arcs lying on opposite sides of the branch cut in Figure 4.12 is therefore

$$F_0(\lambda) = s \int_{C_0''} e^{\lambda h(z)} \tilde{g}(z)(z - z_0)_-^\sigma \, dz - s \int_{C_0''} e^{\lambda h(z)} \tilde{g}(z)(z - z_0)_+^\sigma \, dz \,, \quad (4.37)$$

where $C_0''$ is an integration path of steepest descent for $\Re(h(z))$ beginning at $z = z_0$ and $s = -1$ if $C$ lies to the left of $C_0'$ (as in Figure 4.12) while $s = +1$ if $C$ lies to the right of $C_0'$. Here, the notation $(z - z_0)_\pm^\sigma$ refers to the two distinct values taken by the function $(z - z_0)^\sigma$ on opposite sides of the branch cut; the "plus" (respectively "minus") subscript indicates the boundary value taken on the left (respectively "right") side of $C_0''$ when this contour is traversed according to its orientation in the direction away from $z_0$. Now, the values $(z - z_0)_\pm^\sigma$ are always proportional because going around the branch point from the "minus" side to the "plus" side of the branch cut without crossing the cut means that the argument of $z - z_0$ decreases continuously by $2\pi$. Therefore, we have the relation

$$(z - z_0)_+^\sigma = e^{-2\pi i \sigma}(z - z_0)_-^\sigma \,. \quad (4.38)$$

With the help of this relation, we can reduce (4.37) to a single integral:

$$F_0(\lambda) = s \cdot (1 - e^{-2\pi i \sigma}) \int_{C_0''} e^{\lambda h(z)} \tilde{g}(z)(z - z_0)_-^\sigma \, dz \,. \quad (4.39)$$

The formula for $F_0(\lambda)$ given by (4.39) represents the contribution to $F(\lambda)$ coming from the branch point at $z = z_0$. It is analogous to a residue contribution in the case that $z_0$ is a pole of $g(z)$. However, unlike the case of a pole, the integral in (4.39) cannot usually be evaluated exactly. The asymptotic behavior of $F_0(\lambda)$ when $\lambda$ is large and positive is, however, easy to determine using Watson's Lemma. First, we note that since $C_0''$ is a contour of steepest descent for $\Re(h(z))$, we may introduce a real parametrization

$z = z(t)$ of $C_0''$ by the change of variables

$$h(z) - h(z_0) = -t\,, \tag{4.40}$$

which is an admissible substitution near $z = z_0$ as long as $h'(z_0) \neq 0$. We assume that $h'(z_0) \neq 0$; the case that a singularity of $g(z)$ occurs precisely at the same point where $h(z)$ has a critical point may be regarded as somewhat exceptional, and a modification of this procedure is required to handle this special case. Using (4.40) in (4.39), we get

$$F_0(\lambda) = s \cdot (1 - e^{-2\pi i\sigma})e^{\lambda h(z_0)} \int_0^T e^{-\lambda t}\tilde{g}(z(t))(z(t) - z_0)_-^\sigma z'(t)\,dt\,,$$

where $T$ is the value of $t$ corresponding to the final endpoint of $C_0''$ (we may have $T = \infty$ if $C_0''$ is an infinite contour). Since $z(t)$ is an analytic function in a neighborhood of $t = 0$, so are $\tilde{g}(z(t))$ and $z'(t)$, and we also have $z(t) - z_0 = tr(t)$ where $r(t)$ is an analytic function at $t = 0$ with $r(0) = z'(0) = -1/h'(z_0) \neq 0$. We may then define an analytic function by setting $b(t) = \tilde{g}(z(t))r(t)^\sigma z'(t)$, where $r(t)^\sigma$ is an analytic function in the neighborhood of $t = 0$ since $r(0) \neq 0$, and the branch is concretely specified so that

$$t^\sigma r(t)^\sigma = (z(t) - z_0)_-^\sigma \tag{4.41}$$

holds for $0 < t < T$ and the power function $t^\sigma$ is defined logarithmically as $e^{\sigma \log(t)}$ where $\log(t)$ is *real*. We then have

$$F_0(\lambda) = s \cdot (1 - e^{-2\pi i\sigma})e^{\lambda h(z_0)} \int_0^T e^{-\lambda t} t^\sigma b(t)\,dt\,.$$

Recalling that $\Re(\sigma) > -1$, this is an integral in precisely the form to which Watson's Lemma applies. We therefore obtain the complete asymptotic expansion of $F_0(\lambda)$ as

$$F_0(\lambda) \sim s \cdot (1 - e^{-2\pi i\sigma})e^{\lambda h(z_0)} \sum_{n=0}^{\infty} \frac{\Gamma(\sigma + n + 1)b^{(n)}(0)}{n!\lambda^{\sigma+n+1}} \qquad \text{as } \lambda \to \infty \text{ with } \lambda > 0\,.$$

In particular, regardless of whether the singularity of $g(z)$ at $z = z_0$ is a pole (in which case its contribution can be taken into account exactly by a residue calculation) or a branch point, the asymptotic contribution consists of powers of $\lambda$ multiplied by the exponential factor $e^{\lambda h(z_0)}$. If there are several such contributions to $F(\lambda)$, then all of the terms of the full asymptotic expansion of $F(\lambda)$ will come from the singular point of $g(z)$ for which $\Re(h(z))$ is the largest. All other contributions will be beyond all orders.

◁ *Example.* Consider the integral

$$F(\lambda) := \int_{-\infty}^{\infty} \frac{e^{-\lambda x^2}}{\sqrt{(x-x_0)^2+1}} e^{4i\lambda x}\, dx$$

where the positive square root is meant and $x_0$ is a real parameter. We may rewrite $F(\lambda)$ as a contour integral of the form (4.1) with

$$h(z) := -z^2 + 4iz \quad \text{and} \quad g(z) := \frac{1}{\sqrt{(z-x_0)^2+1}},$$

and the contour $C$ is the real axis in the $z$-plane oriented from left to right. If we complete the square, we see that $h(z) = -(z-2i)^2 - 4$, so there is a single critical point at $z = 2i$ that is a simple saddle point of $\Re(h(z))$. The global steepest descent path over the saddle point is simply the horizontal line $\Im(z) = 2$. Therefore, we would like to deform the given contour $C$ upwards in the complex $z$-plane to agree with this horizontal line.

There is, however, a singularity of $g(z)$ lying between $C$ and the desired steepest descent contour over the saddle point. Indeed, since $(z-x_0)^2+1 = ((z-x_0)-i)((z-x_0)+i)$, there are singularities at $z = x_0 \pm i$ in the complex $z$-plane. Both of these singularities are branch points with exponent $\sigma = -1/2$. It is the singularity at $z = z_0 = x_0 + i$ that obstructs our deformation of $C$ into the global steepest descent path over the saddle point. We want to choose the branch cut for this point to correspond to the steepest descent path for $\Re(h(z))$ from $z_0$. The nature of this path depends on the value of $x_0$. If $x_0 < 0$, then the steepest descent path from $z_0$ is a branch of the hyperbola $\Im(z-2i)\Re(z-2i) = -x_0 > 0$ which begins at $z_0$ and descends to infinity in the left half-plane asymptotically approaching the horizontal line $\Im(z) = 2$. If $x_0 > 0$, then the steepest descent path from $z_0$ is a branch of the hyperbola $\Im(z-2i)\Re(z-2i) = -x_0 < 0$ which begins at $z_0$ and descends to infinity in the right half-plane asymptotically approaching the horizontal line $\Im(z) = 2$. However, if $x_0 = 0$, then the steepest descent path from $z_0$ is upwards along the vertical line $\Re(z) = 0$, and the path therefore runs right into the saddle point at $z = 2i$. These three situations are illustrated in Figure 4.13 along with the corresponding contour deformations $C \to C'$.

If $x_0 \neq 0$, then there are two obvious choices for asymptotic contributions to $F(\lambda)$ as $\lambda \to \infty$ with $\lambda > 0$, namely the branch point singularity of $g(z)$ at $z_0 = x_0 + i$ and the saddle point of $h(z)$ at $z = 2i$. The point at which $\Re(h(z))$ is greatest will give the dominant contribution, with all other contributions being beyond all orders. Since

$$\Re(h(2i)) = -4 \qquad \text{and} \qquad \Re(h(x_0+i)) = -3 - x_0^2,$$

it is easy to see that if $|x_0| > 1$, the saddle point dominates the asymptotics, while if $|x_0| < 1$, the branch point dominates. In the borderline case that

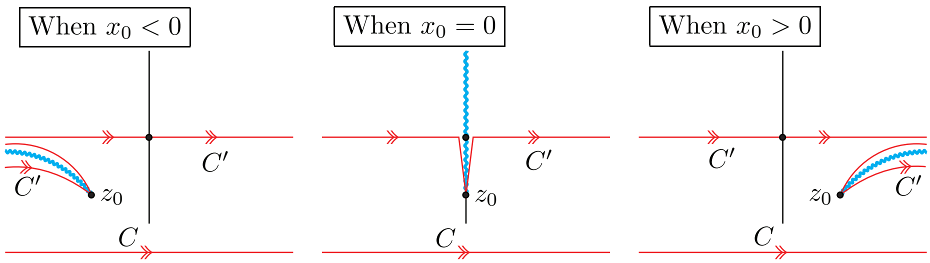


**Figure 4.13.** *The three types of contour deformations for the case of $h(z) = -(z-2i)^2 - 4$ and $g(z) = 1/\sqrt{(z-x_0)^2+1}$. The selected branch cut for $g(z)$ emanating from $z_0 = x_0 + i$ is shown in blue. The horizontal portion of $C'$ always passes over the saddle point at $z = 2i$.*

$x_0 = \pm 1$, both exponential factors $e^{\lambda h(2i)}$ and $e^{\lambda h(x_0+i)}$ have the same modulus and therefore both the branch point and the saddle point contribute terms to the asymptotic expansion of $F(\lambda)$. In the case $x_0 = 0$, it is still true that the branch point dominates, which is good since the deformed contour $C'$ is complicated near the saddle point and the local contribution would be more difficult to approximate asymptotically due to the branch cut of $g(z)$ through the saddle point.

To give some more detail for this example, let us explicitly calculate the leading term of the asymptotic expansion of $F(\lambda)$ in the case $|x_0| < 1$ where it is required to take into account carefully the contribution of the branch point at $z_0 = x_0 + i$. We may choose

$$\tilde{g}(z) := (z - x_0 + i)^{-1/2},$$

where the "usual" branch is meant, *i.e.* $\tilde{g}(z)$ is defined throughout the sector $-\pi < \arg(z - x_0 + i) < \pi$ and is positive real for $\arg(z - x_0 + i) = 0$. Then, we have $g(z) = (z - x_0 - i)^{-1/2}\tilde{g}(z)$, where we choose the steepest descent contour $C_0''$ as illustrated in Figure 4.13 from $x_0 + i$ as the branch cut for the factor $(z - x_0 - i)^{-1/2}$, and then we are forced to choose the sign of the square root (branch) so that $g(z)$ is positive real for $z \in \mathbb{R}$. Here we see that if $(z - x_0 - i)_-^{-1/2}$ denotes the value taken by this factor on the right side of the branch cut as it is traversed from $z_0 = x_0 + i$ to infinity, then this boundary value agrees with the "usual" branch (which is defined for $-\pi < \arg(z - x_0 - i) < \pi$ and is positive real for $\arg(z - x_0 - i) = 0$). One way to see that this must be the case is to consider $z$ tending to infinity along the positive real axis; we need to make sure that $g(z)$ is positive for such $z$, which forces the agreement on the "minus" side of $C_0''$ (from which one may reach large positive $z$ without crossing either our selected branch cut $C_0''$ or the branch cut for the "usual" branch). Using (4.38), we have

$$(z - x_0 - i)_+^{-1/2} = -(z - x_0 - i)_-^{-1/2},$$

because $\sigma = -1/2$. Therefore, the contribution to $F(\lambda)$ coming from the part of $C'$ surrounding the branch cut may be written as (noting $s = +1$ because $C$ lies to the right of $C'$)

$$F_0(\lambda) = 2 \int_{C_0''} e^{\lambda h(z)} \tilde{g}(z)(z - z_0)_-^{-1/2} \, dz .$$

Since $h(z_0) = h(x_0 + i) = -3 - x_0^2 + 2ix_0$, it remains to find the behavior of the function $b(t)$ near $t = 0$. Since we have determined that $(z - z_0)_-^{-1/2}$ agrees with the reciprocal of the "usual" square root function, we see from (4.41) that

$$b(0) = \tilde{g}(x_0 + i) \cdot \left( \frac{-1}{h'(x_0 + i)} \right)^{-1/2} \cdot \frac{-1}{h'(x_0 + i)},$$

where the exponent refers to the "usual" branch. Then, since $h'(x_0 + i) = 2i - 2x_0$ and since

$$\tilde{g}(x_0 + i) = (2i)^{-1/2}$$

(again the "usual" branch), we get finally that

$$b(0) = \frac{1}{2}(1 + ix_0)^{-1/2} .$$

Therefore,

$$F_0(\lambda) = 2e^{2i\lambda x_0} e^{-\lambda(3+x_0^2)} \cdot \frac{\pi^{1/2}}{\lambda^{1/2}} \cdot \frac{1}{2}(1 + ix_0)^{-1/2}(1 + O(\lambda^{-1}))$$
$$\text{as } \lambda \to \infty \text{ with } \lambda > 0.$$

Since for $|x_0| < 1$ the remaining parts of $F(\lambda)$ are beyond all orders, we finally arrive at the formula

$$F(\lambda) = e^{2i\lambda x_0} e^{-\lambda(3+x_0^2)} \cdot \left( \frac{\pi}{(1 + ix_0)\lambda} \right)^{1/2} (1 + O(\lambda^{-1})),$$

again valid as $\lambda \to \infty$ with $\lambda > 0$. ▷

Similar calculations may be carried out for deformations of paths of integration when $g(z)$ contains branch points of logarithmic type. In such cases, the analogue of the formula (4.38) describing the change in $(z - z_0)^\sigma$ as $z$ completes a single circuit about $z_0$ in the counterclockwise direction is the following:

$$\log(z - z_0)_+ = \log(z - z_0)_- - 2\pi i .$$

◀ *Exercise 4.12.* Find the form of the complete asymptotic expansion as $\lambda \to \infty$ with $\lambda > 0$ and the coefficient of the leading term for the integral

$$F(\lambda) := \int_{-\infty}^{\infty} e^{-\lambda x^2} e^{4i\lambda x} \log((x - x_0)^2 + 1)\, dx\,,$$

where $x_0$ is a real parameter. Your expansion may be of a different character depending on the value of $x_0$; make sure you consider all cases. Here the function $\log((x-x_0)^2+1)$ is meant to be positive for all real $x$. Hint: When a branch point gives the dominant contribution, reduce the corresponding integral to the form where Watson's Lemma can be applied with $\sigma = 0$. ▶

**4.8.1. Application: Asymptotics of transform integrals.** In general terms, a *transform* is a "change of variables" at the level of functions. Just as changes of variables sometimes simplify algebraic problems, function transforms can simplify integral and/or differential equations. The price one pays for this simplification is that the transform and its inverse, even when given by explicit integral formulas, rarely admit exact analytical evaluation. This is why asymptotic analysis plays an important role in this field.

As a review, we now define the Fourier and Laplace transform pairs.

**Definition 4.1** (Fourier Transform Pair)**.** *Let $f(x)$ be a function that is square integrable on $(-\infty, \infty)$; that is,*

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty\,.$$

*Then the Fourier Transform of $f$ is a new square integrable function $\hat{f}(k)$ defined for almost all real $k$ by*

$$\hat{f}(k) := \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\, dx\,.$$

*The function $f(x)$ can be reciprocally expressed as the Inverse Fourier Transform of $\hat{f}$ according to the formula*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\, dk\,.$$

*Sometimes we use the notation $\mathcal{F}f := \hat{f}$ and $\mathcal{F}^{-1}\hat{f} := f$.*

**Definition 4.2** (Laplace Transform Pair)**.** *Let $f(t)$ be a function defined for real $t > 0$ so that for some real $C$ we have $f(t) = O(e^{Ct})$ as $t \to \infty$ with $t > 0$. Then the Laplace Transform of $f$ is a new function $F(s)$ analytic for $\Re(s) > C$ and defined by*

$$F(s) := \int_0^{\infty} f(t)e^{-st}\, dt\,.$$

*The function $f(t)$ can be reciprocally expressed as the Inverse Laplace Transform of $F$ by the formula*

$$f(t) = \frac{1}{2\pi i} \int_{D-i\infty}^{D+i\infty} F(s) e^{ts} \, ds$$

*where $D$ is any number greater than $C$. In other words, the path of integration must lie to the right of all singularities of $F(s)$. Sometimes we use the notation $\mathcal{L}f := F$ and $\mathcal{L}^{-1}F := f$.*

The path of integration in the inverse Laplace transform is sometimes called the *Bromwich path*[4]. The fact that both of these transforms are based on exponential functions means that they are well adapted to problems involving differential operators. So, it is easy to see that if $f$ is $n$ times differentiable with all derivatives square integrable, then

$$\mathcal{F} \frac{d^n f}{dx^n} = (ik)^n \mathcal{F} f$$

so differentiation of the function living in the "direct space" is the same thing as multiplication by $ik$ in the "transform space". Similarly for the Laplace transform of a function whose first $n$ derivatives are transformable, we have

$$\mathcal{L} \frac{d^n f}{dt^n} = s^n \mathcal{L} f - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0) \,.$$

The fact that the values of $f(t)$ and its derivatives at $t = 0$ enter this expression suggests that the Laplace transform can be a useful tool in the analysis of initial-value problems.

The utility of facts like these in solving problems can be illustrated by the solution of the diffusion equation in terms of Fourier transforms. Suppose we want to find the solution $\varphi(x,t)$ of

$$\frac{\partial \varphi}{\partial t} = \nu \frac{\partial^2 \varphi}{\partial x^2} \,,$$

subject to the initial condition $\varphi(x, 0) = \varphi_0(x)$, the latter being a square integrable function of $x \in \mathbb{R}$. This is a problem posed in the "direct space". Let us apply the operator $\mathcal{F}$ to the equation, keeping in mind that $t$ is a parameter as far as the transform is concerned, so that $\mathcal{F}$ maps $\varphi(x,t)$ to a new function $\hat{\varphi}(k,t)$. We find that

$$\frac{\partial \hat{\varphi}}{\partial t} = -\nu k^2 \hat{\varphi} \,.$$

The initial condition for this equation is $\hat{\varphi}(k, 0) = \hat{\varphi}_0(k)$. This is the same problem viewed in the "transform space" or the "transform domain". It is

[4] Thomas John l'Anson Bromwich, 1875–1929.

solved independently for each $k$ as a simple ordinary differential equation, and this is the great advantage of working in the transform domain. Thus,

$$\hat{\varphi}(k,t) = \hat{\varphi}_0(k)e^{-\nu k^2 t} .$$

To recover the solution $\varphi(x,t)$ at later times $t$, we must apply the operator $\mathcal{F}^{-1}$ to the result, which amounts to writing the solution as an integral over $k$. The method of solving the equation is expressed in the diagram shown in Figure 4.14.

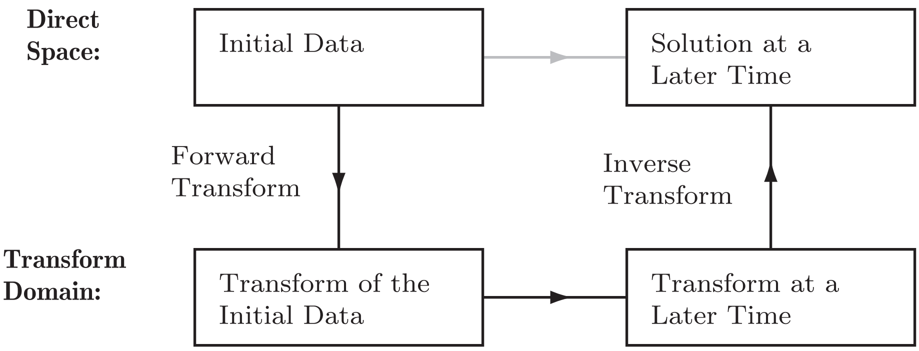


**Figure 4.14.** *The algorithm of solving a problem in the direct space (gray arrow) and in the transform space (black arrows). Although the horizontal part of the solution by transforms is trivial compared to the direct space solution, the extra steps of the forward and inverse transforms involve computing integrals that can rarely be found in closed form.*

The solution of any problem treated by integral transforms like the Fourier and Laplace transforms will ultimately be expressed as an integral that cannot typically be computed in terms of elementary functions. So it is of some interest to calculate asymptotic approximations since they give us a way of extracting concrete information from such integrals. The method of steepest descents is a very useful tool for establishing asymptotic approximations in this context. Branch points frequently play a role in the method.

First let us look at inverse Fourier transform integrals. These are of the form

$$f(x) := \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\,dk ,$$

where we will assume that $\hat{f}$ is an analytic function on the real $k$ axis, a fact that corresponds to exponential decay of the function $f(x)$ for large $x$. The singularities of $\hat{f}(k)$ in the complex $k$-plane can in principle be very complicated, but we assume here that we have a transform $\hat{f}(k)$ for which all singularities are isolated one from another and are of the types we have previously studied: poles, algebraic branch points, or logarithmic branch points. The question is: in this kind of situation what can one learn about $f(x)$? The main tool we have in addressing this question is the freedom to deform the path of integration away from the real axis in an attempt to simplify the integral. There are two obstructions to deforming the contour: the isolated singularities of $\hat{f}(k)$ in the finite $k$-plane and the behavior of the integrand $\hat{f}(k)e^{ikx}$ as $k$ tends to infinity in different sectors. The latter determines our options in moving the infinite "tails" of the contour of integration.

For the present discussion, we also suppose that for some exponent $p < -1$, we have for some real number $x_0$ a bound of the form $\hat{f}(k)e^{ikx_0} = O(k^p)$ as $k \to \infty$ (from any direction in the complex $k$-plane). This means that the factor $e^{ik(x-x_0)}$ completely determines the possibilities for deforming the contour of integration near $k = \infty$, that is, for moving the "tails" of the integration contour. Indeed, suppose that $x < x_0$. Then the exponential factor $e^{ik(x-x_0)}$ is decaying into the lower half $k$-plane. This means that under our assumptions on $\hat{f}(k)$, the real axis in the $k$-plane may, without change of the value of the integral defining $f(x)$ in terms of $\hat{f}(k)$, be replaced with a number of separate inverted U-shaped paths, one belonging to each singularity of $\hat{f}(k)$ as shown in Figure 4.15.

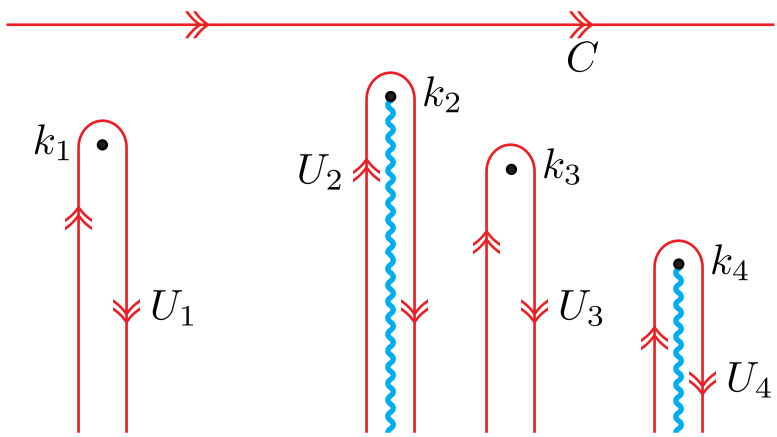


**Figure 4.15.** *The deformation of the contour of an inverse Fourier transform integral into the lower half $k$-plane for $x < x_0$. The original contour $C$ is the real axis in the complex $k$-plane. The new contour $C'$ consists of an isolated inverted U-shaped component for each singularity of $\hat{f}(k)$ in the lower half-plane. If the singularity is a branch point, then the multivaluedness is resolved by placing a branch cut (shown in blue) in the steepest descent direction.*

If a given singularity in the lower half-plane is a pole (as are $k = k_1$ and $k = k_3$ in Figure 4.15), then the Residue Theorem allows the corresponding

U-shaped component of the integration contour to be neglected at the cost of $-2\pi i$ times the residue of the pole. The minus sign comes from the orientation of the contour, which encircles the pole in a clockwise direction. These contributions to the integral coming from poles of $\hat{f}(k)$ are *exact*, since all we are using here is the fact that $x < x_0$. That is, we are not (yet) considering asymptotics as $x \to \infty$ with $x < 0$.

◀ *Exercise 4.13.* Find the exact value of the inverse Fourier transform integral

$$f(x) = \int_{-\infty}^{\infty} \frac{k^2 - 3k + 4}{((k-2)^2 + 4)^2((k+3)^2 + 1)} e^{ikx}\, dk$$

assuming only that $x < 0$. From your exact solution, determine the asymptotic form of $f(x)$ as $x \to \infty$ with $x < 0$. ▶

If, on the other hand, the singularity $k = k_j$ is a branching point, then we cannot usually evaluate the corresponding integral over $U_j$ exactly. However, we can observe that our deformation of the contour of integration into the paths $U_j$ is precisely the steepest descent deformation that is useful for analysis in the limit $x \to \infty$ with $x < 0$. The branch cut for each branch point is chosen to emanate from the singularity in the direction of steepest descent for $\Re(-ik)$ (the minus sign is present because $\lambda = -x$ is our large positive parameter), namely in the vertical direction as shown in Figure 4.15. Let us now describe in some detail how to apply Watson's Lemma to obtain the complete asymptotic expansion of the contribution to $f(x)$ coming from the integral over $U_j$.

Let us assume that, perhaps by integrating by parts several times, $\hat{f}(k)$ is integrable at $k = k_j$, corresponding to a branching exponent $\sigma$ with $\Re(\sigma) > -1$. Then making the change of variables $\kappa = -i(k - k_j)$, we get

$$\int_{U_j} \hat{f}(k)e^{ikx}\, dk = ie^{ik_jx}\left[\int_{-\infty}^{0} \hat{f}(k_j + i(\kappa + i0))e^{-x\kappa}\, d\kappa \right. \\ \left. - \int_{-\infty}^{0} \hat{f}(k_j + i(\kappa - i0))e^{-x\kappa}\, d\kappa\right],$$

where by $\kappa \pm i0$ we mean the boundary value taken on the cut from the upper or lower half-plane for $\kappa$. This change of variables just rotates the U-shaped path and the branch cut about $k_j$ by $-\pi/2$ and centers it at $\kappa = 0$. Now, in some neighborhood of $\kappa = 0$ the function $\hat{f}(k_j + i\kappa)\kappa^{-\sigma}$ is analytic and nonzero, being equal to some function $g(\kappa)$ that has a Taylor series at $\kappa = 0$. Here, by $\kappa^{-\sigma}$ we mean exactly the usual branch of this multivalued function that is cut on the negative real $\kappa$ axis and that is positive for $\kappa$ real and positive. Now for $\kappa < 0$ we have

$$\hat{f}(k_j + i(\kappa \pm i0)) = (\kappa \pm i0)^\sigma g(\kappa) = e^{\pm i\pi\sigma}(-\kappa)^\sigma g(\kappa)\,.$$

So changing variables to $\nu = -\kappa$ we find

$$\begin{aligned}\int_{U_j} \hat{f}(k)e^{ikx}\,dk &= ie^{ik_jx}\left[e^{i\pi\sigma}\int_0^\infty \nu^\sigma g(-\nu)e^{x\nu}\,d\nu \right.\\ &\qquad\left. - e^{-i\pi\sigma}\int_0^\infty \nu^\sigma g(-\nu)e^{x\nu}\,d\nu\right]\\ &= -2e^{ik_jx}\sin(\pi\sigma)\int_0^\infty \nu^\sigma g(-\nu)e^{x\nu}\,d\nu\,.\end{aligned}$$

Applying Watson's Lemma to the latter integral to find the asymptotic behavior as $x \to \infty$ with $x < 0$ yields a complete asymptotic expansion in terms of the Taylor coefficients of $g(-\nu)$ at $\nu = 0$. In particular, we obtain the asymptotic result

$$\int_{U_j} \hat{f}(k)e^{ikx}\,dk = -2e^{ik_jx}\sin(\pi\sigma)g(0)\Gamma(\sigma+1)(-x)^{-(\sigma+1)}(1 + O(|x|^{-1}))$$
$$\text{as } x \to \infty \text{ with } x < 0\,.$$

Notice that whether the contribution to $f(x)$ comes from a pole and is exact for all $x < x_0$ or whether it comes from a branch point and we have to approximate the contribution for large negative $x$, we find that each isolated singularity $k_j$ of $\hat{f}(k)$ results in an exponentially small contribution proportional to $e^{ik_jx}$. Comparing these exponents, we see that as $x \to \infty$ with $x < 0$, only the singularities that are closest to the real axis are important. All other singularities are exponentially small by comparison. So as $x \to \infty$ with $x < 0$, the asymptotic behavior of $f(x)$ is completely determined by the singularity or singularities of $\hat{f}(k)$ in the lower half-plane that have the most positive imaginary part. If there are no singularities at all in the lower half-plane and if the contour may be deformed downward toward $k = -i\infty$ without changing the value of the integral, then $f(x)$ is identically zero for all $x < x_0$.

$\triangleleft$ *Example.* The fact that an inverse Fourier transform integral in which the parameter $x$ appears in the integrand in an analytic fashion can yield a result that is not analytic as a function of $x$ is somewhat counterintuitive and a good example of how nice properties of a family of functions can disappear upon taking limits. Indeed, in the integral

$$f(x) = \int_{-\infty}^{\infty} \frac{e^{ik(x-x_0)}}{(k-(3+2i))(k-(-1+i))}\,dk$$

the integrand is, for each real $k$, an entire analytic function of $x$. However, evaluating the integral using the Residue Theorem, we see that

$$f(x) = \begin{cases} 0\,, & \text{for } x < x_0\,, \\ \dfrac{2\pi i}{4+i}\left[e^{3i(x-x_0)}e^{-2(x-x_0)} - e^{-i(x-x_0)}e^{-(x-x_0)}\right]\,, & \text{for } x > x_0\,. \end{cases}$$

Therefore $f(x)$ is not analytic at $x = x_0$. If we try to evaluate the integral by taking a limit of Riemann sums on $[-L, L]$ in which we refine the partition of $[-L, L]$ and also let $L$ tend to infinity, it is easy to see that for each finite partition of $[-L, L]$ the corresponding Riemann sum is an entire analytic function of $x$. The loss of analyticity comes from the limiting process. ▷

When $x > x_0$, it is not possible to deform the path of integration from the real axis into the lower half-plane. However, deformation into the upper half-plane is certainly possible, and again the contour can often be pushed all the way to infinity in the upper half-plane, leaving only U-shaped components surrounding each singularity of $\hat{f}(k)$ in the upper half-plane. The procedure for finding the contribution of each pole or algebraic branch point is similar as was the case for $x < 0$. Now a singularity $k_j$ in the upper half-plane contributes a term to $f(x)$ of the order of the exponential function $e^{ik_jx}$. Once again, these are all exponentially small, and all contributions are dominated by the singularities closest to the real axis. As $x \to \infty$ with $x > 0$, the dominant behavior of $f(x)$ comes from the singularities of $\hat{f}(k)$ in the upper half-plane with the smallest imaginary part. If there are no singularities of $\hat{f}(k)$, then $f(x)$ vanishes for all $x > x_0$. Note that the dominant contribution fails to have an oscillatory character if and only if the corresponding singularity $k_j$ lies on the imaginary axis.

Exactly the same kinds of ideas apply to inverse Laplace transforms. Consider the integral

$$f(t) := \frac{1}{2\pi i}\int_{D-i\infty}^{D+i\infty} F(s)e^{st}\,ds$$

where $D$ exceeds the real part of all singularities of $F(s)$. Here, we are implicitly taking $F(s)$ to be an analytic function on the contour of integration, and to permit relatively free deformation of the tails of the contour of integration, we will suppose that the analytic continuation of $F(s)$ from the contour behaves sub-exponentially as $s \to \infty$ in any direction; for example we might suppose that $F(s) = O(s^p)$ for some power $p$ as $s \to \infty$. The first observation is that in these circumstances if $t < 0$, then the path of integration may be deformed to the right, and since there are no singularities to stop it, the integral is identically equal to zero. This is consistent with the Laplace transform definition; since $F(s)$ only encodes the values of $f(t)$ for $t > 0$, if we ask what $F(s)$ predicts about $f(t)$ for $t < 0$, we get a simple answer, namely zero! For $t > 0$, we can deform the contour of integration to the left, and now there will be a contribution from each singularity of $F(s)$, which we may take to be isolated points $s = s_j$. If the singularity at $s = s_j$ is a pole, then the contribution can be evaluated exactly and will simply be $2\pi i$ times the residue of $F(s)e^{st}$ at $s_j$. This contribution is proportional to $e^{s_j t}$. Here the sign of the residue contribution is positive because the orientation of the contour is counterclockwise around the pole. This contribution can be exponentially large or small as $t \to \infty$ with $t > 0$ depending on whether $s_j$ lies to the right or the left of the imaginary axis. If, on the other hand, $s_j$ is an algebraic branch point, then the contribution can be approximated as $t \to \infty$ with $t > 0$ using Watson's Lemma. The calculation is very similar to that worked out above for the inverse Fourier transform integral. In particular, the contribution will again be proportional to $e^{s_j t}$. So now comparing the exponents, we see that the dominant contribution to $f(t)$ as $t \to \infty$ with $t > 0$ comes from precisely those singularities of $F(s)$ to the left of the initial contour of integration that have the most positive real parts. Note that in the inverse Laplace transform case, the function $f(t)$ will have a neutrally stable character (no exponential growth or decay) for large $t > 0$ if and only if the dominant singularity lies exactly on the imaginary axis in the complex $s$-plane.

◀ *Exercise 4.14.* Find the leading-order asymptotic behavior of the inverse Laplace transform $f(t) = \mathcal{L}^{-1}F(s)$ as $t \to \infty$ with $t > 0$ (i) when $F(s) = s^{-1}\tanh(s)$ and (ii) when $F(s) = s^{-1/2}e^{-s^{1/2}}$ (here we mean the principal branch of $s^{1/2}$ which you may take to have its branch cut on the negative real axis). In which case(s) is your asymptotic calculation exact? Comment. ▶

**4.8.2. Application: Selection of particular solutions of linear differential equations admitting integral representations.** Consider the third-order differential equation

$$2xw'''(x) + 9w''(x) - 2xw(x) = 0$$

and suppose that we are only interested in those solutions that satisfy $w(x) \to 0$ as $x \to \infty$ with $x < 0$. We can try to obtain solutions in the form of integrals, just as in the analysis of Airy's equation in §4.7.

For some unknown function $q(z)$ and some unknown contour $C$ from $z = a$ to $z = b$, we seek a solution in the form

$$w(x) = \frac{1}{2\pi i}\int_C q(z)e^{zx}\,dz\,.$$

Substituting this expression into the differential equation, differentiating under the integral sign, and integrating by parts to convert the multiplication by $x$ into differentiation with respect to $z$, we get

$$\frac{1}{2\pi i}\int_C e^{zx}\left[3z^2q(z) - 2(z^3-1)q'(z)\right]\,dz + \frac{1}{2\pi i}q(z)e^{zx}(z^3-1)\Big|_a^b = 0\,.$$

So if we can find a function $q(z)$ satisfying the first-order equation

$$2(z^3-1)q'(z) = 3z^2q(z)$$

and then choose a contour $C$ perhaps adapted to the specific function $q(z)$ so that the boundary terms vanish at the endpoints (which may be at infinity in some sectors), then our integral formula will solve the differential equation.

Now, unlike in the analysis of the Airy equation in §4.7, here the auxiliary equation for $q(z)$ does not have any nontrivial solutions that are entire analytic functions of $z$. In fact, the solutions are not even single-valued in the complex $z$-plane. To see this, multiply through by $q(z)$ so that the equation becomes

$$(z^3-1)\frac{d}{dz}q(z)^2 = q(z)^2\frac{d}{dz}(z^3-1)\,.$$

Dividing through by $(z^3-1)^2$ and collecting terms,

$$\frac{d}{dz}\left[\frac{q(z)^2}{(z^3-1)}\right] = 0\,.$$

Therefore, $q(z)^2 = K(z^3-1)$ for any constant $K$. Now since $q(z)^2$ has simple zeros at $z=1$, $z=e^{2\pi i/3}$, and $z=e^{-2\pi i/3}$, $q(z)$ will have a branch point singularity with branching exponent $\sigma = 1/2$ at each of these points. To obtain a concrete function $q(z)$, we of course must choose the constant $K$. But then we must also select a definite value for $q(z)$ at some nonsingular point (there will be two choices, differing by a sign) and then select branch cuts in the complex $z$-plane emanating from each of the three branch points. With these choices made, we will have determined the value of $q(z)$ as an analytic function at all points of the complex $z$-plane with the exception of those points lying along our chosen branch cuts.

We want to think about how to choose the branch cuts for $q(z)$ and an appropriate integration contour $C$ (which should not intersect any branch cuts) so that the boundary terms vanish, at least for all $x$ sufficiently negative (which is our domain of interest) and so that as $x\to\infty$ with $x<0$, the integral $w(x)$ vanishes. First note that the only place where the boundary terms vanish in the finite $z$-plane is at the three branching points. So, for example, we can choose a contour $C$ that joins a pair of branching points. But since we are considering $x<0$, we see that the integration path may also go off to infinity as long as it does so in the right half-plane. Then as $z\to\infty$, the factor $e^{zx}$ will kill all growth associated with $q(z)(z^3-1)$ and the corresponding boundary term will vanish. Choosing the branch cuts to radiate outward to infinity from each branch point, we get a picture like that shown in Figure 4.16. There are evidently three contours, $C_1$, $C_2$, and $C_3$

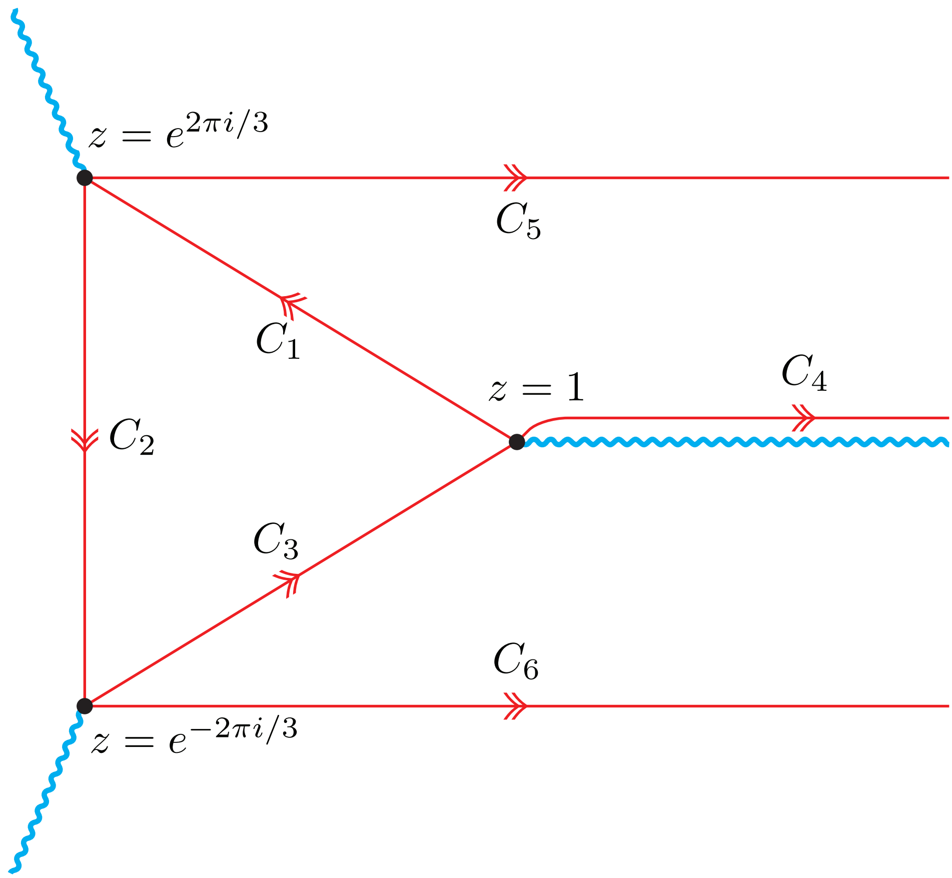


**Figure 4.16.** *A branch of $q(z)$ showing its cuts with wavy blue lines and six admissible contours yielding solutions of the differential equation with $x < 0$.*

that are acceptable for all $x$, since their boundary terms vanish regardless of the sign of $x$. On the other hand, for $x < 0$, we can also use the contours $C_4$, $C_5$, and $C_6$. We are taking $C_4$ to lie just above the branch cut on the positive real axis. If we took it to lie just below instead, this would simply amount to a change of sign in the integral.

Now the equation is only third order, so these cannot all be independent. A third-order linear equation has exactly three linearly independent solutions. So there must be linear relations among the integrals. Let us introduce the specific notation

$$w_k(x) := \frac{1}{2\pi i}\int_{C_k} q(z)e^{zx}\,dz\,, \quad k = 1, 2, \ldots, 6\,.$$

The role of the factor $(2\pi i)^{-1}$ is only traditional since $q(z)$ is determined by the differential equation only up to the arbitrary constant multiplier $K$; when $q(z)$ has poles, the factor $(2\pi i)^{-1}$ ensures that the residue contributions come in without the usual multiple of $2\pi i$. Given this notation, it is easy to see immediately from Figure 4.16 that

$$w_1(x) + w_2(x) + w_3(x) = 0$$

for all $x$, since the sum can be rewritten as a single integral over closed path inside of which the integrand is analytic. Also, since paths of integration may be deformed to infinity in the right half-plane when $x < 0$, we can also get the relations

$$w_1(x) = w_4(x) - w_5(x) \quad \text{and} \quad w_3(x) = w_6(x) + w_4(x) \quad \text{as long as } x < 0\,.$$

For $x < 0$ it looks to be convenient to take the three independent solutions to be $w_4(x)$, $w_5(x)$, and $w_6(x)$, in part because the paths of integration for these three integrals are all straight horizontal half-lines.

Now the final question arises: how do we determine which of these three integrals (or which linear combinations of them) correspond to solutions of the original differential equation that vanish as $x \to \infty$ with $x < 0$? We can asymptotically evaluate each of these integrals in this limit using Watson's Lemma.

For $w_5(x)$, we make the change of variable $t = z - e^{2\pi i/3}$. Then because $\cos(2\pi/3) = -1/2$ and $\sin(2\pi/3) = \sqrt{3}/2$,

$$w_5(x) = e^{-x/2}e^{ix\sqrt{3}/2}\frac{1}{2\pi i}\int_0^\infty q(e^{2\pi i/3} + t)e^{xt}\,dt\,.$$

Now in a neighborhood of $t = 0$, we can write

$$q(e^{2\pi i/3} + t) = t^{1/2}g_5(t)$$

where $g_5(t)$ is an analytic function with $g_5(0) \neq 0$ and $t^{1/2}$ means the usual square root function whose branch cut is on the negative real $t$ axis and that is positive for positive $t$. This is just because $q(t)$ vanishes exactly like a square root at each branching point. This puts the integrand in exactly the form we need to use Watson's Lemma with exponent $\sigma = 1/2$. Using the fact that $\Gamma(3/2) = \Gamma(1/2)/2 = \sqrt{\pi}/2$, we find that

$$w_5(x) = e^{-x/2}e^{ix\sqrt{3}/2}\frac{g_5(0)}{4i\sqrt{\pi}(-x)^{3/2}}\left(1 + O(|x|^{-1})\right)$$

$$\text{as } x \to \infty \text{ with } x < 0\,.$$

Thus, $w_5(x)$ is exponentially large in the limit of interest and certainly does not decay as desired. A similar analysis of $w_6(x)$ using Watson's Lemma gives

$$w_6(x) = e^{-x/2} e^{-ix\sqrt{3}/2} \frac{g_6(0)}{4i\sqrt{\pi}(-x)^{3/2}} \left(1 + O(|x|^{-1})\right)$$
$$\text{as } x \to \infty \text{ with } x < 0\,.$$

Here $g_6(t)$ is the analytic function $q(e^{-2\pi i/3} + t)t^{-1/2}$ in the vicinity of $t = 0$. The value $g_6(0)$ is related to $g_5(0)$ through details of the cut structure of the function $q(z)$. But it is just a number, and again we see that this solution of the differential equation is exponentially large in the limit of interest and in particular does not decay.

To confirm our suspicion that it is in fact $w_4(x)$ that satisfies our auxiliary boundary condition at $x = -\infty$, we use Watson's Lemma to study this integral. Letting $t = z - 1$, we find

$$w_4(x) = \frac{e^x}{2\pi i} \int_0^\infty q(1+t)e^{xt}\, dt\,.$$

Letting $g_4(t)$ be the analytic function $q(1+t)t^{-1/2}$ near $t = 0$ on the contour of integration and keeping in mind that what we mean by $q(1+t)$ in defining $g_4(t)$ is its value infinitessimally above the cut $B_1$, we again apply Watson's Lemma with exponent $\sigma = 1/2$ to find

$$w_4(x) = e^x \frac{g_4(0)}{4i\sqrt{\pi}(-x)^{3/2}} \left(1 + O(|x|^{-1})\right) \quad \text{as } x \to \infty \text{ with } x < 0\,.$$

Thus, $w_4(x)$ and its constant multiples are the only solutions of the differential equation that satisfy our boundary condition at $x = -\infty$.

Note that in this problem the solutions $w_4(x)$, $w_5(x)$, and $w_6(x)$ fail to exist for $x > 0$ because the integrals defining them diverge. On the other hand, for $x > 0$ there are new possibilities for the path of integration that we could not consider for $x < 0$, namely three paths emerging from the branch points and going to infinity in the left half-plane. The fact that the nature of the set of solutions changes suddenly when $x$ passes through zero is related to the fact that $x = 0$ is a *singular point* for the third-order differential equation. In particular, when $x = 0$, the order of the differential equation differs from its value for $x \neq 0$. Such singular points in linear differential equations will be the topic of Chapter 6.

◀ *Exercise 4.15.* Find the dimension of the subspace of the solution space for this third-order differential equation consisting of those solutions that decay as $x \to \infty$ with $x > 0$. ▶

◀ *Exercise 4.16.* Weber's equation for parabolic cylinder functions is the second-order linear differential equation

$$y''(x) - \left(\frac{1}{4}x^2 + a\right) y(x) = 0 ,$$

where $a$ is a real parameter.

Part (a). Make the change of variables $y(x) = e^{-x^2/4} w(x)$ and find the differential equation that $w$ satisfies.

Part (b). Suppose that $a > -1/2$ and find two linearly independent solutions $w_+(x)$ and $w_-(x)$ of the equation you found in part (a) in the form of contour integrals

$$w_\pm(x) = \frac{1}{2\pi i} \int_{C_\pm} q(z) e^{xz} \, dz .$$

Explain how you choose $q(z)$ and the contours $C_\pm$ so that in addition to solving the differential equation, $w_+(x)$ tends to zero as $x \to \infty$ with $x > 0$ and $w_-(x)$ tends to zero as $x \to \infty$ with $x < 0$.

Part (c). Find the leading-order asymptotic behavior of both solutions from part (b) as $x \to \infty$ with $x > 0$ and also as $x \to \infty$ with $x < 0$. Hint: For studying $w_+(x)$ as $x \to \infty$ with $x < 0$ and $w_-(x)$ as $x \to \infty$ with $x > 0$, it is useful to change the integration variable from $z$ to $u = z/x$.

Part (d). Now suppose that $a$ is an odd negative half-integer, that is, $a = -1/2, -3/2, -5/2, \ldots$, and find a nontrivial solution of the equation for $w(x)$ in the form of a contour integral over a path $C$ that is a closed loop. Evaluate the integral using the Residue Theorem for $a = -1/2$ and $a = -3/2$. ▶

## 4.9. Notes and References

The problem of determining the asymptotic behavior of roots of Taylor polynomials (and, more generally, rational Padé approximants) for the exponential function (see Exercise 4.1) is relevant in the design of high-order numerical schemes for time-stepping. For many more details on this interesting problem, see Chapter 4 of Varga's book [**33**].

Weber's equation for parabolic cylinder functions was probably first studied by H. F. Weber [**36**]. A good reference for integral transforms of the type that can be used to study Weber's equation is Hille's text [**15**]. Also, the text by Carrier, Krook, and Pearson [**6**] is a good resource for integral representations of special functions.

show that the wave function has a universal form in the caustic layer that is essentially given by the Airy function.

◀ *Exercise 5.18.* Part (a). Starting from the exact solution formula (5.31), show that if $(x_0, t_0)$ is a typical point on the caustic curve, so that there exists a value $\xi_0$ for which the conditions (5.33) and (5.34) hold, then for each fixed $w \in \mathbb{R}$,

$$\lim_{\substack{\hbar\to 0\\ \hbar>0}} \hbar^{1/6} e^{-i\phi(w)}\psi(x_0+\hbar^{2/3}w, t_0) = \sqrt{\frac{2\pi}{t_0}}\left(\frac{2}{|S'''(\xi_0)|}\right)^{1/3} A(\xi_0) Ai\left(-\left(\frac{2}{S'''(\xi_0)}\right)^{1/3}\frac{w}{t_0}\right),$$

where the phase is

$$\phi(w) := \frac{I(\xi_0; x_0, t_0)}{\hbar} + \frac{x_0-\xi_0}{t_0\hbar^{1/3}}w - \frac{\pi}{4}.$$

As $w$ pans between $-\infty$ and $+\infty$, this gives a profile of the wave function in the transition zone in terms of the Airy function. Hints: Split the integral (5.31) into contributions within and without a $\delta$-neighborhood of the point $\xi = \xi_0$, for some convenient choice of $\delta = \delta(\hbar)$. Consider the two-variable Taylor expansion of $I(\xi; x, t_0)$ with respect to $x$ and $\xi$.

Part (b). Describe the behavior of the wave function $\psi(x,t)$ near a caustic curve; in particular, comment on the thickness of the transition zone and the magnitude of the wave function in the transition zone. Discuss the feasibility of using the asymptotic behavior of the Airy function for large positive and negative arguments to show how the form of the solution near the caustic matches with the asymptotic formulae for $\psi(x,t)$ valid on either side of the caustic. Note that holding $x \neq x_0$ fixed as $\hbar \to 0$ forces $w$ to grow in magnitude in the same limit. ▶

◀ *Exercise 5.19.* Describe how the solutions plotted in Figure 5.7 would be different if instead of $S(x) = 4\,\mathrm{sech}^2(x)$, one had $S(x) = -4\,\mathrm{sech}^2(x)$ with the same amplitude function $A(x)$. Give all details of your analysis. ▶

## 5.7. Multidimensional Integrals

The method of stationary phase can also be adapted to multidimensional integrals. Given a measurable domain $\Omega \subset \mathbb{R}^d$, we consider integrals of the form

$$F(\lambda) := \int_\Omega e^{i\lambda I(\mathbf{x})} g(\mathbf{x})\, d\mathbf{x},$$

where $I(\mathbf{x})$ and $g(\mathbf{x})$ are smooth functions with $I(\mathbf{x})$ real-valued. A point of stationary phase is a point $\mathbf{x}_0 \subset \Omega$ such that

$$\nabla I(\mathbf{x}_0) = 0 \,.$$

Suppose first that $\Omega$ contains no stationary phase points. Then, observe that we may rewrite $F(\lambda)$ in the form

$$F(\lambda) = \frac{1}{i\lambda} \int_\Omega \frac{g(\mathbf{x})}{|\nabla I(\mathbf{x})|^2} \nabla I(\mathbf{x})^T \nabla (e^{i\lambda I(\mathbf{x})}) \, d\mathbf{x} \,.$$

Then, using the vector calculus identity

$$\nabla \cdot (f\mathbf{u}) = \mathbf{u}^T \nabla f + f \nabla \cdot \mathbf{u} \,,$$

we also have

$$F(\lambda) = \frac{1}{i\lambda} \int_\Omega \nabla \cdot \left[ \frac{g(\mathbf{x}) e^{i\lambda I(\mathbf{x})} \nabla I(\mathbf{x})}{|\nabla I(\mathbf{x})|^2} \right] d\mathbf{x} - \frac{1}{i\lambda} \int_\Omega e^{i\lambda I(\mathbf{x})} \nabla \cdot \left[ \frac{g(\mathbf{x}) \nabla I(\mathbf{x})}{|\nabla I(\mathbf{x})|^2} \right] d\mathbf{x} \,.$$

Assuming a smooth boundary $\partial\Omega$ and using the Divergence Theorem, this is the same as

$$F(\lambda) = \frac{1}{i\lambda} \int_{\partial\Omega} e^{i\lambda I(\mathbf{x})} \frac{g(\mathbf{x}) \mathbf{n}(\mathbf{x})^T \nabla I(\mathbf{x})}{|\nabla I(\mathbf{x})|^2} \, dS(\mathbf{x}) - \frac{1}{i\lambda} \int_\Omega e^{i\lambda I(\mathbf{x})} \nabla \cdot \left[ \frac{g(\mathbf{x}) \nabla I(\mathbf{x})}{|\nabla I(\mathbf{x})|^2} \right] d\mathbf{x} \,,$$

where $dS(\mathbf{x})$ denotes an element of surface area on the boundary $\partial\Omega$ and $\mathbf{n}(\mathbf{x})$ is an exterior normal unit vector. Therefore, in the absence of stationary phase points, we immediately see that $F(\lambda) = O(\lambda^{-1})$. The latter integral over $\Omega$ is of the same form as the integral $F(\lambda)$ we began with:

$$F_1(\lambda) := \int_\Omega e^{i\lambda I(\mathbf{x})} g_1(\mathbf{x}) \, d\mathbf{x} \,, \quad g_1(\mathbf{x}) := -\nabla \cdot \left[ \frac{g(\mathbf{x}) \nabla I(\mathbf{x})}{|\nabla I(\mathbf{x})|^2} \right] ,$$

and if $I(\mathbf{x})$ and $g(\mathbf{x})$ have sufficient smoothness, then $F_1(\lambda) = o(1)$ as $\lambda \to \infty$. In fact, if $I(\mathbf{x})$ and $g(\mathbf{x})$ are infinitely differentiable, then the procedure can be repeated any number of times, and we get

$$F(\lambda) \sim \sum_{n=1}^{\infty} \left( \frac{1}{i\lambda} \right)^n \int_{\partial\Omega} e^{i\lambda I(\mathbf{x})} \frac{g_{n-1}(\mathbf{x}) \mathbf{n}(\mathbf{x})^T \nabla I(\mathbf{x})}{|\nabla I(\mathbf{x})|^2} \, dS(\mathbf{x}) \quad \text{as } \lambda \to \infty$$

where the functions $g_n(\mathbf{x})$ are determined iteratively as follows:

$$g_0(\mathbf{x}) := g(\mathbf{x}) \,, \quad g_n(\mathbf{x}) := -\nabla \cdot \left[ \frac{g_{n-1}(\mathbf{x}) \nabla I(\mathbf{x})}{|\nabla I(\mathbf{x})|^2} \right] .$$

In this case, the complete asymptotic expansion of $F(\lambda)$ comes from integrals over the boundary hypersurface. Of course, these surface integrals are also generally oscillatory, and the issue now becomes whether there are any points along $\partial\Omega$ at which $\nabla I$ is normal to the boundary. These points play the role of stationary phase points for the boundary integrals, where admissable infinitessimal variations of $\mathbf{x}$ are necessarily tangent to the boundary. For example, if $\Omega$ is a simply-connected compact subset of $\mathbb{R}^d$ for $d$ odd, with smooth boundary $\partial\Omega$, then there always exist points on $\partial\Omega$ where $\nabla I$ is normal to the boundary. This follows from a deep result of differential topology, the Poincaré-Hopf Index Theorem, which implies that a compact manifold can only support a nonvanishing (tangent) vector field if its Euler characteristic, a topological invariant, is zero. The Euler characteristic of a hypersphere (or any manifold topologically equivalent) in $\mathbb{R}^d$ for $d$ odd is nonzero; therefore every smooth vector field on such a manifold vanishes somewhere. In particular, this applies to the tangent part of $\nabla I$ on $\partial\Omega$. Therefore, in this situation stationary phase points will always exist for the boundary integrals, which will therefore determine the leading-order asymptotic behavior of $F(\lambda)$ when no stationary phase points exist in the interior of $\Omega$.

◁ *Example.* Similar phenomena are possible for boundary hypersurfaces for domains $\Omega$ in even dimensions. Let $\Omega$ be a simply-connected domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. An interesting problem is to determine the number of integer lattice points contained in $\Omega$, that is, to calculate $\#\Omega\cap\mathbb{Z}^2$, where # denotes the cardinality. To give this question an asymptotic aspect, let $\Omega$ be a fixed region and $\lambda$ a positive parameter, and consider the question of determining the asymptotic behavior of $\#\lambda\Omega\cap\mathbb{Z}^2$, that is, the number of integer lattice points contained in dilates of the set $\Omega$ by a factor of $\lambda > 0$, in the limit $\lambda \to \infty$. This is the same as considering the number of lattice points with vertical and horizontal spacing $1/\lambda$ that are contained within the fixed set $\Omega$. Intuitively, it should be clear that

$$\#\lambda\Omega\cap\mathbb{Z}^2 = \lambda^2\int_\Omega d\mathbf{x} + o(\lambda^2) = \lambda^2 \cdot \mathrm{Area}(\Omega) + o(\lambda^2) \tag{5.35}$$

as $\lambda \to \infty$ with $\lambda > 0$.

To prove that (5.35) indeed holds and, more importantly, to calculate higher-order corrections, we need to have an alternative formula for $\#\Omega\cap\mathbb{Z}^2$. If $\phi(\mathbf{x})$ is a *test function* in $\mathbb{R}^d$, that is, an infinitely differentiable function with compact support, then the Poisson Summation Formula holds:

$$\sum_{\mathbf{n}\in\mathbb{Z}^d}\phi(\mathbf{n}) = (2\pi)^d\sum_{\mathbf{m}\in\mathbb{Z}^d}\hat{\phi}(2\pi\mathbf{m})\,, \tag{5.36}$$

where

$$\hat{\phi}(\mathbf{k}) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\mathbf{k}^T\mathbf{x}}\phi(\mathbf{x})\,d\mathbf{x}$$

is the Fourier transform. The *characteristic function* of the set $\Omega \subset \mathbb{R}^2$ is by definition

$$\chi_\Omega(\mathbf{x}) := \begin{cases} 1\,, & \text{for } \mathbf{x} \in \Omega\,, \\ 0\,, & \text{for } \mathbf{x} \notin \Omega\,. \end{cases}$$

Note that if we let $\phi(\mathbf{x}) = \chi_\Omega(\mathbf{x})$ on the left-hand side of (5.36) in dimension $d = 2$, it becomes exactly $\#\Omega \cap \mathbb{Z}^2$. Unfortunately, $\chi_\Omega(\mathbf{x})$ is not smooth enough to be a test function, and so the right-hand side of (5.36) with $\phi(\mathbf{x}) = \chi_\Omega(\mathbf{x})$ does not make sense (it is not an absolutely convergent sum because the corresponding Fourier transform does not decay fast enough). However, $\chi_\Omega(\mathbf{x})$ may be approximated by a sequence $\{\phi_j(\mathbf{x})\}$ of test functions in such a way that

$$\lim_{j\to\infty} \sum_{\mathbf{n}\in\mathbb{Z}^2} \phi_j(\mathbf{n}) = \sum_{\mathbf{n}\in\mathbb{Z}^2} \chi_\Omega(\mathbf{n}) = \#\Omega \cap \mathbb{Z}^2\,.$$

It therefore follows from the Poisson Summation Formula (5.36) that

$$\#\Omega \cap \mathbb{Z}^2 = 4\pi^2 \lim_{j\to\infty} \sum_{\mathbf{m}\in\mathbb{Z}^2} \hat{\phi}_j(2\pi\mathbf{m})\,.$$

Taking the limit $j \to \infty$ is equivalent to any number of different regularization methods for the infinite sum on the right-hand side of (5.36) with $\phi = \chi_\Omega$. For example, we can sum the series by taking an appropriate limit of partial sums:

$$\begin{aligned}\#\Omega \cap \mathbb{Z}^2 &= 4\pi^2 \lim_{R\to\infty} \sum_{\substack{\mathbf{m}\in\mathbb{Z}^2\\ |\mathbf{m}|\le R}} \hat{\chi}_\Omega(2\pi\mathbf{m}) \\ &= \lim_{R\to\infty} \sum_{\substack{\mathbf{m}\in\mathbb{Z}^2\\ |\mathbf{m}|\le R}} \int_{\mathbb{R}^2} e^{-i\mathbf{m}^T\mathbf{x}} \chi_\Omega(\mathbf{x})\, d\mathbf{x} \\ &= \lim_{R\to\infty} \sum_{\substack{\mathbf{m}\in\mathbb{Z}^2\\ |\mathbf{m}|\le R}} \int_{\Omega} e^{-i\mathbf{m}^T\mathbf{x}}\, d\mathbf{x}\,.\end{aligned}$$

This is the desired formula. It tells us how to compute the number of lattice points contained in $\Omega$ by computing a sum of the Fourier transform of the characteristic function. In the case when we replace $\Omega$ by the dilate $\lambda\Omega$, this becomes

$$\begin{aligned}\#\lambda\Omega \cap \mathbb{Z}^2 &= \lim_{R\to\infty} \sum_{\substack{\mathbf{m}\in\mathbb{Z}^2\\ |\mathbf{m}|\le R}} \int_{\lambda\Omega} e^{-i\mathbf{m}^T\mathbf{x}}\, d\mathbf{x} \\ &= \lambda^2 \lim_{R\to\infty} \sum_{\substack{\mathbf{m}\in\mathbb{Z}^2\\ |\mathbf{m}|\le R}} \int_{\Omega} e^{-i\lambda\mathbf{m}^T\mathbf{y}}\, d\mathbf{y}\,,\end{aligned}$$

where we used $\mathbf{x} = \lambda\mathbf{y}$ to change variables in the last step.

Note that the $\mathbf{m} = \mathbf{0}$ term in the sum gives the scaled area. To calculate corrections, we need to consider the asymptotic behavior, as $\lambda \to \infty$ with $\lambda > 0$, of the terms with $\mathbf{m} \neq \mathbf{0}$. This boils down to the consideration of the integrals

$$F_{\mathbf{m}}(\lambda) := \int_{\Omega} e^{-i\lambda \mathbf{m}^T \mathbf{y}} \, d\mathbf{y} \, .$$

These are all oscillatory integrals with phase functions $I_{\mathbf{m}}(\mathbf{y}) := \mathbf{m}^T\mathbf{y}$, so the gradients $\nabla I_{\mathbf{m}}(\mathbf{y}) = \mathbf{m}$ are all constant and nonzero vector fields throughout $\Omega$. Applying the Divergence Theorem once gives the exact result

$$F_{\mathbf{m}}(\lambda) = -\frac{1}{i\lambda|\mathbf{m}|^2} \int_{\partial\Omega} e^{-i\lambda \mathbf{m}^T \mathbf{y}(s)} \mathbf{m}^T \mathbf{n}(s) \, ds \, ,$$

where $s$ is an arc length parameter for $\partial\Omega$. For this one-dimensional oscillatory integral, the phase is stationary for values of $s$ where

$$\mathbf{m}^T \frac{d\mathbf{y}}{ds}(s) = 0 \, .$$

Since $d\mathbf{y}/ds$ is a tangent vector to $\partial\Omega$ at $\mathbf{y}(s)$, the stationary phase points are those at which the boundary $\partial\Omega$ is perpendicular to $\mathbf{m}$. Because $\partial\Omega$ is a closed curve, there are always such points, and therefore the dominant contribution to $F_{\mathbf{m}}(\lambda)$ in the limit $\lambda \to \infty$ with $\lambda > 0$ will arise from such stationary phase points.

If the boundary $\partial\Omega$ has a finite radius of curvature at each point (in particular $\Omega$ is convex), then the stationary phase points will all be simple, because

$$\frac{d^2}{ds^2} \mathbf{m}^T \mathbf{y}(s) = \mathbf{m}^T \frac{d^2}{ds^2} \mathbf{y}(s)$$

and $d^2\mathbf{y}/ds^2$ is always in the normal direction (as is $\mathbf{m}$ at a stationary phase point) and is nonzero (because each point of $\partial\Omega$ has curvature by assumption). Therefore, in this situation, we will have, for each $\mathbf{m} \neq \mathbf{0}$, that $F_{\mathbf{m}}(\lambda) = O(\lambda^{-3/2})$ since a factor of $1/\lambda$ comes from the Divergence Theorem, and an additional factor of $1/\lambda^{1/2}$ comes from the simple stationary phase points of the one-dimensional boundary integral. By an appropriate dominated convergence argument, we then find that

$$\#\lambda\Omega \cap \mathbb{Z}^2 = \lambda^2 \cdot \text{Area}(\Omega) + O(\lambda^{1/2}) \quad \text{as } \lambda \to \infty \text{ with } \lambda > 0$$

for such domains with curvature. The first correction to the area can be asymptotically larger if $\partial\Omega$ has points without curvature, as this leads to higher-order stationary phase points for the boundary integrals. ▷

◀ *Exercise 5.20.* Use the method of stationary phase for one-dimensional integrals to compute the first correction term to the scaled area in the formula for $\#\lambda\Omega \cap \mathbb{Z}^2$ in the case of a simply connected domain $\Omega \in \mathbb{R}^2$ with a curved boundary. ▶

Now we suppose that there exists a unique isolated stationary phase point in the interior of $\Omega$ which we take without loss of generality to be at the origin: $\mathbf{x}_0 = \mathbf{0}$. Let $\delta$ be a small positive number, and let $\mu(\mathbf{x})$ be an infinitely differentiable cutoff function satisfying

$$\mu(\mathbf{x}) = 1 \text{ for } |\mathbf{x}| \leq \delta \text{ and } \mu(\mathbf{x}) = 0 \text{ for } |\mathbf{x}| \geq 2\delta.$$

Then a partition of unity is $\mu(\mathbf{x}) + (1 - \mu(\mathbf{x})) = 1$ and we may write

$$\begin{aligned} F(\lambda) &= \int_\Omega e^{i\lambda I(\mathbf{x})} \mu(\mathbf{x}) g(\mathbf{x})\, d\mathbf{x} + \int_\Omega e^{i\lambda I(\mathbf{x})} (1 - \mu(\mathbf{x})) g(\mathbf{x})\, d\mathbf{x} \\ &= \int_{|\mathbf{x}| \leq 2\delta} e^{i\lambda I(\mathbf{x})} \mu(\mathbf{x}) g(\mathbf{x})\, d\mathbf{x} + O(\lambda^{-1}) \end{aligned}$$

as $\lambda \to \infty$, where the bound on the second integral follows from the above arguments for integrals without stationary phase points. With this method based on partitions of unity, we can isolate small neighborhoods of any number of stationary phase points that may occur in a domain $\Omega$.

As in our study of multidimensional Laplace integrals in §3.7, we will introduce a change of the integration variable near the origin to exactly reduce the exponent function to the quadratic terms of its Taylor expansion about $\mathbf{x} = \mathbf{0}$. In a neighborhood of $\mathbf{x} = \mathbf{0}$ we have (because $\nabla I(\mathbf{0}) = \mathbf{0}$)

$$I(\mathbf{x}) - I(\mathbf{0}) = \frac{1}{2}\mathbf{x}^T \mathbf{H} \mathbf{x} + O(|\mathbf{x}|^3)\,,$$

where $\mathbf{H}$ is the real symmetric Hessian matrix for $I(\mathbf{x})$:

$$H_{jk} := \frac{\partial^2 I}{\partial x_j \partial x_k}\,.$$

The Hessian matrix is diagonalized by an orthogonal matrix $\mathbf{Q}$ so that if $\mathbf{x} = \mathbf{Q}\mathbf{y}$ is the corresponding linear transformation, then

$$I(\mathbf{Q}\mathbf{y}) - I(\mathbf{0}) = \frac{1}{2}\mathbf{y}^T \mathbf{D} \mathbf{y} + O(|\mathbf{y}|^3)\,,$$

where $\mathbf{D}$ is a diagonal matrix of the real eigenvalues of $\mathbf{H}$. In the method of stationary phase, it is not important that the stationary phase point correspond to a maximum of $I$, so the eigenvalues can in principle be any real numbers. We make only the technical assumption that none of the eigenvalues are zero (this is analogous to the generic situation in the one-dimensional case that $I''(t_0) \neq 0$). Therefore, we may represent $\mathbf{D}$ in the form

$$\mathbf{D} = \operatorname{diag}(\nu_1^2, \ldots, \nu_q^2, -\nu_{q+1}^2, \ldots, -\nu_d^2)\,.$$

So, there are $q$ strictly positive eigenvalues and $q' := d - q$ strictly negative eigenvalues. Note that once we decide to put the eigenvalues in this order, we may not be able to arrange $\det(\mathbf{Q}) = 1$, although by orthogonality we will always have $\det(\mathbf{Q}) = \pm 1$. Thus, we have

$$I(\mathbf{Qy}) - I(\mathbf{0}) = \frac{1}{2}\sum_{k=1}^{q} \nu_k^2 y_k - \frac{1}{2}\sum_{k=q+1}^{d} \nu_k^2 y_k + O(|\mathbf{y}|^3)\,.$$

To change the exponent into exactly quadratic form, we seek a change of variables from $\mathbf{y}$ to $\mathbf{u}$ defined near the origin so that

$$I(\mathbf{Qy}) - I(\mathbf{0}) = \frac{1}{2}\sum_{k=1}^{q} \nu_k^2 u_k - \frac{1}{2}\sum_{k=q+1}^{d} \nu_k^2 u_k\,.$$

This requires somewhat of a different strategy than was used in §3.7. To solve this problem, we may begin in the same way that we did in our analysis of multidimensional Laplace integrals, namely by introducing hyperspherical coordinates. Here we introduce such coordinates separately for the variables corresponding to positive and negative eigenvalues of $\mathbf{H}$. Thus, we have four radial variables $r_+$, $r_-$, $\rho_+$, and $\rho_-$ and four sets of hyperspherical angles $\theta_1^+, \ldots, \theta_{q-1}^+$, $\theta_1^-, \ldots, \theta_{q'-1}^-$, $\phi_1^+, \ldots, \phi_{q-1}^+$, and $\phi_1^-, \ldots, \phi_{q'-1}^-$ related to the original coordinates by

$$y_k = \frac{r_+}{\nu_k}\cos(\theta_k^+)\prod_{j=1}^{k-1}\sin(\theta_j^+)\,, \quad k = 1, \ldots, q-1\,,$$

$$y_q = \frac{r_+}{\nu_q}\prod_{j=1}^{q-1}\sin(\theta_j^+)\,,$$

$$y_{q+k} = \frac{r_-}{\nu_{q+k}}\cos(\theta_k^-)\prod_{j=1}^{k-1}\sin(\theta_j^-)\,, \quad k = 1, \ldots, q'-1\,,$$

$$y_d = \frac{r_-}{\nu_d}\prod_{j=1}^{q'-1}\sin(\theta_j^-)\,,$$

$$u_k = \frac{\rho_+}{\nu_k}\cos(\phi_k^+)\prod_{j=1}^{k-1}\sin(\phi_j^+)\,, \quad k = 1, \ldots, q-1\,,$$

$$u_q = \frac{\rho_+}{\nu_q}\prod_{j=1}^{q-1}\sin(\phi_j^+)\,,$$

$$u_{q+k} = \frac{\rho_-}{\nu_{q+k}} \cos(\phi_k^-) \prod_{j=1}^{k-1} \sin(\phi_j^-)\,, \quad k = 1, \ldots, q'-1\,,$$

$$u_d = \frac{\rho_-}{\nu_d} \prod_{j=1}^{q'-1} \sin(\phi_j^-)\,.$$

In terms of these, the equation to be solved takes the form

$$\frac{1}{2}r_+^2 - \frac{1}{2}r_-^2 + E(r_+, \theta_1^+, \ldots, \theta_{q-1}^+, r_-, \theta_1^-, \ldots, \theta_{q'-1}^-) = \frac{1}{2}\rho_+^2 - \frac{1}{2}\rho_-^2\,, \quad (5.37)$$

where $E$ is of cubic or higher order in $r_+$ and $r_-$. Note that the right-hand side factors naturally because

$$\frac{1}{2}\rho_+^2 - \frac{1}{2}\rho_-^2 = \frac{1}{2}(\rho_+ - \rho_-)(\rho_+ + \rho_-)\,.$$

To solve this problem, we therefore will attempt to factor the left-hand side as well; thus we first consider the related equation

$$\frac{1}{2}r_+^2 - \frac{1}{2}r_-^2 + E(r_+, \theta_1^+, \ldots, \theta_{q-1}^+, r_-, \theta_1^-, \ldots, \theta_{q'-1}^-) = 0\,, \quad (5.38)$$

as an equation to be solved for $r_-$ in terms of the other variables. When $r_+ = 0$, this equation is solved by choosing $r_- = 0$ as well, but the Implicit Function Theorem cannot be directly used to obtain $r_-$ for $r_+ \neq 0$ because there are two crossing solution branches that must be separated by blowing up the singularity. Thus, we introduce a new unknown $v$ by $r_- = vr_+$, and then we obtain

$$1 - v^2 + \frac{2}{r_+^2}E(r_+, \theta_1^+, \ldots, \theta_{q-1}^+, vr_+, \theta_1^-, \ldots, \theta_{q'-1}^-) = 0\,,$$

to which the Implicit Function Theorem applies, yielding an infinitely differentiable solution $v$ satisfying $v = 1$ for $r_+ = 0$. Multiplying by $r_+$, we find a solution of (5.38) in the form

$$r_- = f(r_+, \theta_1^+, \ldots, \theta_{q-1}^+, \theta_1^-, \ldots, \theta_{q'-1}^-) = r_+ + O(r_+^2)\,,$$

where the error term is uniform with respect to all $d-2$ angles. The purpose of doing this in connection with the problem that we are trying to solve is that the function

$$F(r_+, \theta_1^+, \ldots, \theta_{q-1}^+, r_-, \theta_1^-, \ldots, \theta_{q'-1}^-) := f(r_+, \theta_1^+, \ldots, \theta_{q-1}^+, \theta_1^-, \ldots, \theta_{q'-1}^-) - r_-$$

is a *divisor* of the left-hand side of (5.37). That is, the equation (5.37) can be written equivalently in the form

$$F(r_+, \theta_1^+, \ldots, \theta_{q-1}^+, r_-, \theta_1^-, \ldots, \theta_{q'-1}^-)Q(r_+, \theta_1^+, \ldots, \theta_{q-1}^+, r_-, \theta_1^-, \ldots, \theta_{q'-1}^-) = \frac{1}{2}(\rho_+ - \rho_-)(\rho_+ + \rho_-)\,,$$

where the quotient $Q := E/F$ is a smooth function of all of its arguments. Our strategy to solve (5.37) for the hyperspherical coordinates associated to $\mathbf{y}$ by a smooth near-identity mapping of a neighborhood of the origin in $\mathbf{u}$ space is then the following. First, we choose

$$\theta_k^+(\rho_+, \phi_1^+, \ldots, \phi_{q-1}^+, \rho_-, \phi_1^-, \ldots, \phi_{q'-1}^-) = \phi_k^+ \,, \quad k = 1, \ldots, q-1 \,, \tag{5.39}$$

and

$$\theta_k^-(\rho_+, \phi_1^+, \ldots, \phi_{q-1}^+, \rho_-, \phi_1^-, \ldots, \phi_{q'-1}^-) = \phi_k^- \,, \quad k = 1, \ldots, q'-1 \,. \tag{5.40}$$

Then, to find $r_+$ and $r_-$, we equate factors by solving simultaneously

$$D := \left[f(r_+, \phi_1^+, \ldots, \phi_{q-1}^+, \phi_1^-, \ldots, \phi_{q'-1}^-) - r_-\right] - [\rho_+ - \rho_-] = 0$$

and

$$S := 2Q(r_+, \phi_1^+, \ldots, \phi_{q-1}^+, r_-, \phi_1^-, \ldots, \phi_{q'-1}^-) - [\rho_+ + \rho_-] = 0 \,.$$

These simultaneous equations can be solved for $r_+$ and $r_-$ in a neighborhood of $\rho_+ = \rho_- = 0$ because the Jacobian matrix is nonsingular:

$$\left.\frac{\partial(D,S)}{\partial(r_+, r_-)}\right|_{r_+=r_-=\rho_+=\rho_-=0} = \left.\begin{bmatrix} \dfrac{\partial D}{\partial r_+} & \dfrac{\partial D}{\partial r_-} \\ \dfrac{\partial S}{\partial r_+} & \dfrac{\partial S}{\partial r_-} \end{bmatrix}\right|_{r_+=r_-=\rho_+=\rho_-=0} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} .$$

◀ *Exercise 5.21.* Verify the calculation of the Jacobian matrix, and show that it follows that the smooth solution obtained in this manner satisfies

$$r_+ = \rho_+ + O(\rho_+^2 + \rho_-^2) \quad \text{and} \quad r_- = \rho_- + O(\rho_+^2 + \rho_-^2)$$

uniformly with respect to the angles $\phi_k^\pm$. This together with the angle relations (5.39) and (5.40) proves that the obtained mapping is a perturbation of the identity mapping near the origin in $\mathbf{u}$ space. ▶

◁ *Example.* Let us find a smooth near-identity transformation $a(u,v)$, $b(u,v)$ defined in a neighborhood of the origin in the $(u,v)$-plane that satisfies the equation

$$y^2 - z^2 + y^2 z + z^3 = u^2 - v^2 \,. \tag{5.41}$$

This is a case for which $q = q' = 1$, and therefore there are no angle variables (this makes the writing simpler, but it does not really make the solution process any less substantial). First, we find (perturbatively) the function $f(y) = y + O(y^2)$ for which $z = f(y)$ is a solution of

$$y^2 - z^2 + y^2 z + z^3 = 0 \,.$$

Substituting $z = sy$ and dividing through by $y^2$, this becomes

$$1 - s^2 + ys + ys^3 = 0\,.$$

By the Implicit Function Theorem, this has a unique solution $s = s(y)$ with $s(0) = 1$, and we may obtain the first few Taylor coefficients by implicit differentiation:

$$s(y) = 1 + y + O(y^2)\,.$$

Therefore the function we seek satisfies $f(y) = y + y^2 + O(y^3)$. Now to satisfy (5.41), we solve simultaneously

$$f(y) - z = u - v \quad \text{and} \quad \frac{y^2 - z^2 + y^2 z + z^3}{f(y) - z} = u + v\,. \tag{5.42}$$

We know that these equations determine a unique solution $y = y(u,v) = u + O(u^2+v^2)$ and $z = z(u,v) = v + O(u^2+v^2)$; let us show how to compute the quadratic terms in the two-variable Taylor expansions of these solution functions. We substitute into (5.42) the following expressions:

$$y = u + Uu^2 + Vuv + Wv^2 + O((u^2+v^2)^{3/2})\,,$$
$$z = v + Xu^2 + Yuv + Zv^2 + O((u^2+v^2)^{3/2})\,,$$

where $U$, $V$, $W$, $X$, $Y$, and $Z$ are coefficients to be determined. Then using the computed terms of $f(y)$, we have

$$f(y) - z = u - v + (U - X + 1)u^2 + (V - Y)uv + (W - Z)v^2 + O((u^2+v^2)^{3/2})\,,$$

so from the first equation in (5.42) we see that

$$U - X + 1 = 0\,, \quad V - Y = 0\,, \quad W - Z = 0\,. \tag{5.43}$$

Now to study the second equation, it is convenient to multiply through by the denominator $f(y)-z$. Thus we are solving $y^2-z^2+y^2z+z^3 = (f(y)-z)(u+v)$ and the left-hand side is

$$y^2 - z^2 + y^2 z + z^3 = u^2 - v^2 + 2Uu^3 + (1-2Z)v^3 + (2V - 2X + 1)u^2 v + (2W - 2Y)uv^2 + O((u^2+v^2)^2)\,.$$

On the other hand, the right-hand side is (using the already obtained relations (5.43))

$$(f(y) - z)(u+v) = u^2 - v^2 + O((u^2+v^2)^2)\,.$$

Therefore we find four more relations:

$$2U = 0\,, \quad 2V - 2X + 1 = 0\,, \quad 2W - 2Y = 0\,, \quad \text{and} \quad 1 - 2Z = 0\,.$$

Now, we have obtained seven equations on six unknown constants. However, these equations are necessarily consistent; this relates to the special choice of the function $f(y)$ in the simultaneous equations (5.42) we are solving. Indeed, the unique solution of the coefficient equations is easily seen to be

$$U = 0\,, \quad V = \frac{1}{2}\,, \quad W = \frac{1}{2}\,,$$
$$X = 1\,, \quad Y = \frac{1}{2}\,, \quad Z = \frac{1}{2}\,.$$

Similar systematic calculations can be used to calculate the coefficients of arbitrarily many terms in the two-variable Taylor expansions of $y(u, v)$ and $z(u, v)$. ▷

We now return to the asymptotic calculation of the integral

$$\begin{aligned} F^{\delta}(\lambda) &:= \int_{|x|\le 2\delta} e^{i\lambda I(\mathbf{x})}\mu(\mathbf{x})g(\mathbf{x})\,d\mathbf{x} \\ &= e^{i\lambda I(\mathbf{0})}\int_{C_\epsilon} \exp\left(\frac{i\lambda}{2}\left[\sum_{k=1}^{q}\nu_k^2 u_k^2 - \sum_{k=q+1}^{d}\nu_k^2 u_k^2\right]\right) G(\mathbf{u})\,d\mathbf{u}\,, \end{aligned}$$

where the region of integration may be taken without loss of generality to be a hypercube $C_\epsilon$ of side length $2\epsilon$ (see (3.40)), on which $G(\mathbf{u})$ is an infinitely differentiable function that vanishes along with all derivatives at the boundary of $C_\epsilon$. Note that $G(\mathbf{u})$ incorporates several factors: the function $g$ evaluated at the corresponding value of $\mathbf{x}$, the cutoff function $\mu$ evaluated at the same point, the Jacobian $\det(\mathbf{Q}) = \pm 1$ for the linear transformation $\mathbf{x} \to \mathbf{y}$, and the Jacobian of the nonlinear near-identity transformation $\mathbf{y} \to \mathbf{z}$. Being infinitely differentiable by assumption, the function $G$ has a Taylor expansion in $d$ variables to any desired (finite) order:

$$G(\mathbf{u}) = \sum_{p=0}^{M}\sum_{p_1+\cdots+p_d=p} G_{p_1,\ldots,p_d}\prod_{k=1}^{d} u_k^{p_k} + r_M(\mathbf{u})\,,$$

where the remainder satisfies $r_M(\mathbf{u}) = O(|\mathbf{u}|^{M+1})$ in a neighborhood of $\mathbf{u} = \mathbf{0}$ and

$$G_{p_1,\ldots,p_d} := \left[\prod_{k=1}^{d}\frac{1}{p_k!}\right]\frac{\partial^p}{\partial u_1^{p_1}\cdots\partial u_d^{p_d}}G(\mathbf{u})\bigg|_{\mathbf{u}=\mathbf{0}}\,.$$

Let $R(u)$ be a polynomial cutoff function in one variable that vanishes to sufficiently high order at $u = \pm\epsilon$ and such that $R(u) - 1$ vanishes to sufficiently high order at $u = 0$. Then, we model $G(\mathbf{u})$ on the hypercube $C_\epsilon$ by a polynomial $P(\mathbf{u})$ that is the product of $R(u_1)\cdots R(u_d)$ with the Taylor polynomial of total degree $M$:

$$P(\mathbf{u}) := \sum_{p=0}^{M}\sum_{p_1+\cdots+p_d=p} G_{p_1,\ldots,p_d}\prod_{k=1}^{d} R(u_k)u_k^{p_k}\,.$$

Writing

$$F_\delta(\lambda) = e^{i\lambda I(\mathbf{0})} J_1(\lambda) + e^{i\lambda I(\mathbf{0})} J_2(\lambda)\,,$$

where

$$J_1(\lambda) := \int_{C_\epsilon} \exp\left( \frac{i\lambda}{2} \left[ \sum_{k=1}^{q} \nu_k^2 u_k^2 - \sum_{k=q+1}^{d} \nu_k^2 u_k^2 \right] \right) P(\mathbf{u})\, d\mathbf{u}$$

and

$$J_2(\lambda) := \int_{C_\epsilon} \exp\left( \frac{i\lambda}{2} \left[ \sum_{k=1}^{q} \nu_k^2 u_k^2 - \sum_{k=q+1}^{d} \nu_k^2 u_k^2 \right] \right) (G(\mathbf{u}) - P(\mathbf{u}))\, d\mathbf{u}\,,$$

we see immediately that the integral $J_1(\lambda)$ breaks into a sum of products of one-dimensional integrals:

$$J_1(\lambda) = \sum_{p=0}^{M} \sum_{p_1+\cdots+p_d=p} G_{p_1,\ldots,p_d} \cdot \prod_{k=1}^{q} \int_{-\epsilon}^{\epsilon} e^{i\lambda \nu_k^2 u^2/2} R(u) u^{p_k}\, du \prod_{\ell=q+1}^{d} \int_{-\epsilon}^{\epsilon} e^{-i\lambda \nu_\ell^2 u^2/2} R(u) u^{p_\ell}\, du\,.$$

Steepest descent analysis of the one-dimensional integrals may then be applied term-by-term in this finite sum. Furthermore, since

$$\nabla \frac{1}{2} \left[ \sum_{k=1}^{q} \nu_k^2 u_k^2 - \sum_{k=q+1}^{d} \nu_k^2 u_k^2 \right] = (\nu_1^2 u_1, \ldots, \nu_q^2 u_q, -\nu_{q+1}^2 u_{q+1}, \ldots, -\nu_d^2 u_d)^T$$

vanishes for $\mathbf{u} = \mathbf{0}$ only, and exactly to first order, while $G(\mathbf{u}) - P(\mathbf{u})$ vanishes to high order both for $\mathbf{u} = \mathbf{0}$ as well as on the boundary of the hypercube $C_\epsilon$, repeated use of the Divergence Theorem to "integrate by parts" shows that $J_2(\lambda)$ is asymptotically small compared to the meaningful terms in $J_1(\lambda)$. Therefore we may assemble these results for any $M$ in the statement

$$F^\delta(\lambda) \sim e^{i\pi(q-q')/4} \frac{e^{i\lambda I(\mathbf{0})}}{\sqrt{|\det \mathbf{H}|}} \left( \frac{2\pi}{\lambda} \right)^{d/2} \sum_{m=0}^{\infty} c_m \lambda^{-m} \qquad \text{as } \lambda \to \infty \text{ with } \lambda > 0\,,$$

where

$$c_m := \frac{1}{2^m} \sum_{\substack{p_1+\cdots+p_d=2m \\ p_k \text{ all even}}} e^{i\pi((p_1+\cdots+p_q)-(p_{q+1}+\cdots+p_d))/4} G_{p_1,\ldots,p_d} \prod_{k=1}^{d} \frac{p_k!}{\nu_k^{p_k} (p_k/2)!}\,.$$

# Weakly Nonlinear Waves

While the method of multiple scales can be usefully applied to certain problems of singular asymptotics for ordinary differential equations, it is also well known as a method that applies equally well to certain analogous problems related to partial differential equations. This is especially true for nonlinear partial differential equations describing the propagation of waves, in which case one approach of interest is to view the influence of nonlinear terms as a perturbation. This is the subject of weakly nonlinear waves.

## 10.1. Derivation of Universal Partial Differential Equations Using the Method of Multiple Scales

Recall that in applying the method of multiple scales to ordinary differential equations, one does not obtain the asymptotic form of the solution valid over long time scales in a single step. Rather, one first obtains a new differential equation for the complex amplitude $A$ as a function of the slow time $T_1$ from the solvability condition. An intermediate step to determining the asymptotic form of the weakly nonlinear oscillations valid for times $t$ proportional to $1/\epsilon$ is then the solution of this new differential equation for $A(T_1)$.

In the study of weakly nonlinear waves by the method of multiple scales, the same multi-step approach will apply; however in place of the ordinary differential equation for the complex amplitude $A$ that is used to avoid secular terms, we will obtain partial differential equations with respect to the slow variables. However, the emphasis now shifts, because we will be less interested in solving these new equations directly and more interested in observing the way that the same few model partial differential equations arise again and again from very different problems as solvability conditions. These model equations are consequently very important in their own right since each solution of the model equation implies facts about every individual nonlinear wave problem in which the same model arises from a perturbation procedure. For this reason, we view these model equations as being *universal*.

The notion of the same model equation appearing in many different problems in an asymptotic limit is perhaps not so surprising. For example, recall that in turning point problems (see §7.2.2), no matter what the exact coefficients in the equation are, Airy's differential equation describes the local transition as long as the turning point is nondegenerate (a generic condition).

**10.1.1. Modulated wavetrains with dispersion and nonlinear effects. The cubic nonlinear Schrödinger equation.** One nonlinear wave equation (among many different ones) to which the method of multiple scales can be applied to deduce simple universal model equations for certain weakly nonlinear motions is the nonlinear partial differential equation

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + \sin(\varphi) = 0\,, \tag{10.1}$$

which is called the *sine-Gordon equation*. The sine-Gordon equation (10.1) describes directly at least two distinct physical phenomena:

① The mechanical oscillations of a chain of coupled pendula or any system that can be approximately modeled in this way, for example the torsional vibrations of base-pairs on RNA molecules. The chain consists of pendula each of which is indexed by an integer $n$ and has angular displacement $\varphi_n(t)$ and the corresponding gravitational restoring force $-\sin(\varphi_n(t))$. In the chain of pendula, each pendulum is coupled to its left and right nearest neighbors by linear springs, so there is also a contribution to the force on the $n$th pendulum from each neighbor. Therefore, Newton's second law of motion takes the form

$$\frac{d^2\varphi_n}{dt^2} = -\sin(\varphi_n) + k(\varphi_{n+1} - \varphi_n) + k(\varphi_{n-1} - \varphi_n)\,. \tag{10.2}$$

Here $k$ is a normalized spring constant of the coupling. If the displacements of neighboring pendula are close, then a continuum limit approach makes sense. In such an approach, we suppose the existence of some smooth function $\varphi(x,t)$ of which $\varphi_n(t)$ is an approximate sampled value for $x = nh$:

$$\varphi_n(t) \approx \varphi(nh, t)\,.$$

Here $h$ is the *lattice spacing*. Setting $k = 1/h^2$, Taylor expansion of $\varphi(x,t)$ in $x$ shows that (10.2) becomes the sine-Gordon equation (10.1) in the *continuum limit* of $h \to 0$. This mechanical analogy is a good way to visualize the solutions of the sine-Gordon equation.

② The dynamics of the relative phase $\varphi(x,t)$ of quantum mechanical wavefunctions on either side of a long superconducting Josephson junction.

Since $\sin(0) = 0$, the function $\varphi(x,t) \equiv 0$ is an exact solution of (10.1). This seems uninteresting, except that it means that we can in fact obtain a great deal of information about nontrivial solutions of the sine-Gordon equation using perturbation theory based on the trivial solution $\varphi \equiv 0$. In other words, what we want to do is to consider solutions of (10.1) for which $\varphi(x,t)$ is small. This can be done by introducing a small perturbation parameter $\epsilon$ and setting $\varphi = \epsilon u$. The sine-Gordon equation (10.1) thus becomes, exactly,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{1}{\epsilon}\sin(\epsilon u) = 0\,. \tag{10.3}$$

Now since the sine is not a linear function, the equation for $u$ contains $\epsilon$ in an essential way, and consequently $u = u(x,t;\epsilon)$ will depend on $\epsilon$ too. But since as $\epsilon \to 0$ we have the Taylor expansion

$$\frac{1}{\epsilon}\sin(\epsilon u) = u - \frac{\epsilon^2}{6}u^3 + \cdots = u + O(\epsilon^2)\,, \tag{10.4}$$

we see that the equation (10.3) for $u(x,t;\epsilon)$ is a small perturbation of a linear equation (the reduced equation) when $\epsilon$ is small. Now on this basis we would like to suppose that $u(x,t;\epsilon)$ has an asymptotic expansion in powers of $\epsilon$. Since only even powers of $\epsilon$ appear in the Taylor expansion (10.4), it might seem that a series in positive integer powers of $\delta := \epsilon^2$ would be appropriate. However, it turns out that there is some advantage to using odd powers of $\epsilon$ too; in any case, if these orders were not necessary, we would find out later by deducing that the corresponding coefficients in the asymptotic series were zero. So let us proceed by substituting

$$u(x,t;\epsilon) = u_0(x,t) + \epsilon u_1(x,t) + \epsilon^2 u_2(x,t) + O(\epsilon^3) \tag{10.5}$$

into (10.3) and collecting powers of $\epsilon$ to determine $u_0(x,t)$, $u_1(x,t)$, and $u_2(x,t)$.

The terms independent of $\epsilon$ in (10.3) give the reduced equation

$$\frac{\partial^2 u_0}{\partial t^2} - \frac{\partial^2 u_0}{\partial x^2} + u_0 = 0\,, \tag{10.6}$$

which is a linear constant-coefficient partial differential equation for $u_0(x,t)$. Recall from §5.5 that the basic solutions of this linear equation are exponential traveling waves $e^{i(kx-\omega t)}$ where the wavenumber $k$ and angular frequency

$\omega$ are parameters that are not independent, being linked by the dispersion relation

$$\omega^2 = k^2 + 1\,. \tag{10.7}$$

The corresponding functions $\omega_+(k)$ and $\omega_-(k)$ are thus the upper and lower branches of a hyperbola, as illustrated in Figure 10.1. We can pick concrete

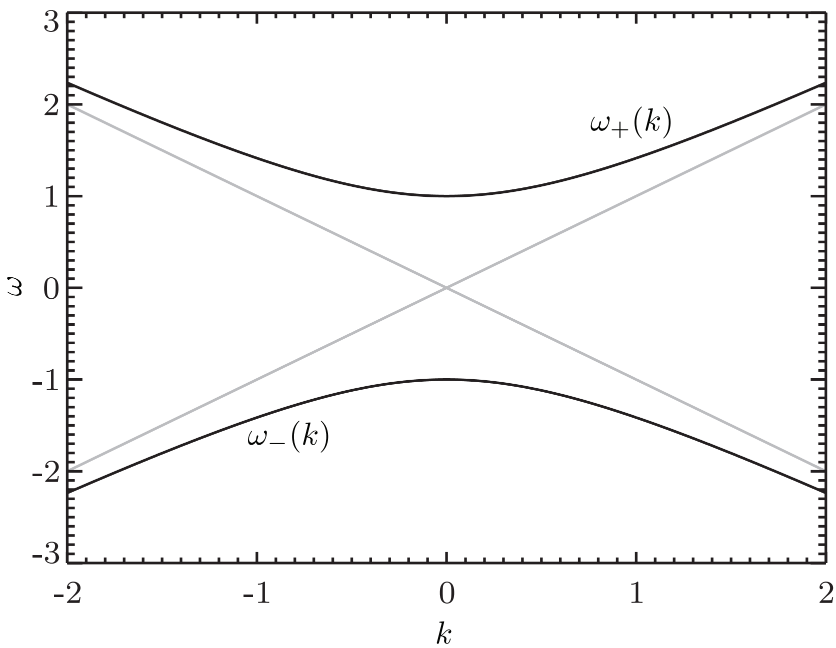


**Figure 10.1.** *The two branches $\omega_+(k)$ and $\omega_-(k)$ of the dispersion relation for the linearized sine-Gordon equation (10.6), as functions of k, with the hyperbolic asymptotes shown in gray.*

values of $k$ and $\omega$ satisfying (10.7) and obtain a real-valued solution for $u_0(x,t)$ by writing

$$u_0(x,t) = Ae^{i(kx-\omega t)} + A^* e^{-i(kx-\omega t)}\,, \tag{10.8}$$

where $A$ is a complex constant and $A^*$ is the complex-conjugate of $A$. Note that both exponential functions appearing in (10.8) are admitted as solutions of (10.6) (and therefore so is the linear combination (10.8)) because the dispersion relation (10.7) is symmetric under the transformation $k \to -k$ and $\omega \to -\omega$.

Experience gained from studying weakly nonlinear oscillations in the context of ordinary differential equations (see Chapter 9) suggests that upon seeking higher-order corrections $u_1$, $u_2$, and so on, terms analogous to secular terms will appear unless some mechanism is introduced to prevent them. To see what the correct analogue of secular terms should be and to understand what makes these terms appear, we continue to calculate the higher-order terms. The partial differential equation for $u_1$ is in this case

$$\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} + u_1 = 0$$

which may be satisfied by taking $u_1 \equiv 0$ (this is a consequence of only even powers of $\epsilon$ appearing in (10.3)), and then the equation governing $u_2$ is

$$\begin{aligned}\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} + u_2 &= \frac{1}{6}u_0^3 \\ &= \frac{A^3}{6}e^{3i\theta} + \frac{A^2A^*}{2}e^{i\theta} + \circledast\,.\end{aligned} \tag{10.9}$$

In writing this, we have introduced two new notations that will be convenient throughout this chapter:

① The *phase* is defined as $\theta := kx - \omega t$.

② The symbol "$\circledast$" denotes the complex conjugate of the term(s) preceding. Thus, $x + \circledast$ is shorthand for $x + x^*$, no matter how complicated an expression $x$ is.

The right-hand side of (10.9) consists of a linear combination of terms of the form $e^{in\theta}$ for $n \in \mathbb{Z}$. Such terms are called *harmonics*. We refer to the terms $e^{\pm i\theta}$ as *fundamental harmonics* and the terms $e^{\pm in\theta}$ for $n \neq 0, 1, -1$ as *overtones* or *higher harmonics*. If present, a constant term (that is, a harmonic with $n = 0$) is called a *mean* term. In general, a harmonic is called *resonant* if it is annihilated by the linear operator appearing on the left-hand side of the equation. Resonant harmonics give rise to secular terms, as the following exercise shows.

◀ *Exercise 10.1.* The reason that having exponentials on the right hand side whose wavenumbers and frequencies satisfy (10.7) leads to secular terms is easy to see in this case. Let $k$ and $\omega$ satisfy the dispersion relation (10.7) and let $B$ be a complex constant. Consider the linear equation

$$\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} + u_2 = Be^{i\theta} \tag{10.10}$$

with $\theta = kx - \omega t$. Use (10.7) to verify that a particular solution is given by

$$u_2(x,t) = \frac{Bit}{2\omega}e^{i\theta}$$

which grows linearly in $t$. An arbitrary solution $u_2$ of (10.10) will differ from this growing solution by a homogeneous solution, which can be obtained by Fourier transforms. The general homogeneous solution that is bounded in $x$ is also bounded in $t$. Consequently, all solutions that are bounded in $x$ must be linearly growing in $t$. ▶

The fundamental harmonics $e^{\pm i\theta}$ on the right-hand side of (10.9) are automatically resonant because $u_0(x,t)$ satisfies the reduced equation (10.6). At this point, we need to know whether any other harmonics appearing on the right-hand side of (10.9) are also resonant. The following exercise considers whether the third harmonics $e^{\pm 3i\theta}$ can be resonant.

◀ *Exercise 10.2.* Part (a). Show that the third harmonics appearing on the right-hand side of (10.9) are not resonant by checking that the dispersion relation (10.7) does not admit the pair $(\pm 3k, \pm 3\omega)$ if it admits the pair $(k,\omega)$. Hint: Eliminate the frequency $\omega$.

Part (b). For some dispersion relations it is possible for the third harmonics to be resonant. Consider the dispersion relation $\omega = k - k^3 + k^5$. Show that the third harmonics $e^{\pm 3i\theta}$ are resonant if and only if the fundamental wavenumber $k$ satisfies $k = 0$ or $k = \pm\sqrt{10}$. This is typical in the sense that for most dispersion relations the third harmonics are only resonant if the fundamental wavenumber has certain values. Sometimes such resonances are called *accidental* resonances. ▶

Therefore, we indeed see that because resonant harmonics appear on the right-hand side of (10.9), secular terms will appear in $u_2(x,t)$, and the corresponding asymptotic series expansion of $u(x,t)$ will therefore become invalid when $t$ becomes as large as $1/\epsilon^2$. The strategy of the method of multiple scales is to introduce multiple time scales $T_0 = t$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$ and to choose the "constant" $A$ to depend on these slow time scales in such a way that the resonant harmonics no longer appear on the right-hand side of (10.9). The method of multiple scales requires one further modification in the context of partial differential equations, namely the introduction of multiple spatial scales. Recall from Chapter 9 that in the context of weakly nonlinear oscillations described by ordinary differential equations the introduction of dependence on slow time scales (as in the method of multiple scales) is equivalent to slightly adjusting the fundamental frequency $\omega$ of oscillation (as in the Poincaré-Lindstedt method). This correspondence also makes sense in the context of partial differential equations; however, according to the dispersion relation (10.7) the wavenumber $k$ and the frequency $\omega$ are linked, and therefore any adjustment of the frequency $\omega$ implies a corresponding adjustment of the wavenumber $k$, to maintain consistency. If adjusting frequencies is equivalent to introducing multiple time scales, then adjusting wavenumbers is equivalent to introducing multiple spatial scales. It turns out that introducing one additional spatial scale is sufficient for many applications. We thus introduce the variables $X_0 = x$ and $X_1 = \epsilon x$.

To implement the method of multiple scales to avoid the secular terms appearing in a direct asymptotic expansion of weakly nonlinear waves in the sine-Gordon equation (10.3), suppose that

$$u(x,t;\epsilon) = U(X_0, X_1, T_0, T_1, T_2; \epsilon)$$

and insert into (10.3) using the chain rule to compute the derivatives:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 U}{\partial T_0^2} + 2\epsilon \frac{\partial^2 U}{\partial T_0 \partial T_1} + \epsilon^2 \left[ \frac{\partial^2 U}{\partial T_1^2} + 2\frac{\partial^2 U}{\partial T_0 \partial T_2} \right], \\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 U}{\partial X_0^2} + 2\epsilon \frac{\partial^2 U}{\partial X_0 \partial X_1} + \epsilon^2 \frac{\partial^2 U}{\partial X_1^2}. \end{aligned} \tag{10.11}$$

We attempt to find $U$ in the form of an asymptotic expansion analogous to (10.5):

$$\begin{aligned} U(X_0, X_1, T_0, T_1, T_2; \epsilon) = U_0&(X_0, X_1, T_0, T_1, T_2) \\ &+ \epsilon U_1(X_0, X_1, T_0, T_1, T_2) \\ &+ \epsilon^2 U_2(X_0, X_1, T_0, T_1, T_2) + O(\epsilon^3). \end{aligned} \tag{10.12}$$

The terms in (10.3) using (10.11) and (10.12) that are independent of $\epsilon$ amount to the linear equation (10.6) written in terms of the fast variables:

$$\frac{\partial^2 U_0}{\partial T_0^2} - \frac{\partial^2 U_0}{\partial X_0^2} + U_0 = 0,$$

which has the wavetrain solution

$$U_0 = Ae^{i\theta} + \circledast$$

where the phase is written in terms of the fast variables as $\theta = kX_0 - \omega T_0$ and $k$ and $\omega$ satisfy the dispersion relation (10.7). The complex amplitude $A = A(X_1, T_1, T_2)$ is a function of the slow variables, but it is independent of $X_0$ and $T_0$.

The terms in (10.3) proportional to $\epsilon$ give an equation for $U_1$:

$$\begin{aligned} \frac{\partial^2 U_1}{\partial T_0^2} - \frac{\partial^2 U_1}{\partial X_0^2} + U_1 &= -2\frac{\partial^2 U_0}{\partial T_0 \partial T_1} + 2\frac{\partial^2 U_0}{\partial X_0 \partial X_1} \\ &= 2i\left[\omega \frac{\partial A}{\partial T_1} + k\frac{\partial A}{\partial X_1}\right] e^{i\theta} + \circledast. \end{aligned} \tag{10.13}$$

Because the fundamental harmonics $e^{\pm i\theta}$ are resonant, we will have secular terms in $U_1$ unless we choose the dependence of $A$ on $X_1$ and $T_1$ according to

$$\frac{\partial A}{\partial T_1} + \frac{k}{\omega}\frac{\partial A}{\partial X_1} = 0. \tag{10.14}$$

Since $\omega$ depends on $k$ according to the dispersion relation (10.7), we can differentiate (10.7) implicitly with respect to $k$. Regardless of whether we are on the branch $\omega = \omega_+(k)$ or $\omega = \omega_-(k)$, we find that

$$\frac{k}{\omega_\pm(k)} = \omega'_\pm(k).$$

Consequently, the general solution of (10.14) can be written in the form

$$A = f(X_1 - \omega'_\pm(k)T_1)$$

where $f(\xi)$ is an arbitrary function. Thus we learn that on the time scale $T_1$, the modulation of the wave amplitude amounts to a rigid translation to the right with a speed equal to the group velocity $v_g := \omega'_\pm(k)$. With the complex amplitude $A$ chosen to satisfy (10.14), the right-hand side of (10.13) is identically zero. We therefore choose[1] the zero solution for $U_1$.

With $U_1 \equiv 0$, the equation determining $U_2$ is

$$\begin{aligned}\frac{\partial^2 U_2}{\partial T_0^2} - \frac{\partial^2 U_2}{\partial X_0^2} + U_2 &= -\frac{\partial^2 U_0}{\partial T_1^2} - 2\frac{\partial^2 U_0}{\partial T_0 \partial T_2} + \frac{\partial^2 U_0}{\partial X_1^2} + \frac{1}{6}U_0^3 \\ &= \left[-\frac{\partial^2 A}{\partial T_1^2} + 2i\omega\frac{\partial A}{\partial T_2} + \frac{\partial^2 A}{\partial X_1^2} + \frac{1}{2}A^2A^*\right]e^{i\theta} \\ &\quad + \frac{1}{6}A^3 e^{3i\theta} + \circledast\,.\end{aligned} \tag{10.15}$$

Since in this case the harmonics $e^{\pm 3i\theta}$ are not resonant, the solvability condition for a bounded solution $U_2$ is the equation

$$-\frac{\partial^2 A}{\partial T_1^2} + 2i\omega\frac{\partial A}{\partial T_2} + \frac{\partial^2 A}{\partial X_1^2} + \frac{1}{2}A^2A^* = 0 \tag{10.16}$$

(and its complex conjugate, which is equivalent). With this choice of dependence of $A$ on $T_2$, we are guaranteed that the solution of (10.15) for $U_2$ will not contain any secular terms. On the other hand, unlike $U_1$ the correction $U_2$ cannot be taken to be zero, since the equation (10.15) subject to (10.16) is not homogeneous. There are nonresonant harmonics $e^{\pm 3i\theta}$ on the right-hand side that will give rise to contributions to $U_2$ proportional to these same harmonics.

The two solvability conditions (10.14) and (10.16) can be put into a more transparent form with a change of independent variables. Namely, based on the discussion following (10.14), we anticipate the utility of the change of variables $(X_1, T_1) \to (\xi, \tau)$ given by

$$\xi = X_1 - \omega'_\pm(k)T_1 \quad \text{and} \quad \tau = T_1\,.$$

Indeed from the Jacobian of this transformation, we immediately find that (10.14) becomes simply

$$\frac{\partial A}{\partial \tau} = 0 \tag{10.17}$$

[1] While consistent, this choice is not necessary. One could choose any homogeneous solution for $U_1$, which amounts to correcting $U_0$ (already a homogeneous solution) with more of the same. Unless there is a specific reason to do this (for example, initial conditions that are dependent on $\epsilon$ in an appropriate way), it is usually sufficient to choose the particular solution $U_1 \equiv 0$.

or, in words, in the frame of reference moving with the group velocity, the amplitude $A$ is independent of $\tau = T_1$. With the same change of variables and using (10.17), we find that (10.16) becomes

$$2i\omega_\pm(k)\frac{\partial A}{\partial T_2} + \left(1 - [\omega_\pm'(k)]^2\right)\frac{\partial^2 A}{\partial \xi^2} + \frac{1}{2}A^2A^* = 0\,.$$

Once again, some useful information may be obtained from implicitly differentiating the dispersion relation (10.7). Namely, we find that

$$[\omega_\pm'(k)]^2 + \omega_\pm(k)\omega_\pm''(k) = 1\,.$$

Therefore, we may finally write (10.16) in the form

$$i\frac{\partial A}{\partial T_2} + \frac{\omega_\pm''(k)}{2}\frac{\partial^2 A}{\partial \xi^2} + \beta|A|^2A = 0\,, \tag{10.18}$$

where in this case the *nonlinear coefficient* is given by

$$\beta := \frac{1}{4\omega_\pm(k)}\,. \tag{10.19}$$

Thus, while the shape of the wave envelope $A$ is stationary in the group velocity frame over time scales of length $1/\epsilon$ (in the original $t$ variable, since $T_1$ being order one means $t = O(1/\epsilon)$), on time scales of length $1/\epsilon^2$, the envelope is evolving in the group velocity frame according to the nonlinear partial differential equation (10.18).

The equation (10.18) governing the complex amplitude $A$ is called the *cubic nonlinear Schrödinger equation*. This is the first example we have of a universal amplitude equation arising from solvability conditions in a perturbation expansion based on the method of multiple scales. The universal nature of the cubic nonlinear Schrödinger equation is hinted at by the perturbative nature of the analysis leading to its derivation, as the following exercise shows.

◀ *Exercise 10.3.* Carry out a parallel analysis of small solutions $\varphi = \epsilon u$ of the nonlinear Klein-Gordon equation

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + F(\varphi) = 0\,,$$

where $F(\varphi)$ is an arbitrary odd (that is, $F(-\varphi) = -F(\varphi)$ holds) analytic function of $\varphi$ with $F'(0) \neq 0$ and $F'''(0) \neq 0$. Derive the cubic nonlinear Schrödinger equation (10.18) and find the appropriate formula for the nonlinear coefficient $\beta$ in terms of $F'(0)$ and $F'''(0)$. ▶

In fact, the universality class of nonlinear wave equations for which the cubic nonlinear Schrödinger equation (10.18) is a model for the dynamics of slowly modulated waves is far broader than the class of nonlinear Klein-Gordon equations. We will encounter other examples in §10.2 and §10.3, but for now it should be said clearly that whenever weakly nonlinear, nearly monochromatic, one-dimensional waves with wavenumber $k$ and frequency $\omega$ are propagating in the presence of strong dispersion (meaning $\omega''(k) \neq 0$) and in the absence of dissipation or accidental resonances, the complex wave amplitude $A$ will satisfy an equation of the general form (10.18), where $T_2 = \epsilon^2 t$, $\xi = \epsilon(x - \omega'(k)t)$, and where the parameter $\beta$ encodes the nonlinear terms in the wave equation under study and must be determined by a multiple scales analysis (it is generally not given by the formula (10.19) but rather its form depends on the problem).

One of the important properties of the cubic nonlinear Schrödinger equation is that it selects a particular profile for a "finite energy" traveling wave $f(\xi)$ moving with the group velocity, as the following exercise shows.

◀ *Exercise 10.4.* Part (a). Suppose that $\beta\omega''(k) > 0$, which is known as the *focusing* case. Consider solutions of (10.18) of the form $A = e^{i\Omega T_2} f(\xi)$, where $\Omega$ is a real frequency parameter and $f$ is real. Show that nonzero profiles $f$ that decay to zero as $\xi \to \pm\infty$ may only arise if $\Omega\omega''(k) > 0$ and that in this case $f(\xi)$ necessarily has the form $f(\xi) = a\,\mathrm{sech}(b(\xi - \xi_0))$. Are the parameters $a$, $b$, and $\Omega$ independent? If not, what is the relationship between them?

Part (b). Suppose that $\beta\omega''(k) < 0$, which is known as the *defocusing* case. Consider again solutions of (10.18) of the form $A = e^{i\Omega T_2} f(\xi)$ with $\Omega$ and $f$ real. Show that nonzero profiles $f$ that decay to constant values as $\xi \to \pm\infty$ may only arise if $\Omega\omega''(k) < 0$ and that in this case $f(\xi)$ necessarily has the form $f(\xi) = a\tanh(b(\xi - \xi_0))$. Are the parameters $a$, $b$, and $\Omega$ independent? If not, what is the relationship between them? ▶

These special solutions are called *solitons.* When it is important to distinguish the cases, the solitons of the focusing nonlinear Schrödinger equation are sometimes called "bright" solitons while those of the defocusing equation are called "dark" solitons. This terminology arises from applications in nonlinear fiber optics.

The cubic nonlinear Schrödinger equation also has many other different exact solutions. This is the first of many remarkable properties of this equation that go beyond its universality as a model amplitude equation for complex amplitudes $A$ of weakly nonlinear waves.

**10.1.2. Spontaneous excitation of a mean flow.** Sometimes when nonlinear terms are neglected, the wave equation of interest admits nontrivial solutions that are independent of $x$ and $t$. This occurs when the dispersion relation admits the solution $(k, \omega) = (0, 0)$ and the corresponding harmonic (constant solution) is called a *mean flow.* One implication of this is that no matter what harmonic $e^{i\theta}$ with $\theta = kx - \omega t$ one chooses as a starting point for perturbation theory, the possibility exists that nonlinear terms will generate from the fundamental harmonic a mean term which, being resonant, will result in secular terms. While generally we have viewed such resonances as accidental, the case of a resonant mean is much more common, since if the dispersion relation admits a mean flow, the appearance of secular terms is independent of the fundamental wavenumber $k$.

To see how to handle this situation in the framework of the method of multiple scales, consider the partial differential equation

$$\frac{\partial^2\varphi}{\partial t^2} - c^2\frac{\partial^2\varphi}{\partial x^2} + \alpha^2\frac{\partial^4\varphi}{\partial x^4} + \sigma\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2} = 0\,,$$

which is known as the Boussinesq equation (see §10.1.4). Here $c$ is a real parameter (the sound speed), $\alpha$ is a real parameter measuring the dispersion in the system, and $\sigma$ is a real parameter measuring the nonlinear effects. To view this as a weakly nonlinear problem, we introduce a small parameter $\epsilon$ and set $\varphi = \epsilon u$. Therefore, the rescaled equation is

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} + \alpha^2\frac{\partial^4 u}{\partial x^4} + \epsilon\sigma\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} = 0\,. \tag{10.20}$$

To carry out the weakly nonlinear analysis using the method of multiple scales, we introduce fast scales $X_0 = x$ and $T_0 = t$ and slow scales $X_1 = \epsilon x$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$, and we write $u(x,t;\epsilon) = U(X_0, X_1, T_0, T_1, T_2; \epsilon)$. Expanding $U$ as in (10.12), we obtain first the reduced equation for $U_0$:

$$\frac{\partial^2 U_0}{\partial T_0^2} - c^2\frac{\partial^2 U_0}{\partial X_0^2} + \alpha^2\frac{\partial^4 U_0}{\partial X_0^4} = 0\,.$$

If we try a solution of the form

$$U_0 = Ae^{i\theta} + \circledast\,, \tag{10.21}$$

where $\theta = kX_0 - \omega T_0$ with $k$ and $\omega$ satisfying the dispersion relation

$$\omega^2 = c^2k^2 + \alpha^2k^4 \tag{10.22}$$

and with $A = A(X_1, T_1, T_2)$ being an undetermined complex amplitude, then the corresponding equation governing $U_1$ is

$$\begin{aligned}
&\frac{\partial^2 U_1}{\partial T_0^2} - c^2\frac{\partial^2 U_1}{\partial X_0^2} + \alpha^2\frac{\partial^4 U_1}{\partial X_0^4} \\
&= -2\frac{\partial^2 U_0}{\partial T_0\partial T_1} + 2c^2\frac{\partial^2 U_0}{\partial X_0\partial X_1} - 4\alpha^2\frac{\partial^4 U_0}{\partial X_0^3\partial X_1} - \frac{\sigma}{2}\frac{\partial}{\partial X_0}\left(\frac{\partial U_0}{\partial X_0}\right)^2 \\
&= i\sigma k^3A^2e^{2i\theta} + \left[2i\omega\frac{\partial A}{\partial T_1} + (2ic^2k + 4i\alpha^2k^3)\frac{\partial A}{\partial X_1}\right]e^{i\theta} + \circledast\,.
\end{aligned} \tag{10.23}$$

The fundamental harmonics $e^{\pm i\theta}$ are resonant, and to avoid the corresponding secular terms in $U_1$, we need to impose the solvability condition

$$\frac{\partial A}{\partial T_1} + \frac{c^2 k + 2\alpha^2 k^3}{\omega}\frac{\partial A}{\partial X_1} = 0\,, \tag{10.24}$$

which indicates the rigid translation of the profile of the complex amplitude $A$ to the right with the group velocity since by implicit differentiation of (10.22) we see easily that whatever branch of the dispersion relation corresponds to the fundamental harmonic,

$$v_g := \omega'(k) = \frac{c^2 k + 2\alpha^2 k^3}{\omega(k)}\,.$$

With the solvability condition satisfied, the correction $U_1$ may be found as a bounded function of $X_0$ and $T_0$ since secular growth has been suppressed. The only terms that remain on the right-hand side of (10.23) are the second harmonics, and since

$$\begin{aligned}\left[\frac{\partial^2}{\partial T_0^2} - c^2\frac{\partial^2}{\partial X_0^2} + \alpha^2\frac{\partial^4}{\partial X_0^4}\right] e^{\pm 2i\theta} &= \left[-4\omega^2 + 4c^2k^2 + 16\alpha^2 k^4\right] e^{\pm 2i\theta} \\ &= 12\alpha^2 k^4 e^{\pm 2i\theta}\end{aligned}$$

(the condition that the second harmonics are not resonant is therefore that the fundamental wavenumber $k$ is nonzero), a particular solution for $U_1$ is

$$U_1 = \frac{i\sigma A^2}{12\alpha^2 k} e^{2i\theta} + \circledast\,. \tag{10.25}$$

The equation determining $U_2$ is then

$$\begin{aligned}&\frac{\partial^2 U_2}{\partial T_0^2} - c^2\frac{\partial^2 U_2}{\partial X_0^2} + \alpha^2\frac{\partial^4 U_2}{\partial X_0^4} \\ &\quad = -2\frac{\partial^2 U_0}{\partial T_0 \partial T_2} - \frac{\partial^2 U_0}{\partial T_1^2} - 2\frac{\partial^2 U_1}{\partial T_0 \partial T_1} \\ &\qquad + c^2\frac{\partial^2 U_0}{\partial X_1^2} - 6\alpha^2\frac{\partial^4 U_0}{\partial X_0^2 \partial X_1^2} + 2c^2\frac{\partial^2 U_1}{\partial X_0 \partial X_1} - 4\alpha^2\frac{\partial^4 U_1}{\partial X_0^3 \partial X_1} \\ &\qquad - \sigma\frac{\partial U_0}{\partial X_1}\frac{\partial^2 U_0}{\partial X_0^2} - 2\sigma\frac{\partial U_0}{\partial X_0}\frac{\partial^2 U_0}{\partial X_0 \partial X_1} - \sigma\frac{\partial U_1}{\partial X_0}\frac{\partial^2 U_0}{\partial X_0^2} - \sigma\frac{\partial U_0}{\partial X_0}\frac{\partial^2 U_1}{\partial X_0^2} \\ &\quad = B_3 e^{3i\theta} + B_2 e^{2i\theta} + B_1 e^{i\theta} + \circledast + B_0\,,\end{aligned} \tag{10.26}$$

where

$$B_3 := -\frac{\sigma^2 k^2}{2\alpha^2} A^3 \,, \tag{10.27}$$

$$B_2 := -\frac{\sigma}{3\alpha^2}\left[\frac{\omega}{k} A \frac{\partial A}{\partial T_1} + \left(2c^2 + 7k^2\alpha^2\right) A \frac{\partial A}{\partial X_1}\right] , \tag{10.28}$$

$$B_1 := 2i\omega \frac{\partial A}{\partial T_2} - \frac{\partial^2 A}{\partial T_1^2} + \left[c^2 + 6\alpha^2 k^2\right] \frac{\partial^2 A}{\partial X_1^2} + \frac{\sigma^2 k^2}{6\alpha^2} |A|^2 A \,, \tag{10.29}$$

and

$$B_0 := -\sigma k^2 \frac{\partial}{\partial X_1} |A|^2 \,. \tag{10.30}$$

The fundamental harmonics $e^{\pm i\theta}$ are always resonant, and the corresponding secular terms may be avoided by taking the coefficients of the fundamental harmonics on the right-hand side of (10.26) to be zero, which apparently imposes certain slow dynamics on the amplitude $A$; one thus deduces a cubic nonlinear Schrödinger equation evidently satisfied by $A$. The second and third harmonics $e^{\pm 2i\theta}$ and $e^{\pm 3i\theta}$ are not resonant, as can easily be checked. However, the mean term $B_0$ is resonant because the dispersion relation (10.22) admits $(k, \omega) = (0, 0)$. Avoiding the secular terms in $U_2$ originating from the resonant mean term is problematic, since this appears to require taking $|A|^2$ to be independent of $X_1$. Doing so is not consistent, since the evolution of $A$ is already governed by a cubic nonlinear Schrödinger equation, whose solutions having $|A|^2$ independent of $X_1$ correspond to very special initial conditions.

The rescaled equation (10.20) of course has nontrivial solutions when $\epsilon$ is small, and we should view the failure of our approach as an indication that an assumption leading to the problematic conclusion is incorrect. To see what needs to be corrected, note that if $|A|^2$ is not constant in $X_1$, the solution of the equation (10.26) contains secular terms that grow linearly in $T_0$ but that are constant in $X_0$. In other words, $U_2$ considered as a function of $X_0$ appears to develop a nonzero mean value rapidly. This is a clue that the leading-order solution $U_0$ should contain a component of the mean.

Based on this reasoning, we may try, in place of (10.21), a leading-order solution of the form

$$U_0 = Ae^{i\theta} + \circledast + M \,,$$

where $M$ is a real function of the slow variables $X_1$, $T_1$, and $T_2$. Then, because derivatives of $U_0$ with respect to $X_0$ and $T_0$ are unchanged by adding the mean $M$, (10.23) remains the same, and we again insist that the solvability condition (10.24) holds and then take $U_1$ to be given by (10.25). The equation for $U_2$ is still (10.26), and $B_3$ and $B_2$ are still given by (10.27) and

(10.28), respectively, but in place of (10.29) we have

$$B_1 := 2i\omega\frac{\partial A}{\partial T_2} - \frac{\partial^2 A}{\partial T_1^2} + \left[c^2 + 6\alpha^2 k^2\right]\frac{\partial^2 A}{\partial X_1^2} + \frac{\sigma^2 k^2}{6\alpha^2}|A|^2 A + \sigma k^2 A\frac{\partial M}{\partial X_1}\,, \quad (10.31)$$

and in place of (10.30) we have

$$B_0 := -\frac{\partial^2 M}{\partial T_1^2} + c^2\frac{\partial^2 M}{\partial X_1^2} - \sigma k^2\frac{\partial}{\partial X_1}|A|^2\,. \quad (10.32)$$

The solvability conditions are therefore $B_0 = 0$ and $B_1 = 0$, which can now both be satisfied because the dependence of the mean $M$ on the slow variables is a new unknown. The solvability conditions should be viewed as a coupled system of equations governing the dependence of the complex amplitude $A$ and the real mean $M$ on the slow variables.

In this case, an equation for $A$ alone can be derived by eliminating $\partial M/\partial X_1$ as follows. First, note that by differentiating the solvability condition $B_0 = 0$ with respect to $X_1$, we get

$$\frac{\partial^2}{\partial T_1^2}\frac{\partial M}{\partial X_1} - c^2\frac{\partial^2}{\partial X_1^2}\frac{\partial M}{\partial X_1} + \sigma k^2\frac{\partial^2}{\partial X_1^2}|A|^2 = 0\,. \quad (10.33)$$

Now, multiplication of (10.24) by $A^*$ and adding the resulting equation to its complex conjugate implies that

$$\frac{\partial}{\partial T_1}|A|^2 + \omega'(k)\frac{\partial}{\partial X_1}|A|^2 = 0\,,$$

from which it follows that $|A|^2$ also satisfies

$$\frac{\partial^2}{\partial T_1^2}|A|^2 - \omega'(k)^2\frac{\partial^2}{\partial X_1^2}|A|^2 = 0\,.$$

If $\omega'(k)^2 \neq c^2$, the latter can be rearranged to read

$$\frac{\partial^2}{\partial X_1^2}|A|^2 = \frac{1}{\omega'(k)^2 - c^2}\cdot\left[\frac{\partial^2}{\partial T_1^2}|A|^2 - c^2\frac{\partial^2}{\partial X_1^2}|A|^2\right]\,,$$

so that (10.33) can be written as

$$\frac{\partial^2}{\partial T_1^2}\frac{\partial M}{\partial X_1} - c^2\frac{\partial^2}{\partial X_1^2}\frac{\partial M}{\partial X_1} = -\frac{\sigma k^2}{\omega'(k)^2 - c^2}\left[\frac{\partial^2}{\partial T_1^2}|A|^2 - c^2\frac{\partial^2}{\partial X_1^2}|A|^2\right]\,.$$

A particular solution for $\partial M/\partial X_1$ is therefore proportional to $|A|^2$, and the homogeneous solutions are of the form $f(X_1 - cT_1)$ and $g(X_1 + cT_1)$ for general functions $f$ and $g$. Therefore, the general solution of (10.33) is

$$\frac{\partial M}{\partial X_1} = -\frac{\sigma k^2|A|^2}{\omega'(k)^2 - c^2} + f(X_1 - cT_1; T_2) + g(X_1 + cT_1; T_2)\,. \quad (10.34)$$

Inserting this relation into the solvability condition $B_1 = 0$ with $B_1$ given by (10.31) gives a closed equation for the complex amplitude $A$:

$$i\frac{\partial A}{\partial T_2} - \frac{1}{2\omega(k)}\frac{\partial^2 A}{\partial T_1^2} + \frac{c^2 + 6\alpha^2 k^2}{2\omega(k)}\frac{\partial^2 A}{\partial X_1^2} + \beta|A|^2 A$$
$$= -\frac{\sigma k^2}{2\omega(k)}\left[f(X_1 - cT_1; T_2) + g(X_1 + cT_1; T_2)\right] A\,,$$

where

$$\beta := \frac{1}{2\omega(k)}\left[\frac{\sigma^2 k^2}{6\alpha^2} - \frac{\sigma^2 k^4}{\omega'(k)^2 - c^2}\right]. \tag{10.35}$$

With the change of variables $(X_1, T_1) \to (\xi, \tau)$ given by $\xi = X_1 - \omega'(k)T_1$ and $\tau = T_1$, this becomes

$$i\frac{\partial A}{\partial T_2} + \frac{\omega''(k)}{2}\frac{\partial^2 A}{\partial \xi^2} + \beta|A|^2 A$$
$$= -\frac{\sigma k^2}{2\omega(k)}\left[f(\xi + (\omega'(k) - c)\tau; T_2) + g(\xi + (\omega'(k) + c)\tau; T_2)\right] A\,. \tag{10.36}$$

Unless $f$ and $g$ depend on $T_2$ only, this equation appears to be inconsistent with the solvability condition (10.24) which reads $\partial A/\partial\tau = 0$ in these variables. However, if one considers the initial-value problem where $A$ is given at $t = 0$ as a function of $\xi$ that decays exponentially to zero as $|\xi| \to \infty$ (for example, $A$ proportional to $\operatorname{sech}(\xi)$) and if $M$ and $\partial M/\partial\tau$ vanish identically at $t = 0$, then according to (10.34), $f$ and $g$ will also be exponentially decaying functions. The condition $\omega'(k)^2 \neq c^2$ then implies that the product on the right-hand side of (10.36) will quickly tend to zero because on the time scale $\tau$, which is fast on the time scale $T_2$, the complex amplitude $A$ remains stationary while the profiles of $f$ and $g$ propagate with nonzero velocities. Indeed, when $T_2$ becomes large compared to $\epsilon$ the profiles of $f$ and $g$ will move by a distance $\xi$ that is large compared with 1 while the profile of $A$ remains nearly fixed; in this way the product on the right-hand side of (10.36) becomes uniformly exponentially small as $\epsilon \to 0$ for each fixed $T_2 > 0$. This argument justifies the description of the dynamics of the complex amplitude $A$ by a cubic nonlinear Schrödinger equation:

$$i\frac{\partial A}{\partial T_2} + \frac{\omega''(k)}{2}\frac{\partial^2 A}{\partial \xi^2} + \beta|A|^2 A = 0\,. \tag{10.37}$$

Note that taking account of the mean flow is not just a technical matter in order to ensure that $U_2$ contains no secular terms; without taking account of the mean flow, one arrives at the incorrect value of $\beta$ by missing the second term in (10.35). Also, (10.34) shows that even if the mean $M$ vanishes initially, it has nontrivial dynamics consisting of components propagating with three distinct velocities: the two hyperbolic velocities $\pm c$ and the group velocity $\omega'(k)$ of the complex amplitude $A$. Therefore even if there is no mean flow present initially, the mean will evolve away from the quiescent state if $A$ is nonzero.

◀ *Exercise 10.5.* Consider the weakly nonlinear Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + \epsilon u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

and the weakly nonlinear *modified* Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + \epsilon^2 u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0\,.$$

(These arise from the Korteweg-de Vries equation $\varphi_t + \varphi\varphi_x + \varphi_{xxx} = 0$ and the modified Korteweg-de Vries equation $\varphi_t + \varphi^2\varphi_x + \varphi_{xxx} = 0$, respectively, by the substitution $\varphi = \epsilon u$.) In the linear limit ($\epsilon = 0$) both of these equations degenerate to the same equation having the dispersion relation $\omega + k^3 = 0$. This dispersion relation admits the pair $(k, \omega) = (0, 0)$ and therefore there is the possibility that a derivation of an equation governing the complex amplitude $A$ of a wavepacket with a wavenumber $k \neq 0$ will require the introduction of a slowly-varying mean flow $M$.

Part (a). By seeking a solution of the weakly nonlinear Korteweg-de Vries equation having the form

$$u = A(X_1, T_1, T_2)e^{i(kX_0 + k^3 T_0)} + \circledast$$
$$+ \epsilon U_1(X_0, X_1, T_0, T_1, T_2) + \epsilon^2 U_2(X_0, X_1, T_0, T_1, T_2) + O(\epsilon^3)\,,$$

show that to avoid secular terms in $U_1$, one must take

$$\frac{\partial A}{\partial T_1} - 3k^2 \frac{\partial A}{\partial X_1} = 0$$

and that then

$$U_1 = \frac{A^2}{6k^2} e^{i(2kX_0 + 2k^3 T_0)} + \circledast + \text{arbitrary homogeneous solution.}$$

Then show that to avoid secular terms in $U_2$, one may take a slowly-varying mean flow $M(X_1, T_1, T_2)$ as the arbitrary homogeneous solution in $U_1$. Thus derive the equations

$$\frac{\partial M}{\partial T_1} + \frac{\partial}{\partial X_1}|A|^2 = 0$$

and

$$i\frac{\partial A}{\partial T_2} - 3k\frac{\partial^2 A}{\partial X_1^2} - \frac{1}{6k}|A|^2 A - ikMA = 0\,.$$

Finally, obtain the nonlinear Schrödinger equation

$$i\frac{\partial A}{\partial T_2} - 3k\frac{\partial^2 A}{\partial \xi^2} + \frac{1}{6k}|A|^2 A = 0$$

by going into the group velocity frame with $\xi = X_1 + 3k^2T_1$ and $\tau = T_1$. Verify that this is a nonlinear Schrödinger equation of *defocusing* type (see Exercise 10.4).

Part (b). Carry out a similar perturbative procedure as in part (a) for the weakly nonlinear modified Korteweg-de Vries equation. Show that to obtain the correct nonlinear Schrödinger equation governing $A$:

$$i\frac{\partial A}{\partial T_2} - 3k\frac{\partial^2 A}{\partial \xi^2} - k|A|^2 A = 0\,,$$

where again $\xi = X_1 + 3k^2T_1$ and $\tau = T_1$, it is not necessary to introduce a slowly-varying mean flow at all even though it is admitted by the linear dispersion relation. Check that this nonlinear Schrödinger equation is of *focusing* type (see Exercise 10.4). ▶

**10.1.3. Multiple wave resonances.** A frequent characterization of nonlinear dynamics in wave mechanics is that the function of the nonlinear terms in a wave equation is to transport energy from one wave number to another in the Fourier spectrum. We have already encountered this concept in our multiple-scale analysis of weakly nonlinear waves, where we have seen that the effect of the nonlinear terms is to generate higher harmonics of a given fundamental harmonic. If nonresonant, these higher harmonics appear in the first nonzero correction to the leading term $U_0$. At the next order the nonlinear terms will combine the fundamental harmonics present in the leading term with those present in the first correction, producing yet higher harmonics in the higher-order corrections.

Nonlinear terms can do more than produce multiples of a given harmonic phase $\theta$. They can also mix energy among several different harmonic phases. For example, a quadratic term can combine two different phases $\theta_1$ and $\theta_2$ to produce the combinations $\theta_1 \pm \theta_2$.

Of course the harmonics produced by nonlinear terms have little consequence in perturbation theory unless they are resonant. While we have seen that multiples of a given fundamental harmonic $\theta$ are only unusually resonant (this of course depends on the fundamental wavenumber $k$ and the dispersion relation $\omega = \omega(k)$ for the problem at hand, and more generally one must be aware of the excitation of a mean flow), the situation is more favorable for the production of resonant harmonics if more waves are present in the leading term of the asymptotic expansion. Indeed, if we suppose that

$$U_0 = \sum_{n=1}^{N} \left[ A_n e^{i\theta_n} + \circledast \right] , \tag{10.38}$$

where $\theta_n = k_n X_0 - \omega_n T_0$ are phases such that the pairs $(k_n, \omega_n)$ all satisfy the dispersion relation for the problem, then the nonlinear terms will produce cross terms involving linear combinations of the fundamental harmonics $\theta_n$ as well as simple multiples. With more new harmonics generated from nonlinear terms acting on $U_0$, there is a greater chance that some of these will coincide with the fundamental harmonics $\theta_n$ themselves and therefore will be resonant. A situation in which $U_0$ contains several different waves that combine under the nonlinear terms in the problem to produce terms proportional to these same waves is called a *multiple wave resonance*. The solvability conditions that are required to avoid secular terms from appearing in the asymptotic expansion generally impose coupled dynamics on the complex amplitudes $A_n$. This dynamical process of slow exchange of energy among waves making up a multiple wave resonance is called *wave mixing*.

*Quadratic nonlinearities. Resonant triads.* Quadratic nonlinearities can mix energy among three different fundamental harmonics. In this case, $N = 3$ in (10.38), and the three waves making up $U_0$ are said to make up a *resonant triad*. The condition for triad resonance can be written in the form

$$k_1 + k_2 + k_3 = 0 \quad \text{and} \quad \omega_1 + \omega_2 + \omega_3 = 0\,.$$

As an example of a dispersion relation supporting triad resonances, consider

$$\omega^2 = 1 + 2k^2 + k^4 = (1 + k^2)^2\,. \qquad (10.39)$$

Suppose that $(k_1, \omega_1 := 1 + k_1^2)$ and $(k_2, \omega_2 := 1 + k_2^2)$ are given pairs satisfying the dispersion relation (10.39). The third wave in the resonant triad should correspond to $k_3 = -(k_1 + k_2)$ and $\omega_3 = -(\omega_1 + \omega_2)$. The question is whether the pair $(k_3, \omega_3)$ also satisfies the same dispersion relation.

◀ *Exercise 10.6.* Show that the pair $(k_3, \omega_3)$ completing the resonant triad satisfies the dispersion relation (10.39) if and only if $2k_1k_2 = 1$. ▶

This exercise shows that there is a one-parameter family of resonant triads for the dispersion relation (10.39) since the wavenumber $k_1$ can be chosen arbitrarily. An equation with the dispersion relation (10.39) and weak quadratic nonlinearity is

$$\frac{\partial^2 u}{\partial t^2} - 2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} + u + \epsilon u^2 = 0\,. \qquad (10.40)$$

Let us introduce the time scales $T_0 = t$ and $T_1 = \epsilon t$ and the spatial scales $X_0 = x$ and $X_1 = \epsilon x$. Supposing that $u(x, t; \epsilon) = U(X_0, X_1, T_0, T_1; \epsilon)$ and that $U$ has an asymptotic expansion in integer powers of $\epsilon$ with coefficients

$U_0$, $U_1$, and so on, one finds for the leading term the reduced equation

$$\frac{\partial^2 U_0}{\partial T_0^2} - 2\frac{\partial^2 U_0}{\partial X_0^2} + \frac{\partial^4 U_0}{\partial X_0^4} + U_0 = 0\,. \tag{10.41}$$

Let us take the solution of this equation in the form of a resonant triad:

$$U_0 = A_1 e^{i\theta_1} + A_2 e^{i\theta_2} + A_3 e^{i\theta_3} + \circledast\,,$$

where the complex amplitudes $A_n$ are functions of the slow variables $X_1$ and $T_1$. Note that because $\theta_1 + \theta_2 + \theta_3 = 0$,

$$U_0^2 = 2A_2^* A_3^* e^{i\theta_1} + 2A_1^* A_3^* e^{i\theta_2} + 2A_1^* A_2^* e^{i\theta_3} + \circledast + \text{other harmonics}\,,$$

where the other harmonics are generally nonresonant. Therefore, the equation for $U_1$ is

$$\begin{aligned}
\frac{\partial^2 U_1}{\partial T_0^2} &- 2\frac{\partial^2 U_1}{\partial X_0^2} + \frac{\partial^4 U_1}{\partial X_0^4} + U_1 \\
&= -2\frac{\partial^2 U_0}{\partial T_0 \partial T_1} + 4\frac{\partial^2 U_0}{\partial X_0 \partial X_1} - 4\frac{\partial^4 U_0}{\partial X_0^3 \partial X_1} + U_0^2 \\
&= \left[2i\omega_1 \frac{\partial A_1}{\partial T_1} + \left(4ik_1 + 4ik_1^3\right)\frac{\partial A_1}{\partial X_1} + 2A_2^* A_3^*\right] e^{i\theta_1} \\
&\quad + \left[2i\omega_2 \frac{\partial A_2}{\partial T_1} + \left(4ik_2 + 4ik_2^3\right)\frac{\partial A_2}{\partial X_1} + 2A_1^* A_3^*\right] e^{i\theta_2} \\
&\quad + \left[2i\omega_3 \frac{\partial A_3}{\partial T_1} + \left(4ik_3 + 4ik_3^3\right)\frac{\partial A_3}{\partial X_1} + 2A_1^* A_2^*\right] e^{i\theta_3} \\
&\quad + \circledast + \text{nonresonant harmonics}\,.
\end{aligned}$$

There are evidently three solvability conditions for avoiding secular terms in $U_1$. Noting that by differentiation of (10.39), $(2k + 2k^3)/\omega$ is the group velocity $v_g$ associated with the wave having wavenumber $k$ and frequency $\omega$, these may be written in the form

$$\begin{aligned}
\frac{\partial A_1}{\partial T_1} + v_{g,1}\frac{\partial A_1}{\partial X_1} + \frac{1}{i\omega_1}A_2^* A_3^* &= 0\,, \\
\frac{\partial A_2}{\partial T_1} + v_{g,2}\frac{\partial A_2}{\partial X_1} + \frac{1}{i\omega_2}A_1^* A_3^* &= 0\,, \\
\frac{\partial A_3}{\partial T_1} + v_{g,3}\frac{\partial A_3}{\partial X_1} + \frac{1}{i\omega_3}A_1^* A_2^* &= 0\,.
\end{aligned} \tag{10.42}$$

Here the group velocity is $v_{g,n} = (2k_n + 2k_n^3)/\omega_n$ for $n = 1, 2, 3$. This system of equations is called the *3-wave interaction equations.*

*Cubic nonlinearities. Resonant quartets.* Frequently quadratic nonlinearities are absent due to basic symmetries of the system under consideration. For example, if the equation (10.40) should include nonlinear terms but should be invariant under the transformation taking $u$ to $-u$, then in place of $\epsilon u^2$ the simplest admissable term is a cubic term $\epsilon u^3$. Whereas quadratic nonlinearities can serve to mix three fundamental waves making up a resonant triad, cubic nonlinearities can mix four fundamental waves making up a *resonant quartet.* The resonance condition for quartets is

$$k_1 + k_2 + k_3 + k_4 = 0 \quad \text{and} \quad \omega_1 + \omega_2 + \omega_3 + \omega_4 = 0\,. \tag{10.43}$$

Of course the corresponding pairs $(k_n, \omega_n)$ all have to satisfy the dispersion relation for the problem at hand.

◀ *Exercise 10.7.* Show that for the dispersion relation (10.39) there is a two-parameter family of resonant quartets, parametrized by the independent wavenumbers $k_1$ and $k_2$. ▶

This exercise shows that for the dispersion relation (10.39), resonant quartets are more common than resonant triads. This situation is typical, since in each case the resonance condition places one additional condition on a collection of wavenumbers that otherwise are only constrained in that their sum is zero. The more fundamental wavenumbers one admits, the more resonances are possible.

To see the effect of cubic nonlinearities on resonant quartets, consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - 2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} + u + \epsilon u^3 = 0\,,$$

and suppose that $u(x, t; \epsilon) = U(X_0, X_1, T_0, T_1; \epsilon)$ and that $U$ has an asymptotic expansion in integer powers of $\epsilon$ with coefficients $U_0$, $U_1$, and so on. Then, the leading term $U_0$ satisfies the partial differential equation (10.41), and to study the resonant quartet, we suppose a solution of the form

$$U_0 = A_1 e^{i\theta_1} + A_2 e^{i\theta_2} + A_3 e^{i\theta_3} + A_4 e^{i\theta_4} + \circledast\,,$$

where $\theta_n = k_n X_0 - \omega_n T_0$ and $(k_n, \omega_n)$ are pairs satisfying the dispersion relation (10.39) and the resonance condition (10.43). Therefore $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 0$, and if we multiply out the cubic nonlinear term acting on $U_0$, we will find

$$
\begin{aligned}
U_0^3 = {} & \Big(6A_2^*A_3^*A_4^* + \Big[3|A_1|^2 + 6|A_2|^2 + 6|A_3|^2 + 6|A_4|^2\Big] A_1\Big) e^{i\theta_1} \\
& + \Big(6A_1^*A_3^*A_4^* + \Big[6|A_1|^2 + 3|A_2|^2 + 6|A_3|^2 + 6|A_4|^2\Big] A_2\Big) e^{i\theta_2} \\
& + \Big(6A_1^*A_2^*A_4^* + \Big[6|A_1|^2 + 6|A_2|^2 + 3|A_3|^2 + 6|A_4|^2\Big] A_3\Big) e^{i\theta_3} \\
& + \Big(6A_1^*A_2^*A_3^* + \Big[6|A_1|^2 + 6|A_2|^2 + 6|A_3|^2 + 3|A_4|^2\Big] A_4\Big) e^{i\theta_4} \\
& \qquad + \circledast + \text{other harmonics}\,,
\end{aligned}
$$

where the other harmonics are not generally resonant.

◀ *Exercise 10.8.* Verify the above, including the nonresonance of the omitted harmonics for general choices of $k_1$ and $k_2$. ▶

Since the equation governing $U_1$ is

$$
\begin{aligned}
& \frac{\partial^2 U_1}{\partial T_0^2} - 2\frac{\partial^2 U_1}{\partial X_0^2} + \frac{\partial^4 U_1}{\partial X_0^4} + U_1 \\
& \qquad = -2\frac{\partial^2 U_0}{\partial T_0 \partial T_1} + 4\frac{\partial^2 U_0}{\partial X_0 \partial X_1} - 4\frac{\partial^4 U_0}{\partial X_0^3 \partial X_1} + U_0^3\,,
\end{aligned}
$$

it is easy to see that avoiding secular terms in $U_1$ requires imposing four solvability conditions that can be written as a coupled system governing the slow evolution of the four complex amplitudes $A_1$, $A_2$, $A_3$, and $A_4$:

$$
\begin{aligned}
& \frac{\partial A_1}{\partial T_1} + v_{g,1}\frac{\partial A_1}{\partial X_1} \\
& \qquad + \frac{3}{i\omega_1}A_2^*A_3^*A_4^* + \frac{3}{i\omega_1}\left[\frac{1}{2}|A_1|^2 + |A_2|^2 + |A_3|^2 + |A_4|^2\right] A_1 = 0\,, \\
& \frac{\partial A_2}{\partial T_1} + v_{g,2}\frac{\partial A_2}{\partial X_1} \\
& \qquad + \frac{3}{i\omega_2}A_1^*A_3^*A_4^* + \frac{3}{i\omega_2}\left[|A_1|^2 + \frac{1}{2}|A_2|^2 + |A_3|^2 + |A_4|^2\right] A_2 = 0\,, \\
& \frac{\partial A_3}{\partial T_1} + v_{g,3}\frac{\partial A_3}{\partial X_1} \\
& \qquad + \frac{3}{i\omega_3}A_1^*A_2^*A_4^* + \frac{3}{i\omega_3}\left[|A_1|^2 + |A_2|^2 + \frac{1}{2}|A_3|^2 + |A_4|^2\right] A_3 = 0\,, \\
& \frac{\partial A_4}{\partial T_1} + v_{g,4}\frac{\partial A_4}{\partial X_1} \\
& \qquad + \frac{3}{i\omega_4}A_1^*A_2^*A_3^* + \frac{3}{i\omega_4}\left[|A_1|^2 + |A_2|^2 + |A_3|^2 + \frac{1}{2}|A_4|^2\right] A_4 = 0\,.
\end{aligned}
\tag{10.44}
$$

Here the quantities $v_{g,n}$ are the group velocities associated with the waves of respective wavenumbers $k_n$ and frequencies $\omega_n$.

The system of equations (10.44) describes the way that a weak cubic nonlinearity couples together the dynamics of four fundamental waves making up a resonant quartet. Generally, this process is called *four-wave mixing.*

*Manley-Rowe relations and modal interactions.* If we begin with either (10.42) or (10.44) and multiply the equation governing $A_n$ by $A_n^*$ and then add the complex conjugate, we obtain a system of equations for the square moduli $|A_n|^2$ that has a common form:

$$\frac{\partial}{\partial T_1}|A_n|^2 + v_{g,n}\frac{\partial}{\partial X_1}|A_n|^2 = \frac{K}{\omega_n}\Im\left[\prod_{j=1}^{N} A_j\right], \quad \text{for } n = 1, \ldots, N, \tag{10.45}$$

where for (10.42) we have $N = 3$ and $K = 2$ while for (10.44) we have $N = 4$ and $K = 6$.

The main observation is that the right-hand side of (10.45) is, up to a constant factor, independent of $n$. Therefore, we may form various simpler relations by taking linear combinations of the $N$ equations making up the system (10.45). Such relations constrain the flow of energy among the fundamental waves and are called *Manley-Rowe relations.* The simplest Manley-Rowe relation comes from multiplying each equation in (10.45) through by $\omega_n^2$ and summing the equations. Setting

$$I(X_1, T_1) := \sum_{n=1}^{N} \omega_n^2 |A_n|^2,$$

this Manley-Rowe relation reads

$$\frac{\partial I}{\partial T_1} + \frac{\partial}{\partial X_1}\sum_{n=1}^{N} v_{g,n}\omega_n^2|A_n|^2 = 0. \tag{10.46}$$

The terms on the right-hand side of (10.45) do not appear here because $\omega_1 + \cdots + \omega_N = 0$ due to the resonance condition. The relation (10.46) is a conservation law in local form. Integrating with respect to $X_1$ over $\mathbb{R}$, assuming vanishing boundary conditions for the amplitudes $A_n$, we find

$$\frac{d}{dT_1}\int_{-\infty}^{\infty} I(X_1, T_1)\, dX_1 = 0,$$

so the integral of $I$ remains constant. This Manley-Rowe relation has the interpretation of conservation of the total energy in the system. While this is the simplest Manley-Rowe relation, there are $N - 1$ linearly independent ones, since this is the number of independent linear combinations of the equations in (10.45) for which the right-hand side vanishes.

◀ *Exercise 10.9.* For $N = 3$ and $N = 4$ find a complete basis for the Manley-Rowe relations implied by (10.45). Hint: Consider linear combinations of the products $\omega_n|A_n|^2$. ▶

The Manley-Rowe relations are constraints on the way that energy may flow among the fundamental waves. In the case of the four-wave mixing system (10.44), the "in phase" nonlinear terms proportional to $A_n$ in the differential equation for $A_n$ evidently drop out altogether in the calculation leading to (10.45). Such terms cannot appear in the Manley-Rowe relations, and therefore they cannot influence the flow of energy among the fundamental waves. These terms serve only to modulate the phase of $A_n$ but do not mix any energy directly. These terms characterize what is called a *modal interaction.*

**10.1.4. Long wave asymptotics. The Boussinesq equation and the Korteweg-de Vries equation.** In systems whose governing equations admit constant solutions, so that in particular the dispersion relation is satisfied by the pair $(k, \omega) = (0, 0)$, a useful asymptotic limit to consider is one in which solutions are slowly varying in space and time. When combined with a small-amplitude limit making the system weakly nonlinear, such a *long-wave limit* can also lead to universal equations that govern the asymptotic dynamics for a broad class of models.

*Systems with left/right symmetry. The Boussinesq equation.* Many physical systems of wave motion can be written in the form

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial x} F[u] = 0\,, \tag{10.47}$$

where $F[u]$ denotes an expression in $u$ and its derivatives with respect to $x$. Physical symmetries further constrain the possible form of $F$. If the system is symmetric with respect to reflections about the origin in $x$, the equation should be invariant under the change $x \to -x$, and if $u$ has the interpretation of a velocity, then the change $u \to -u$ should go hand-in-hand. In such a situation, we should restrict our attention to those $F[u]$ that are invariant under the change $(x, u) \to (-x, -u)$, which if $p$ is odd leaves $\partial^p u/\partial x^p$ invariant, but if $p$ is even, it changes the sign of $\partial^p u/\partial x^p$. If we are interpreting $u$ as a velocity, then Galilean invariance implies that the wave equation should be invariant under the substitution $u \to u + c$ where $c$ is an arbitrary constant. This symmetry further requires that $F[u]$ depend on $u$ only through derivatives with respect to $x$.

Such functions $F[u]$ are necessarily of the form

$$F[u] = -c^2 \frac{\partial u}{\partial x} + a \frac{\partial^3 u}{\partial x^3} + b \left(\frac{\partial u}{\partial x}\right)^2 + \text{other terms}\,, \tag{10.48}$$

where the other terms are of total degree[2] six or more. Setting $T = \epsilon t$ and $X = \epsilon x$ and setting $U = \epsilon u$, the wave equation (10.47) subject to (10.48) becomes

$$\epsilon^3 \left[ \frac{\partial^2 U}{\partial T^2} - c^2 \frac{\partial^2 U}{\partial X^2} \right] + \epsilon^5 \left[ a \frac{\partial^4 U}{\partial X^4} + 2b \frac{\partial U}{\partial X} \frac{\partial^2 U}{\partial X^2} \right] + O(\epsilon^7) = 0 .$$

Dividing by $\epsilon^3$ and neglecting terms of size $O(\epsilon^4)$, we arrive at the *Boussinesq equation*[3]

$$\frac{\partial^2 U}{\partial T^2} - c^2 \frac{\partial^2 U}{\partial X^2} + \epsilon^2 \left[ a \frac{\partial^4 U}{\partial X^4} + 2b \frac{\partial U}{\partial X} \frac{\partial^2 U}{\partial X^2} \right] = 0 . \tag{10.49}$$

This equation represents a perturbation of the hyperbolic linear wave equation. The perturbation includes the most important dispersive and nonlinear terms admitted by the symmetries of the system under a weakly nonlinear long-wave scaling.

*Right-going waves. The Korteweg-de Vries equation.* The Boussinesq equation allows for waves propagating in two opposite directions according to the two characteristic velocities $\pm c$ of the unperturbed problem ($\epsilon = 0$). As is well known, the general solution of the unperturbed problem resolves, according to the d'Alembert formula, into a sum of two disturbances propagating with speeds $\pm c$. After some time, these two disturbances will become separated from each other, and the perturbative influence that one may have on the other due to the correction terms in (10.49) will be much smaller than the effect that each disturbance has on itself due to these terms. It therefore seems possible that a simpler model could be obtained if we consider primarily the effect of disturbances propagating in one direction.

Without loss of generality, let us consider waves propagating to the right with speed $c > 0$. To go into the frame of reference moving with this velocity, consider the change of independent variables $(X, T) \to (\xi, \tau)$, where

$$\xi = X - cT \qquad \text{and} \qquad \tau = \epsilon^2 T .$$

Then, if $V(\xi, \tau) = U(X, T)$, the Boussinesq equation (10.49) becomes (after dividing through by $\epsilon^2$):

$$\epsilon^2 \frac{\partial^2 V}{\partial \tau^2} - 2c \frac{\partial^2 V}{\partial \xi \partial \tau} + a \frac{\partial^4 V}{\partial \xi^4} + 2b \frac{\partial V}{\partial \xi} \frac{\partial^2 V}{\partial \xi^2} = 0 .$$

[2] By total degree, we mean the total number of $x$-derivatives plus the number of factors, that is, the exponent of $\epsilon$ upon replacing $u$ by $\epsilon u$ and $\partial/\partial x$ by $\epsilon \partial/\partial x$. Thus $\partial^3 u/\partial x^3$ has total degree four, as does $(\partial u/\partial x)^2$.

[3] Valentin Joseph Boussinesq, 1842–1929.

Neglecting the formally small term proportional to $\epsilon^2$, the resulting equation only involves $W := \partial V/\partial\xi$:

$$-2c\frac{\partial W}{\partial\tau} + a\frac{\partial^3 W}{\partial\xi^3} + 2bW\frac{\partial W}{\partial\xi} = 0\,. \tag{10.50}$$

By a simple rescaling of $W$ and $\tau$ by constant coefficients, this equation can be written in a canonical form:

$$\frac{\partial W}{\partial\tau} + W\frac{\partial W}{\partial\xi} + \frac{\partial^3 W}{\partial\xi^3} = 0\,. \tag{10.51}$$

◀ *Exercise 10.10.* Carry out explicitly the rescaling necessary to convert (10.50) into (10.51). ▶

The equation (10.51) is another universal nonlinear partial differential equation known as the *Korteweg-de Vries equation*[4]. The Korteweg-de Vries equation also has remarkable mathematical structure. The simplest aspect of this structure is illustrated in the following exercise.

◀ *Exercise 10.11.* Find relations between the constants $a$, $b$, and $c$ such that the expression $W(\xi,\tau) := a\,\mathrm{sech}^2(b(\xi - c\tau - \xi_0))$ is a solution of (10.51). These solutions are called *solitons* of the Korteweg-de Vries equation. ▶

The Korteweg-de Vries equation (10.51) arises frequently as a model when weak dispersion and weak nonlinearity are combined as a perturbative effect acting on waves propagating along a characteristic velocity of an underlying hyperbolic system (which may or may not have the symmetries of the Boussinesq equation; the equation (10.51) can be derived from many problems without using a Boussinesq equation as an intermediate step).

## 10.2. Waves in Molecular Chains

To illustrate the application of the method of multiple scales to the derivation of universal model equations, consider the problem of describing weakly nonlinear vibrations in a monatomic chain. The setting is the classical mechanics of a chain of particles of equal mass, say $m$, each interacting with its two nearest neighbors by an interatomic force law that one can visualize as a nonlinear spring. See Figure 10.2. This sort of model arises in several branches of applied mathematics. For example, the study of deformations of $\alpha$-helix protein molecules can be reduced (under certain conditions) to a problem of this type. In this case, the individual masses are the peptide groups in the chain, which are connected one to the next by hydrogen bonds. It is also a basic model in solid-state physics.

[4] Diederik Johannes Korteweg, 1848–1941, and Gustav de Vries, 1866–1934.

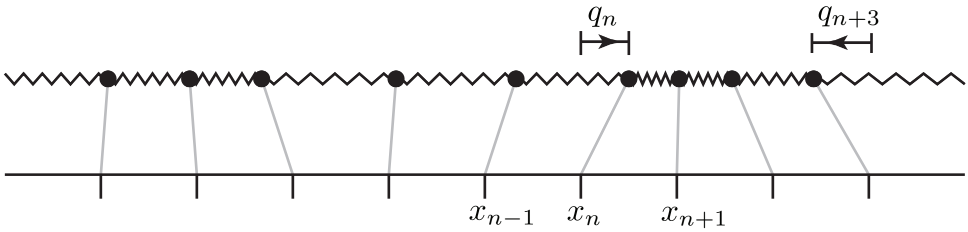


**Figure 10.2.** *The molecular chain showing displacements $q_n$ of the particles from equally spaced equilibrium positions $x_n$.*

**10.2.1. The Fermi-Pasta-Ulam model.** A very famous story concerns three physicists, E. Fermi[5], J. Pasta[6], and S. Ulam[7], who proposed a model of this type in an attempt to explain the process of *thermalization* in solid materials. Thermalization refers to the way that energy that is in the system at time $t = 0$ ultimately becomes equally distributed among all possible vibrational modes in accordance with the laws of statistical thermodynamics; it is the reason why hitting a piece of metal repeatedly for a long time with a hammer makes the metal start to feel hot to the touch. Fermi, Pasta, and Ulam knew that if the springs were all linear, then any energy put into a particular vibrational mode will stay there forever, and consequently they thought that thermalization must be the result of weakly nonlinear effects in the springs, which serve to mix the energy among the linear vibrational modes. In fact their (very primitive by today's standards) numerical simulations seemed to show otherwise, namely that the energy dispersed among the various vibrational modes for a short time but then spontaneously came back into just a few modes of vibration some time later! In other words, their simulations indicated that the energy does not thermalize at all in spite of the nonlinearities in the springs. This was sufficiently surprising that it warranted further investigation, and it seems that the nonlinear molecular chain, sometimes called the *Fermi-Pasta-Ulam model*, has not at this time been used to successfully explain thermalization. On the other hand, it has been a prototype for studying unexpectedly coherent phenomena ever since.

The total energy of this mechanical system is a sum of the total kinetic energy $T$ and the total potential energy $V$. The kinetic energy is related to the motion of the particles. If we assign to the particle whose displacement from equilibrium at time $t$ is $q_n(t)$ the momentum

$$p_n(t) := m\frac{dq_n}{dt}(t)\,,$$

[5]Enrico Fermi, 1901–1954.
[6]John R. Pasta, 1918–1981.
[7]Stanislaw Ulam, 1909–1984.

then the kinetic energy of this particle is $p_n^2/2m$. Consequently the total kinetic energy of the whole chain is a sum:

$$T = \sum_{n=-\infty}^{\infty} \frac{p_n^2}{2m}\,.$$

We assume that particles are really moving very much only in some finite part of the chain, so that the infinite sum converges. On the other hand, the potential energy is stored in the compressed or extended springs. Let $V_n$ be the potential energy of the spring connecting the particles with displacements $q_n$ and $q_{n-1}$. We normalize $V_n$ to be zero when $q_n - q_{n-1} = 0$, that is, when the spring is neither stretched nor compressed relative to its equilibrium configuration. Assuming the springs are all identical, then $V_n$ is just a function $V(s_n)$ of the stretching, $s_n := q_n - q_{n-1}$, that vanishes when $s_n = 0$. For simplicity, we will also assume that the function $V(\cdot)$ is analytic. The total potential energy of the system is then the sum

$$V = \sum_{n=-\infty}^{\infty} V(q_n - q_{n-1})\,.$$

Once again, we assume that $s_n \to 0$ for large $n$ fast enough given the function $V(\cdot)$ so that the sum converges. This means that far enough away from the center of all the action, the masses are not moving much and are approximately equally spaced with spacing $\Delta x$.

The equations of motion for the masses are Hamilton's equations corresponding to the total energy $H = T + V$. These equations are

$$\frac{dq_n}{dt} = \frac{\partial H}{\partial p_n} = \frac{p_n}{m}$$

and

$$\frac{dp_n}{dt} = -\frac{\partial H}{\partial q_n} = V'(q_{n+1} - q_n) - V'(q_n - q_{n-1})\,.$$

Eliminating the momenta $p_n$, we can combine these into a system of coupled second-order equations, Newton's equations, for the dynamical variables $q_n(t)$:

$$m\frac{d^2 q_n}{dt^2} = V'(q_{n+1} - q_n) - V'(q_n - q_{n-1})\,,$$

for $n \in \mathbb{Z}$. Note that if all the particles are equally spaced with spacing $\Delta x$, the force terms on the right-hand side become simply $V'(0) - V'(0) = 0$, so the system is indeed in equilibrium in such a configuration.

**10.2.2. Derivation of the cubic nonlinear Schrödinger equation.** For weakly nonlinear oscillations, we should suppose that the displacements from equilibrium are very small, so that we may take into account nonlinear effects perturbatively. Thus we set $q_n = \epsilon Q_n$ where $\epsilon$ is a small parameter.

Making this substitution and expanding $V$ in Taylor series about $s_n = 0$ for $\epsilon$ small, the equations of motion become

$$
\begin{aligned}
m\frac{d^2Q_n}{dt^2} &= V''(0)\left[Q_{n+1} - 2Q_n + Q_{n-1}\right] \\
&\quad + \epsilon\frac{V'''(0)}{2}\left[(Q_{n+1} - Q_n)^2 - (Q_n - Q_{n-1})^2\right] \\
&\quad + \epsilon^2\frac{V^{(IV)}(0)}{6}\left[(Q_{n+1} - Q_n)^3 - (Q_n - Q_{n-1})^3\right] + O(\epsilon^3)\,.
\end{aligned}
\tag{10.52}
$$

This is evidently of the form of a formally small perturbation of a system of linear ordinary differential equations for the rescaled displacements $Q_n(t)$.

We anticipate that as we try to compute higher-order terms in an asymptotic expansion in powers of $\epsilon$, we will encounter secular terms that will ruin the asymptotic nature of our subsequent approximations; for a direct exposition we will introduce the slow spatial and temporal scales from the very beginning. Let the time scales be $T_0 = t$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$. Introduce also the spatial scale $X_1 = \epsilon n$. Even though $n$ is a discrete variable, we will view $X_1$ as a continuous variable, as will be made clear below. Thus, we are interested in seeking a solution of the form

$$
\begin{aligned}
Q_n(t) &= Q_n^{(0)}(X_1, T_0, T_1, T_2) + \epsilon Q_n^{(1)}(X_1, T_0, T_1, T_2) \\
&\quad + \epsilon^2 Q_n^{(2)}(X_1, T_0, T_1, T_2) + O(\epsilon^3)\,.
\end{aligned}
$$

We are thinking of $n$ and $X_1$ as independent spatial variables and of $T_0$, $T_1$, and $T_2$ as independent time scales. We know how to expand the time derivatives using the chain rule. In order to rewrite the discrete differences with respect to $n$, we use Taylor expansion. More precisely, using the definition of $X_1$, we see that

$$
\begin{aligned}
Q_{n\pm1}(t) &= Q_{n\pm1}^{(0)}(X_1 \pm \epsilon, T_0, T_1, T_2) + \epsilon Q_{n\pm1}^{(1)}(X_1 \pm \epsilon, T_0, T_1, T_2) \\
&\quad + \epsilon^2 Q_{n\pm1}^{(2)}(X_1 \pm \epsilon, T_0, T_1, T_2) + O(\epsilon^3) \\
&= Q_{n\pm1}^{(0)}(X_1, T_0, T_1, T_2) \\
&\quad + \epsilon\left[Q_{n\pm1}^{(1)}(X_1, T_0, T_1, T_2) \pm \frac{\partial Q_{n\pm1}^{(0)}}{\partial X_1}(X_1, T_0, T_1, T_2)\right] \\
&\quad + \epsilon^2\left[Q_{n\pm1}^{(2)}(X_1, T_0, T_1, T_2) \pm \frac{\partial Q_{n\pm1}^{(1)}}{\partial X_1}(X_1, T_0, T_1, T_2)\right. \\
&\qquad \left. + \frac{1}{2}\frac{\partial^2 Q_{n\pm1}^{(0)}}{\partial X_1^2}(X_1, T_0, T_1, T_2)\right] + O(\epsilon^3)\,.
\end{aligned}
$$

We may then collect the coefficients of powers of $\epsilon$ order by order. At leading order, we find the reduced equation:

$$m\frac{\partial^2 Q_n^{(0)}}{\partial T_0^2} = V''(0)\left[Q_{n+1}^{(0)} - 2Q_n^{(0)} + Q_{n-1}^{(0)}\right] . \tag{10.53}$$

Elementary traveling wave solutions can be sought in the form

$$Q_n^{(0)} = e^{i(\delta n - \omega T_0)} .$$

It is immediate to check that these will solve (10.53) if $\delta$ and $\omega$ satisfy the dispersion relation:

$$D(\delta,\omega) := m\omega^2 + V''(0)\left[e^{i\delta} - 2 + e^{-i\delta}\right] = 0 \quad \text{or} \quad \omega^2 = \frac{2V''(0)}{m}\left[1 - \cos(\delta)\right] . \tag{10.54}$$

Evidently, for bounded motions, we need to have $V''(0) > 0$. This ensures that the equilibrium configuration is *stable*. The appearance of trigonometric functions in the dispersion relation is a new feature and is the hallmark of mechanical systems with discrete translational symmetry like crystals or other periodic molecules. The two branches of the dispersion relation are displayed in Figure 10.3. We look for a solution that is a modulated wave

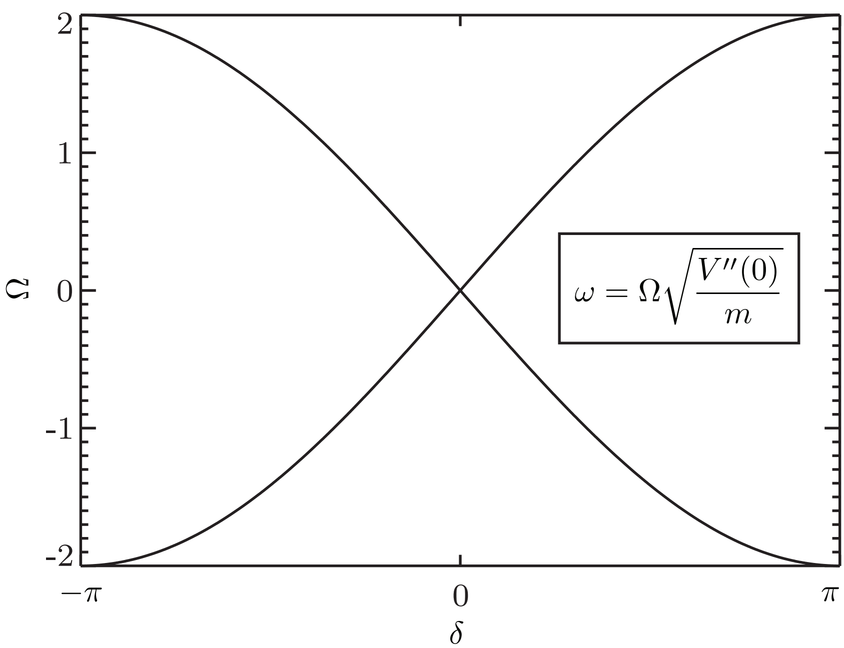


**Figure 10.3.** *The two branches of the discrete dispersion relation for linear motions of the lattice. The frequency is normalized to* $\sqrt{V''(0)/m}$.

with discrete wavenumber $\delta \in (-\pi, \pi)$, so with a corresponding value of $\omega$

(upper or lower branch) we have a real solution in the form

$$Q_n^{(0)} = Ae^{i\theta} + \circledast\,, \tag{10.55}$$

where $\theta = \delta n - \omega t$. In general, since the dispersion relation admits the pair $(\delta, \omega) = (0, 0)$, we would have to consider the possibility that a mean flow is spontaneously excited by nonlinear interactions, but it turns out that the structure of the nonlinear terms in the lattice model under consideration prevents this from occurring. (The situation is similar to that in part (b) of Exercise 10.5.) Thus, it is not necessary to include in $Q_n^{(0)}$ any component proportional to the mean.

With the solution (10.55) at leading order, the terms proportional to $\epsilon$ yield the equation

$$\begin{aligned} L[Q^{(1)}] &= -2m\frac{\partial^2 Q_n^{(0)}}{\partial T_0 \partial T_1} + V''(0)\left[\frac{\partial Q_{n+1}^{(0)}}{\partial X_1} - \frac{\partial Q_{n-1}^{(0)}}{\partial X_1}\right] \\ &\quad + \frac{V'''(0)}{2}\left[(Q_{n+1}^{(0)} - Q_n^{(0)})^2 - (Q_n^{(0)} - Q_{n-1}^{(0)})^2\right] \\ &= \left[2im\omega\frac{\partial A}{\partial T_1} + 2iV''(0)\sin(\delta)\frac{\partial A}{\partial X_1}\right]e^{i\theta} \\ &\quad + iA^2V'''(0)\left[\sin(2\delta) - 2\sin(\delta)\right]e^{2i\theta} + \circledast\,, \end{aligned} \tag{10.56}$$

where for general $f$, $L[f]$ denotes the linear expression

$$L[f] := m\frac{\partial^2 f_n}{\partial T_0^2} - V''(0)\left[f_{n+1} - 2f_n + f_{n-1}\right]$$

and where we have used the explicit form (10.55) of the leading-order solution to compute the right-hand side. The second harmonic terms on the right-hand side of (10.56) are nonresonant, since $D(2\delta, 2\omega)$ (see the dispersion relation (10.54)) is not zero given that $D(\delta, \omega) = 0$ and $\delta \neq 0$. On the other hand, the fundamental harmonics $e^{\pm i\theta}$ are resonant, and their presence will lead to secular terms unless we choose $A$ so as to remove the corresponding coefficient. Therefore we take $A$ to satisfy the solvability condition

$$2im\omega\frac{\partial A}{\partial T_1} + 2iV''(0)\sin(\delta)\frac{\partial A}{\partial X_1} = 0\,.$$

Differentiating the dispersion relation (10.54) implicitly with respect to $\delta$, we see that this equation can equivalently be written as

$$\frac{\partial A}{\partial T_1} + \omega'(\delta)\frac{\partial A}{\partial X_1} = 0\,, \tag{10.57}$$

which indicates rigid motion of the wave envelope at the group velocity $v_g := \omega'(\delta)$. With $A$ chosen to satisfy (10.57), the right-hand side of (10.56) will contain only nonresonant second harmonics. We can find a particular solution for (10.56) simply by assuming the form

$$Q_n^{(1)} = Be^{2i\theta} + \circledast$$

for some complex constant $B$. Substituting into (10.56), we find that

$$B = -\frac{iA^2V'''(0)}{D(2\delta, 2\omega)}\left[\sin(2\delta) - 2\sin(\delta)\right] .$$

The nonresonance condition $D(2\delta, 2\omega) \neq 0$ is clearly crucial here. Although we can add to our particular solution for $Q_n^{(1)}$ an arbitrary homogeneous solution, which is a superposition of waves satisfying the dispersion relation (10.54), we take the point of view that we have already selected such a wave at leading order and that adding something of the same type at this order would be contributing nothing interesting. Therefore we have the solution

$$Q_n^{(1)} = -\frac{iA^2V'''(0)}{D(2\delta, 2\omega)}\left[\sin(2\delta) - 2\sin(\delta)\right]e^{2i\theta} + \circledast .$$

The terms proportional to $\epsilon^2$ yield an equation of motion governing $Q_n^{(2)}$:

$$\begin{aligned} L[Q^{(2)}] = &\left[2im\omega\frac{\partial A}{\partial T_2} - m\frac{\partial^2 A}{\partial T_1^2} + V''(0)\cos(\delta)\frac{\partial^2 A}{\partial X_1} - K|A|^2A\right]e^{i\theta} \\ &+ \circledast + \text{second and third harmonics}\,, \end{aligned} \tag{10.58}$$

where

$$K := 2V^{(IV)}(0)(\cos(\delta) - 1)^2 + \frac{2[V'''(0)]^2(2\sin(\delta) - \sin(2\delta))^2}{D(2\delta, 2\omega)} .$$

As was the case with the second harmonics, the third harmonics are not resonant. Indeed, according to (10.54) we have $D(3\delta, 3\omega) \neq 0$ if $D(\delta, \omega) = 0$ and $\delta \neq 0$. Thus the terms that must be eliminated to avoid secular terms in $Q^{(2)}$ are those proportional to the fundamental harmonic and its complex conjugate. The solvability condition is thus

$$2im\omega\frac{\partial A}{\partial T_2} - m\frac{\partial^2 A}{\partial T_1^2} + V''(0)\cos(\delta)\frac{\partial^2 A}{\partial X_1} - K|A|^2A = 0 . \tag{10.59}$$

With this condition satisfied, we may solve for $Q_n^{(2)}$ which we will find has components proportional to the second and third harmonics $e^{\pm 2i\theta}$ and $e^{\pm 3i\theta}$, but as desired, it is a bounded function.

There is some utility in the change of variables

$$\tau = T_1 \quad \text{and} \quad \xi = X_1 - \omega'(\delta)T_1 .$$

With this change of variables, the solvability condition (10.57) takes the simple form

$$\frac{\partial A}{\partial \tau} = 0$$

and (10.59) can be written as

$$i\frac{\partial A}{\partial T_2} + \frac{\omega''(\delta)}{2}\frac{\partial^2 A}{\partial \xi^2} + \beta |A|^2 A = 0\,, \tag{10.60}$$

where the real parameter $\beta$ is given by

$$\beta := -\frac{m\omega^3}{4[V''(0)]^2}\left(V^{(IV)}(0) + 4[V'''(0)]^2 \sin^2(\delta)\right)\,.$$

Thus we have derived from the discrete molecular chain model the cubic nonlinear Schrödinger equation. The behavior of solutions of (10.60) is totally different depending on whether $\beta\omega''(\delta) > 0$ (the so-called "focusing" case) or $\beta\omega''(\delta) < 0$ (the so-called "defocusing" case). In the focusing case, broad packets ($A$ nearly constant in $\xi$) are *modulationally unstable* and break up into smaller packets (solitons). On the other hand, broad packets are stable in the defocusing case. Now from looking at Figure 10.3 or by using the exact formula (10.54), we see that $\omega''(\delta)$ and $\omega(\delta)$ always have opposite signs. Thus, we see that whether we are in the stable or unstable case depends only on the sign of $V^{(IV)}(0)$ and its size relative to $4[V'''(0)]^2\sin^2(\delta)$. If $V^{(IV)}(0)$ is positive, meaning that the nonlinear springs become "harder" with larger deviations from equilibrium, then we are necessarily in the focusing case. On the other hand, if $V^{(IV)}(0)$ is sufficiently negative given $\delta$, which corresponds to springs that become significantly "softer" when the amplitude increases, then we are in the defocusing case.

◀ *Exercise 10.12.* In the nonlinear Schrödinger equation, solutions that are independent of $\xi$ represent uniform wavetrains of the underlying problem. Therefore the dynamical stability of the $\xi$-independent solutions determines the modulational stability of broad wavepackets. Explicitly obtain the $\xi$-independent solutions of the nonlinear Schrödinger equation (in both focusing and defocusing cases). Then linearize the equation about these solutions and analyze their stability. Confirm the modulational instability in the focusing case. ▶

**10.2.3. Derivation of the Boussinesq and Korteweg-de Vries equations.** An important point to make is that the dispersion $\omega''(\delta)$ vanishes on either branch if $\delta = 0$. In this case, the nonlinear Schrödinger equation (10.60) is the wrong model, and we should modify our assumptions. Here the idea is that $\delta = 0$ corresponds to *long waves.* When $\delta$ is close to zero, the displacement $Q_n^{(0)}$ is not changing much at all from particle to particle in the lattice. So suppose we return to the original equations of motion (10.52) for $Q_n(t)$ and assume that

$$Q_n(t) = Q(X_1, T_1; \epsilon)\,,$$

that is, a smooth function of $X_1$ and $T_1$. Inserting into (10.52), we find that

$$m\frac{\partial^2 Q}{\partial T_1^2} = V''(0)\frac{\partial^2 Q}{\partial X_1^2} + \epsilon^2\left[\frac{V''(0)}{12}\frac{\partial^4 Q}{\partial X_1^4} + V'''(0)\frac{\partial Q}{\partial X_1}\frac{\partial^2 Q}{\partial X_1^2}\right] + O(\epsilon^3)\,.$$

If we neglect the terms of order $O(\epsilon^3)$, this is a Boussinesq equation for $Q$, admitting waves propagating in two opposite directions. Making the change of variables

$$\tau = \epsilon^2 T_1 \quad \text{and} \quad \xi = X_1 - \sqrt{\frac{V''(0)}{m}}T_1 = X_1 - \omega'(0)T_1\,,$$

we go into the frame of reference moving with the right-going waves, and the equation becomes

$$-2\sqrt{mV''(0)}\frac{\partial^2 Q}{\partial\tau\partial\xi} = \frac{V''(0)}{12}\frac{\partial^4 Q}{\partial\xi^4} + V'''(0)\frac{\partial Q}{\partial\xi}\frac{\partial^2 Q}{\partial\xi^2} + O(\epsilon)\,.$$

Therefore, under the assumption that

$$q_n(t) = \epsilon Q(\xi, \tau; \epsilon) \quad \text{with} \quad \xi = \epsilon\left(n - \sqrt{\frac{V''(0)}{m}}t\right) \quad \text{and} \quad \tau = \epsilon^3 t\,,$$

the function $F := \partial Q/\partial\xi$ asymptotically satisfies the Korteweg-de Vries equation

$$2\sqrt{mV''(0)}\frac{\partial F}{\partial\tau} + V'''(0)F\frac{\partial F}{\partial\xi} + \frac{V''(0)}{12}\frac{\partial^3 F}{\partial\xi^3} = 0\,,$$

the universal model for weakly nonlinear, weakly dispersive long waves. The constant coefficients can be normalized away by rescaling $F$ and $\tau$ by $\epsilon$-independent factors. Thus,

$$\frac{\partial F}{\partial\tau} + F\frac{\partial F}{\partial\xi} + \frac{\partial^3 F}{\partial\xi^3} = 0$$

holds for the rescaled $F$ with the rescaled time $\tau$.

## 10.3. Water Waves

Consider an irrotational, inviscid fluid moving in a plane perpendicular to the surface of the earth with horizontal coordinate $x$ and vertical coordinate $y$. The fluid is confined by gravity to lie between the surface of the earth, which we take to be $y = -h_0$, and the fluid/air interface, which is a curve $y = h(x,t)$. We suppose that as $|x| \to \infty$, $h(x,t) \to 0$, so $h(x,t)$ represents a localized disturbance of a fluid of undisturbed depth $h_0$. More generally, flows over nontrivial topography can be handled by replacing the constant $h_0$ with an appropriate fixed function $h_0(x)$ modeling the landscape features.

Because the flow is irrotational, the fluid velocity at any point in the region $-h_0 < y < h(x,t)$ can be obtained as the gradient of a potential function $\phi(x,t)$ satisfying Laplace's equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0\,, \qquad -h_0 < y < h(x,t)\,. \tag{10.61}$$

The fluid velocity should be purely horizontal at $y = -h_0$, which means that Laplace's equation should be considered along with the boundary condition

$$\frac{\partial \phi}{\partial y} = 0\,, \quad \text{at } y = -h_0\,. \tag{10.62}$$

There are two boundary conditions associated with the fluid/air interface:

$$\frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x} = \frac{\partial \phi}{\partial y}\,, \quad \text{at } y = h(x,t)\,, \tag{10.63}$$

and

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial y}\right)^2 + gh = 0\,, \quad \text{at } y = h(x,t)\,, \tag{10.64}$$

where $g$ is the gravitational acceleration near the surface of the earth. In a sense, only one boundary condition should be necessary to solve Laplace's equation, but since the surface itself is undetermined, a second condition is also required. The geometry is illustrated in Figure 10.4.

We may nondimensionalize the problem by selecting appropriate units of time and length. In this problem there are two numerical quantities with units: the undisturbed depth $h_0$ and the gravitational acceleration $g$. The former gives a natural length scale

$$\lambda := h_0\,,$$

while the latter has units of length per unit time squared, so a natural time scale is given by

$$\tau := \sqrt{\frac{h_0}{g}}\,.$$

Clearly the interface height function $h(x,t)$ has units of length, and since the velocity potential $\phi(x,y,t)$ has derivatives with respect to $x$ and $y$ that are velocities, $\phi$ itself has units of length squared per unit time. Therefore, the problem will become nondimensional if we introduce new variables:

$$\overline{x} := \frac{x}{\lambda}\,, \quad \overline{y} := \frac{y}{\lambda}\,, \quad \overline{h} := \frac{h}{\lambda}\,,$$
$$\overline{t} := \frac{t}{\tau}\,, \quad \overline{\phi} := \frac{\tau}{\lambda^2}\phi\,.$$

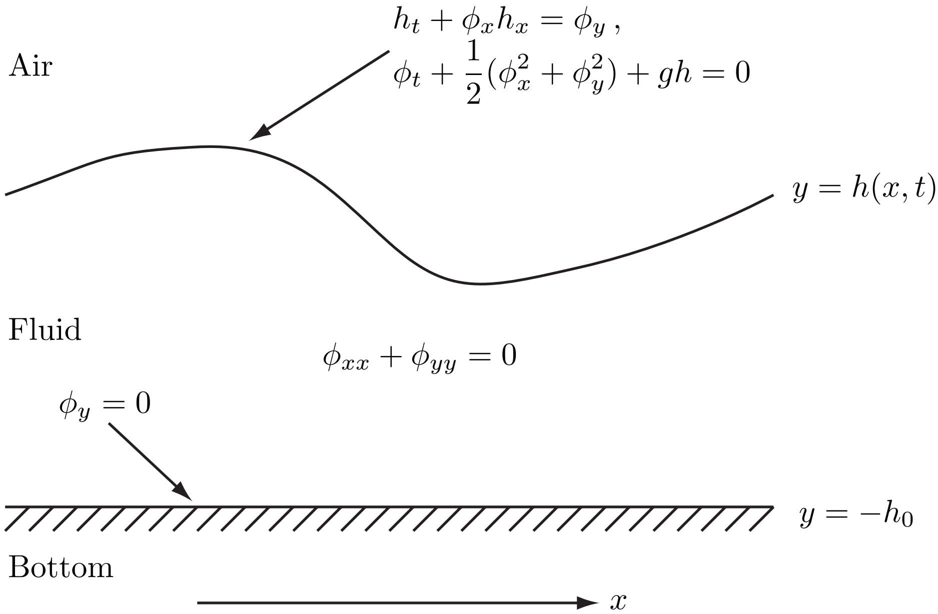


**Figure 10.4.** *The geometry of the water wave problem.*

From these we may derive from (10.61)–(10.64) the corresponding equations governing the potential $\overline{\phi}(\overline{x}, \overline{y}, \overline{t})$ and the interface $\overline{h}(\overline{x}, \overline{t})$. Dropping the bars for notational convenience, the resulting equations of motion are exactly the same as (10.61)–(10.64) but with $h_0 = 1$ and $g = 1$. We will restrict our attention to this nondimensional case.

A nondimensional measure of the size of the initial disturbance of the surface is the number

$$\epsilon := \sup_{x \in \mathbb{R}} |h(x,0)| .$$

Physically, $\epsilon < 1$ is the ratio of the magnitude of the initial disturbance measured in the supremum norm to the undisturbed depth of the fluid. If we rescale the dependent variables of the problem by setting

$$h(x,t) := \epsilon H(x,t) , \quad \phi(x,y,t) := \epsilon \Phi(x,t) ,$$

then the supremum norm of $H(x,0)$ is one, and the nondimensionalized problem becomes

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 , \quad -1 < y < \epsilon H(x,t) , \tag{10.65}$$

subject to the bottom boundary condition

$$\frac{\partial \Phi}{\partial y} = 0 , \quad \text{at } y = -1 ,$$

and the top boundary conditions

$$\frac{\partial H}{\partial t} + \epsilon \frac{\partial \Phi}{\partial x}\frac{\partial H}{\partial x} = \frac{\partial \Phi}{\partial y}\,, \quad \text{at } y = \epsilon H(x,t)\,, \tag{10.66}$$

and

$$\frac{\partial \Phi}{\partial t} + \frac{\epsilon}{2}\left(\frac{\partial \Phi}{\partial x}\right)^2 + \frac{\epsilon}{2}\left(\frac{\partial \Phi}{\partial y}\right)^2 + H = 0\,, \quad \text{at } y = \epsilon H(x,t)\,. \tag{10.67}$$

The wave height $H(x,t)$ may be eliminated between these latter two boundary conditions to obtain a single condition relating partial derivatives of the potential $\Phi(x,y,t)$ on the interface, as the following exercise shows.

◀ *Exercise 10.13.* By differentiating (10.67) partially with respect to $x$ and $t$, eliminate $\partial H/\partial t$ and $\partial H/\partial x$ from (10.66) to obtain in its place

$$\frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial \Phi}{\partial y} + 2\epsilon\left[\frac{\partial \Phi}{\partial x}\frac{\partial^2 \Phi}{\partial x \partial t} + \frac{\partial \Phi}{\partial y}\frac{\partial^2 \Phi}{\partial y \partial t}\right] + \epsilon^2\left[\left(\frac{\partial \Phi}{\partial x}\right)^2 \frac{\partial^2 \Phi}{\partial x^2} + 2\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial y}\frac{\partial^2 \Phi}{\partial x \partial y} + \left(\frac{\partial \Phi}{\partial y}\right)^2 \frac{\partial^2 \Phi}{\partial y^2}\right] = 0\,, \tag{10.68}$$

evaluated for $y = \epsilon H(x,t)$. Hint: Use the chain rule to take into account that $y = \epsilon H(x,t)$. ▶

We regard (10.67) and (10.68) as a useful form of the boundary conditions at the interface $y = \epsilon H(x,t)$.

**10.3.1. Derivation of the cubic nonlinear Schrödinger equation.** Small amplitude waves are those for which $\epsilon$ is a small number, and we will now consider the limit $\epsilon \to 0$ and derive a nonlinear Schrödinger equation as a consequence of a multiple scales analysis. Assume that the potential $\Phi(x,y,t)$ can be extended as a harmonic function to $y = 0$. This is not an issue if the interface $H(x,t)$ is everywhere positive, describing a so-called *wave of elevation*, but if the interface ever falls below the undisturbed height of $y = 0$ (a *wave of depression*), it becomes a nontrivial assumption. In any case, we may then expand derivatives of $\Phi(x, \epsilon H(x,t), t)$ about $y = 0$ in Taylor series to any desired order, and we expect to be able to control the errors in the small amplitude limit. Thus, assuming $H$ remains $O(1)$ for all time, from (10.67) we obtain

$$\frac{\partial \Phi}{\partial t} + \epsilon \frac{\partial^2 \Phi}{\partial y \partial t} H + \frac{\epsilon}{2}\left(\frac{\partial \Phi}{\partial x}\right)^2 + \frac{\epsilon}{2}\left(\frac{\partial \Phi}{\partial y}\right)^2 + H + O(\epsilon^2) = 0\,, \quad \text{at } y = 0\,.$$

Solving for $H$, this becomes

$$\begin{aligned} H &= -\left(1+\epsilon\frac{\partial^2\Phi}{\partial y\partial t}\right)^{-1}\left[\frac{\partial\Phi}{\partial t}+\frac{\epsilon}{2}\left(\frac{\partial\Phi}{\partial x}\right)^2+\frac{\epsilon}{2}\left(\frac{\partial\Phi}{\partial y}\right)^2+O(\epsilon^2)\right] \\ &= -\frac{\partial\Phi}{\partial t}+\epsilon\left[\frac{\partial\Phi}{\partial t}\frac{\partial^2\Phi}{\partial y\partial t}-\frac{1}{2}\left(\frac{\partial\Phi}{\partial x}\right)^2-\frac{1}{2}\left(\frac{\partial\Phi}{\partial y}\right)^2\right]+O(\epsilon^2) \end{aligned} \tag{10.69}$$

as $\epsilon \to 0$, where all derivatives of $\Phi$ are evaluated at $y = 0$. Likewise, expanding (10.68) around $y = 0$, we find

$$\begin{aligned} &\frac{\partial^2\Phi}{\partial t^2}+\frac{\partial\Phi}{\partial y}+\epsilon\left[\frac{\partial^3\Phi}{\partial y\partial t^2}H-\frac{\partial^2\Phi}{\partial x^2}H+2\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x\partial t}+2\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y\partial t}\right] \\ &\qquad+\epsilon^2\left[-\frac{1}{2}\frac{\partial^4\Phi}{\partial x^2\partial t^2}H^2-\frac{1}{2}\frac{\partial^3\Phi}{\partial x^2\partial y}H^2\right. \\ &\quad+2\frac{\partial^2\Phi}{\partial x\partial y}\frac{\partial^2\Phi}{\partial x\partial t}H+2\frac{\partial\Phi}{\partial x}\frac{\partial^3\Phi}{\partial x\partial y\partial t}H-2\frac{\partial^2\Phi}{\partial x^2}\frac{\partial^2\Phi}{\partial y\partial t}H-2\frac{\partial\Phi}{\partial y}\frac{\partial^3\Phi}{\partial x^2\partial t}H \\ &\quad\left.+\left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^2\Phi}{\partial x^2}+2\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial x\partial y}-\left(\frac{\partial\Phi}{\partial y}\right)^2\frac{\partial^2\Phi}{\partial x^2}\right]+O(\epsilon^3)=0\,, \end{aligned} \tag{10.70}$$

where derivatives of $\Phi$ are again evaluated at $y = 0$ and we have used (10.65) to eliminate $\partial^2\Phi/\partial y^2$. Substituting (10.69) into (10.70) gives a boundary condition at $y = 0$ phrased only in terms of the potential $\Phi$ through terms proportional to $\epsilon^2$:

$$\begin{aligned} &\frac{\partial^2\Phi}{\partial t^2}+\frac{\partial\Phi}{\partial y}+\epsilon\left[2\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x\partial t}+2\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y\partial t}+\frac{\partial\Phi}{\partial t}\frac{\partial^2\Phi}{\partial x^2}-\frac{\partial\Phi}{\partial t}\frac{\partial^3\Phi}{\partial y\partial t^2}\right] \\ &\quad+\epsilon^2\left[-\frac{1}{2}\left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^3\Phi}{\partial y\partial t^2}-\frac{1}{2}\left(\frac{\partial\Phi}{\partial y}\right)^2\frac{\partial^3\Phi}{\partial y\partial t^2}+\frac{\partial\Phi}{\partial t}\frac{\partial^2\Phi}{\partial y\partial t}\frac{\partial^3\Phi}{\partial y\partial t^2}\right. \\ &\quad+\frac{3}{2}\left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^2\Phi}{\partial x^2}-\frac{1}{2}\left(\frac{\partial\Phi}{\partial y}\right)^2\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial\Phi}{\partial t}\frac{\partial^2\Phi}{\partial x^2}\frac{\partial^2\Phi}{\partial y\partial t} \\ &\quad-\frac{1}{2}\left(\frac{\partial\Phi}{\partial t}\right)^2\frac{\partial^4\Phi}{\partial x^2\partial t^2}-\frac{1}{2}\left(\frac{\partial\Phi}{\partial t}\right)^2\frac{\partial^3\Phi}{\partial x^2\partial y}-2\frac{\partial\Phi}{\partial t}\frac{\partial^2\Phi}{\partial x\partial y}\frac{\partial^2\Phi}{\partial x\partial t} \\ &\quad\left.-2\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial t}\frac{\partial^3\Phi}{\partial x\partial y\partial t}+2\frac{\partial\Phi}{\partial y}\frac{\partial\Phi}{\partial t}\frac{\partial^3\Phi}{\partial x^2\partial t}+2\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial x\partial y}\right]+O(\epsilon^3)=0\,. \end{aligned}$$

To analyze the problem perturbatively for small $\epsilon$, we introduce time scales $T_0 = t$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$ and horizontal spatial scales $X_0 = x$ and $X_1 = \epsilon x$. The only vertical scale will be $y$, and we seek an asymptotic expansion of the potential

$$\Phi(x,y,t;\epsilon) \sim \sum_{n=0}^{\infty} \Phi_n(X_0,X_1,y,T_0,T_1,T_2)\epsilon^n \quad \text{as } \epsilon \to 0.$$

The boundary conditions on $\Phi_n(X_0,X_1,y,T_0,T_1,T_2)$ will be imposed at $y = 0$ and $y = -1$. An important difference between the water wave problem and other problems in which we have derived envelope equations is that in this case it will be the boundary condition at $y = 0$ that determines the dependence of the coefficients $\{\Phi_n\}$ on the slow scales $X_1$, $T_1$, and $T_2$, rather than solvability conditions required to remove secular terms.

The leading terms will determine a problem satisfied by $\Phi_0$:

$$\frac{\partial^2 \Phi_0}{\partial X_0^2} + \frac{\partial^2 \Phi_0}{\partial y^2} = 0\,, \qquad -1 < y < 0\,, \tag{10.71}$$

$$\frac{\partial \Phi_0}{\partial y} = 0\,, \quad \text{at } y = -1\,, \tag{10.72}$$

and

$$\frac{\partial^2 \Phi_0}{\partial T_0^2} + \frac{\partial \Phi_0}{\partial y} = 0\,, \quad \text{at } y = 0\,. \tag{10.73}$$

We seek a solution representing a wavetrain with some nonzero wavenumber $k$ in the horizontal direction. Such a solution of (10.71) satisfying the boundary condition (10.72) at $y = -1$ is

$$\Phi_0 = a\cosh(k(y+1))e^{ikX_0} + \circledast\,,$$

where $a$ is complex and independent of $X_0$ and $y$. Applying to this expression the boundary condition (10.73) at $y = 0$, we find that it is satisfied if

$$\cosh(k)\frac{\partial^2 a}{\partial t^2} + k\sinh(k)a = 0\,,$$

which may be solved by taking

$$a = A(X_1,T_1,T_2)e^{-i\omega T_0}\,,$$

where $A$ is a complex amplitude whose dependence on the slow scales $X_1$, $T_1$, and $T_2$ is to be determined and where the frequency $\omega$ and wavenumber $k$ are related by the water wave dispersion relation

$$\omega^2 = k\tanh(k)\,. \tag{10.74}$$

The branches of this dispersion relation are plotted in Figure 10.5. Note that the relation (10.74) is transcendental, which indicates that this is not the dispersion relation of any partial differential equation in $X_0$ and $T_0$; indeed it arises here from a kind of reduction of a higher-dimensional problem involving the additional variable $y$. Note also that the dispersion relation admits the pair $(k,\omega) = (0,0)$. Thus the possibility exists that nonlinear effects will excite a mean flow. Unlike the case of weakly nonlinear vibrations

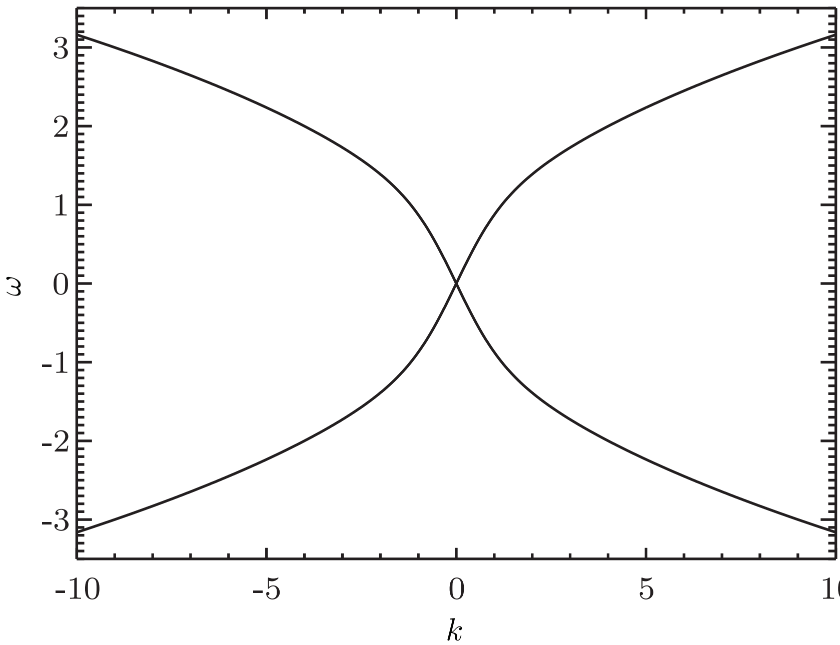


**Figure 10.5.** *The dispersion relation for water waves.*

of molecular chains, where a mean flow is admitted by the dispersion relation but ultimately is not excited, it turns out that for water waves we will indeed need to take the mean into account to derive the envelope equation satisfied by $A(X_1, T_1, T_2)$. Noting that a real constant $M = M(X_1, T_1, T_2)$ is a solution of Laplace's equation (10.71) and also the boundary conditions (10.72) and (10.73), we take as our leading-order solution a superposition of the mean flow $M$ and a wavetrain with horizontal wavenumber $k$:

$$\Phi_0 = A \cosh(k(y+1))e^{i\theta} + \circledast + M\,, \tag{10.75}$$

where we have introduced a phase by writing $\theta := kX_0 - \omega T_0$.

Our aim is to determine the dependence of the complex amplitude $A$ and the real mean $M$ on the slow scales $X_1$, $T_1$, and $T_2$. This will follow from our attempts to compute higher-order corrections to $\Phi_0$. The problem satisfied by $\Phi_1$ consists of the partial differential equation

$$\begin{aligned}\frac{\partial^2 \Phi_1}{\partial X_0^2} + \frac{\partial^2 \Phi_1}{\partial y^2} &= -2\frac{\partial^2 \Phi_0}{\partial X_0 \partial X_1} \\ &= -2ik\frac{\partial A}{\partial X_1}\cosh(k(y+1))e^{i\theta} + \circledast\,, \quad -1 < y < 0\,,\end{aligned} \tag{10.76}$$

and the boundary conditions

$$\frac{\partial \Phi_1}{\partial y} = 0\,, \quad \text{at } y = -1\,, \tag{10.77}$$

and (grouping together in braces the linear and quadratic terms in $\Phi_0$ and $\Phi_1$)

$$\left[\frac{\partial^2 \Phi_1}{\partial T_0^2} + \frac{\partial \Phi_1}{\partial y} + 2\frac{\partial^2 \Phi_0}{\partial T_0 \partial T_1}\right] + \left[2\frac{\partial \Phi_0}{\partial X_0}\frac{\partial^2 \Phi_0}{\partial X_0 \partial T_0}\right.$$

$$\left. + 2\frac{\partial \Phi_0}{\partial y}\frac{\partial^2 \Phi_0}{\partial y \partial T_0} - \frac{\partial \Phi_0}{\partial T_0}\frac{\partial^3 \Phi_0}{\partial y \partial T_0^2} - \frac{\partial \Phi_0}{\partial T_0}\frac{\partial^2 \Phi_0}{\partial y^2}\right] = 0\,, \quad \text{at } y = 0\,.$$

Using the expression (10.75) for $\Phi_0$ and the dispersion relation (10.74), this can be rewritten as

$$\left[\frac{\partial^2 \Phi_1}{\partial T_0^2} + \frac{\partial \Phi_1}{\partial y}\right]\Bigg|_{y=0} = 2i\omega\frac{\partial A}{\partial T_1}\cosh(k)e^{i\theta} - 3ik^2\omega A^2 e^{2i\theta} + \circledast\,. \tag{10.78}$$

The general solution of (10.76) is

$$\Phi_1 = \left[-i\frac{\partial A}{\partial X_1}(y+1)\sinh(k(y+1))e^{i\theta} + \circledast\right] + \tilde{\Phi}_1\,,$$

where $\tilde{\Phi}_1$ is an arbitrary solution of the associated homogeneous equation (an arbitrary harmonic function of $X_0$ and $y$). Since the explicit terms already satisfy the boundary condition (10.77), whatever we might add must satisfy (10.77) as well. The guiding principle is that we should not add anything that is not necessary; noticing the second harmonics $e^{\pm 2i\theta}$ appearing in the boundary condition (10.78), we will need something to balance these so we introduce a second harmonic term satisfying the boundary condition (10.77):

$$\Phi_1 = -i\frac{\partial A}{\partial X_1}(y+1)\sinh(k(y+1))e^{i\theta} + B\cosh(2k(y+1))e^{2i\theta} + \circledast\,.$$

Applying the boundary condition (10.78), we then obtain equations by separating the coefficients of the linearly independent fundamentals $e^{\pm i\theta}$ and second harmonics $e^{\pm 2i\theta}$ and setting them to zero. From the coefficients of the second harmonics, we find

$$B = \frac{3ik\omega\cosh(k)}{4\sinh^3(k)}A^2$$

(along with the complex conjugate of this relation), and then from the coefficients of the fundamentals, we get (again with the conjugate)

$$\frac{\partial A}{\partial T_1} + \frac{\tanh(k) + k\,\mathrm{sech}^2(k)}{2\omega}\frac{\partial A}{\partial X_1} = 0\,.$$

Here we have used the dispersion relation (10.74). If we implicitly differentiate (10.74) with respect to $k$, we arrive at a result we might have anticipated:

$$\frac{\partial A}{\partial T_1} + \omega'(k)\frac{\partial A}{\partial X_1} = 0\,. \tag{10.79}$$

This equation expresses the effect, evident on the time scale $T_1$, of the rigid translation of the wave envelope $A$ to the right with the group velocity $\omega'(k)$. To obtain further information about the dependence of the envelope $A$ on the slower time scale $T_2$, we must continue the calculation to the next order.

The correction $\Phi_2$ satisfies the boundary-value problem consisting of the partial differential equation

$$\frac{\partial^2 \Phi_2}{\partial X_0^2} + \frac{\partial^2 \Phi_2}{\partial y^2} = -2\frac{\partial^2 \Phi_1}{\partial X_0 \partial X_1} - \frac{\partial^2 \Phi_0}{\partial X_1^2}\,, \qquad -1 < y < 0\,, \tag{10.80}$$

and the boundary conditions

$$\frac{\partial \Phi_2}{\partial y} = 0\,, \quad \text{at } y = -1\,, \tag{10.81}$$

and (grouping together in braces the linear, quadratic, and cubic terms in $\Phi_0$, $\Phi_1$, and $\Phi_2$)

$$\begin{aligned}
&\left[\frac{\partial^2 \Phi_2}{\partial T_0^2} + 2\frac{\partial^2 \Phi_1}{\partial T_0 \partial T_1} + 2\frac{\partial^2 \Phi_0}{\partial T_0 \partial T_2} + \frac{\partial^2 \Phi_0}{\partial T_1^2} + \frac{\partial \Phi_2}{\partial y}\right] + \left[2\frac{\partial \Phi_0}{\partial X_0}\frac{\partial^2 \Phi_1}{\partial X_0 \partial T_0}\right. \\
&\quad + 2\frac{\partial \Phi_1}{\partial X_0}\frac{\partial^2 \Phi_0}{\partial X_0 \partial T_0} + 2\frac{\partial \Phi_0}{\partial y}\frac{\partial^2 \Phi_1}{\partial y \partial T_0} + 2\frac{\partial \Phi_1}{\partial y}\frac{\partial^2 \Phi_0}{\partial y \partial T_0} + \frac{\partial \Phi_0}{\partial T_0}\frac{\partial^2 \Phi_1}{\partial X_0^2} \\
&\quad + \frac{\partial \Phi_1}{\partial T_0}\frac{\partial^2 \Phi_0}{\partial X_0^2} - \frac{\partial \Phi_0}{\partial T_0}\frac{\partial^3 \Phi_1}{\partial y \partial T_0^2} - \frac{\partial \Phi_1}{\partial T_0}\frac{\partial^3 \Phi_0}{\partial y \partial T_0^2} + 2\frac{\partial \Phi_0}{\partial X_1}\frac{\partial^2 \Phi_0}{\partial X_0 \partial T_0} \\
&\quad + 2\frac{\partial \Phi_0}{\partial X_0}\frac{\partial^2 \Phi_0}{\partial X_1 \partial T_0} + 2\frac{\partial \Phi_0}{\partial X_0}\frac{\partial^2 \Phi_0}{\partial X_0 \partial T_1} + 2\frac{\partial \Phi_0}{\partial y}\frac{\partial^2 \Phi_0}{\partial y \partial T_1} + \frac{\partial \Phi_0}{\partial T_1}\frac{\partial^2 \Phi_0}{\partial X_0^2} \\
&+ 2\frac{\partial \Phi_0}{\partial T_0}\frac{\partial^2 \Phi_0}{\partial X_0 \partial X_1} - \frac{\partial \Phi_0}{\partial T_1}\frac{\partial^3 \Phi_0}{\partial y \partial T_0^2} - \left. 2\frac{\partial \Phi_0}{\partial T_0}\frac{\partial^3 \Phi_0}{\partial y \partial T_0 \partial T_1}\right] + \left[-\frac{1}{2}\left(\frac{\partial \Phi_0}{\partial X_0}\right)^2 \frac{\partial^3 \Phi_0}{\partial y \partial T_0^2}\right. \\
&\quad - \frac{1}{2}\left(\frac{\partial \Phi_0}{\partial y}\right)^2 \frac{\partial^3 \Phi_0}{\partial y \partial T_0^2} + \frac{\partial \Phi_0}{\partial T_0}\frac{\partial^2 \Phi_0}{\partial y \partial T_0}\frac{\partial^3 \Phi_0}{\partial y \partial T_0^2} + \frac{3}{2}\left(\frac{\partial \Phi_0}{\partial X_0}\right)^2 \frac{\partial^2 \Phi_0}{\partial X_0^2} \\
&\quad - \frac{1}{2}\left(\frac{\partial \Phi_0}{\partial y}\right)^2 \frac{\partial^2 \Phi_0}{\partial X_0^2} + \frac{\partial \Phi_0}{\partial T_0}\frac{\partial^2 \Phi_0}{\partial X_0^2}\frac{\partial^2 \Phi_0}{\partial y \partial T_0} - \frac{1}{2}\left(\frac{\partial \Phi_0}{\partial T_0}\right)^2 \frac{\partial^4 \Phi_0}{\partial X_0^2 \partial T_0^2} \\
&- \frac{1}{2}\left(\frac{\partial \Phi_0}{\partial T_0}\right)^2 \frac{\partial^3 \Phi_0}{\partial X_0^2 \partial y} - 2\frac{\partial \Phi_0}{\partial T_0}\frac{\partial^2 \Phi_0}{\partial X_0 \partial y}\frac{\partial^2 \Phi_0}{\partial X_0 \partial T_0} - 2\frac{\partial \Phi_0}{\partial X_0}\frac{\partial \Phi_0}{\partial T_0}\frac{\partial^3 \Phi_0}{\partial X_0 \partial y \partial T_0} \\
&\qquad + 2\frac{\partial \Phi_0}{\partial y}\frac{\partial \Phi_0}{\partial T_0}\frac{\partial^3 \Phi_0}{\partial X_0^2 \partial T_0} + \left. 2\frac{\partial \Phi_0}{\partial X_0}\frac{\partial \Phi_0}{\partial y}\frac{\partial^2 \Phi_0}{\partial X_0 \partial y}\right] = 0\,, \quad \text{at } y = 0\,.
\end{aligned}$$

With the help of the previously obtained expressions for $\Phi_0$ and $\Phi_1$, this is seen to have the form

$$\left[\frac{\partial^2\Phi_2}{\partial T_0^2}+\frac{\partial\Phi_2}{\partial y}\right]\Bigg|_{y=0} = G_3(k,\omega)e^{3i\theta}+G_2(k,\omega)e^{2i\theta}+G_1(k,\omega)e^{i\theta}+\circledast+G_0(k,\omega)\,, \quad (10.82)$$

where the $G_j(k,\omega)$ are easily obtained by direct substitution ($G_0(k,\omega)$ is real).

◀ *Exercise 10.14.* Find explicitly $G_0(k,\omega)$ and $G_1(k,\omega)$. ▶

Similarly, the partial differential equation (10.80) to be solved becomes

$$\begin{aligned}\frac{\partial^2\Phi_2}{\partial X_0^2}+\frac{\partial^2\Phi_2}{\partial y^2} &= -\frac{\partial^2 A}{\partial X_1^2}\left[2k(y+1)\sinh(k(y+1))+\cosh(k(y+1))\right]e^{i\theta}\\ &\quad+6k^2\omega\frac{\cosh(k)}{\sinh^3(k)}A\frac{\partial A}{\partial X_1}\cosh(2k(y+1))e^{2i\theta}+\circledast-\frac{\partial^2 M}{\partial X_1^2}\,.\end{aligned} \quad (10.83)$$

The right-hand side of (10.83) forces $\Phi_2$ to have some terms proportional to the fundamental harmonics $e^{\pm i\theta}$, the second harmonics $e^{\pm 2i\theta}$, and a constant term. It will also be necessary to include some homogeneous solutions proportional to second and third harmonics to balance with corresponding automatically generated harmonics present in the boundary condition (10.82) to be imposed at $y=0$. The appropriate solution of (10.83) satisfying the boundary condition (10.81) at $y=-1$ has the form

$$\begin{aligned}\Phi_2 = &-\frac{1}{2}\frac{\partial^2 A}{\partial X_1^2}(y+1)^2\cosh(k(y+1))e^{i\theta}+Ce^{2i\theta}+De^{3i\theta}+\circledast\\ &-\frac{1}{2}(y+1)^2\frac{\partial^2 M}{\partial X_1^2}\,,\end{aligned}$$

where $C$ and $D$ are arbitrary. We now impose the boundary condition (10.82) at $y=0$ by separately equating to zero the coefficients of the fundamental, second, and third harmonics, as well as the constant term. The equations resulting from the second and third harmonics determine the constants $C$ and $D$ in terms of $A$ and $M$. The equations resulting from the fundamental harmonics and the constant term are more interesting as these amount to equations satisfied by $A$ and $M$.

From the constant terms in (10.82), we obtain (with the help of the dispersion relation (10.74) to simplify the resulting expressions) the equation

$$\frac{\partial^2 M}{\partial T_1^2} - \frac{\partial^2 M}{\partial X_1^2} = \left(k^2 - 2k^2\cosh^2(k)\right)\frac{\partial}{\partial T_1}|A|^2 + \left(k\omega + k\omega\cosh^2(k) - k^2\omega\tanh(k)\right)\frac{\partial}{\partial X_1}|A|^2 .$$

By differentiation of this equation with respect to $X_1$ and $T_1$ using (10.79) and its complex conjugate, we get

$$\frac{\partial^2}{\partial T_1^2}\frac{\partial M}{\partial X_1} - \frac{\partial^2}{\partial X_1^2}\frac{\partial M}{\partial X_1} = C(k,\omega)\left(\frac{\partial^2}{\partial T_1^2}|A|^2 - \frac{\partial^2}{\partial X_1^2}|A|^2\right)$$

and also

$$\frac{\partial^2}{\partial T_1^2}\frac{\partial M}{\partial T_1} - \frac{\partial^2}{\partial X_1^2}\frac{\partial M}{\partial T_1} = -\omega'(k)C(k,\omega)\left(\frac{\partial^2}{\partial T_1^2}|A|^2 - \frac{\partial^2}{\partial X_1^2}|A|^2\right) ,$$

where

$$C(k,\omega) := \frac{1}{\omega'(k)^2 - 1}\left(\frac{k^2\omega}{2\cosh(k)\sinh(k)} + \left(\frac{1}{2} + 2\cosh^2(k)\right)k\omega\right) .$$

Since the same differential operator appears on both sides, a particular solution for $M(X_1, T_1, T_2)$ satisfies

$$\frac{\partial M}{\partial X_1} = C(k,\omega)|A|^2 \tag{10.84}$$

and

$$\frac{\partial M}{\partial T_1} = -\omega'(k)C(k,\omega)|A|^2 . \tag{10.85}$$

Now we examine the coefficient of the fundamental harmonic $e^{i\theta}$ in the boundary condition (10.82) at $y = 0$. With the help of the dispersion relation (10.74) and some identities among hyperbolic functions, we find

$$i\frac{\partial A}{\partial T_2} + \tanh(k)\frac{\partial^2 A}{\partial X_1 \partial T_1} - \frac{1}{2\omega}\frac{\partial^2 A}{\partial T_1^2} + \frac{1}{2\omega}\frac{\partial^2 A}{\partial X_1^2} - \frac{\cosh(6k) + 2\cosh(4k) + 13\cosh(2k) + 20}{16\omega\cosh^2(k)\sinh^2(k)}k^4|A|^2 A - kA\frac{\partial M}{\partial X_1} + \frac{k^2}{2\omega}\operatorname{sech}^2(k)A\frac{\partial M}{\partial T_1} = 0 .$$

Using (10.79) and implicit differentiation of the dispersion relation (10.74), the linear terms simplify:

$$i\frac{\partial A}{\partial T_2} + \frac{\omega''(k)}{2}\frac{\partial^2 A}{\partial X_1^2} \\ - \frac{\cosh(6k) + 2\cosh(4k) + 13\cosh(2k) + 20}{16\omega\cosh^2(k)\sinh^2(k)}k^4|A|^2 A \\ - kA\frac{\partial M}{\partial X_1} + \frac{k^2}{2\omega}\mathrm{sech}^2(k)A\frac{\partial M}{\partial T_1} = 0\,.$$

Finally, substituting from (10.84) and (10.85), we obtain a closed nonlinear Schrödinger equation for $A$:

$$i\frac{\partial A}{\partial T_2} + \frac{\omega''(k)}{2}\frac{\partial^2 A}{\partial X_1^2} + \beta|A|^2 A = 0\,,$$

in which the nonlinear coefficient is

$$\beta := \frac{k^4 N(k)}{16\omega\sinh^2(k)\left[2k\sinh(4k) - \sinh^2(2k) - 4k^2\right]}\,,$$

where

$$\begin{aligned} N(k) := &-4k\left[\sinh(8k) + 4\sinh(4k) + 24\sinh(2k)\right] \\ &+ 5\cosh(8k) + 8\cosh(6k) + (16k^2 - 18)\cosh(4k) \\ &- 8\cosh(2k) + 128k^2 + 13\,. \end{aligned}$$

The product $\beta\omega$ is plotted as a function of $k$ in Figure 10.6. This plot indicates a sign change occurring at $|k| = k_{\mathrm{crit}} \approx 1.36278$. In fact, $k = \pm k_{\mathrm{crit}}$ are the only two roots of $\beta\omega$. For water waves, the dispersion $\omega''(k)$ always has the opposite sign of $\omega(k)$, so we see that $\omega''(k)$ and $\beta$ have the same sign if $|k| > k_{\mathrm{crit}}$ and they have the opposite sign if $|k| < k_{\mathrm{crit}}$. This is a physically relevant prediction of the weakly nonlinear theory of water waves. Indeed, long waves (those with $|k| < k_{\mathrm{crit}}$ or waves of wavelength longer than $2\pi/k_{\mathrm{crit}} \approx 4.61056$ times the undisturbed depth) are modulationally stable since the wave packet amplitudes are governed by a defocusing nonlinear Schrödinger equation. On the other hand, waves shorter than this critical wavelength are modulationally unstable because the nonlinear Schrödinger equation is of focusing type.

**10.3.2. Derivation of the Korteweg-de Vries equation.** The analysis that leads to the cubic nonlinear Schrödinger equation breaks down if $k = 0$, because the dispersion vanishes (that is, $\omega''(0) = 0$). To analyze the $k = 0$ case really means that instead of looking at slow modulations of wavepackets, we are looking at long waves. Generally, long-wave limits lead to universal equations of a different type.

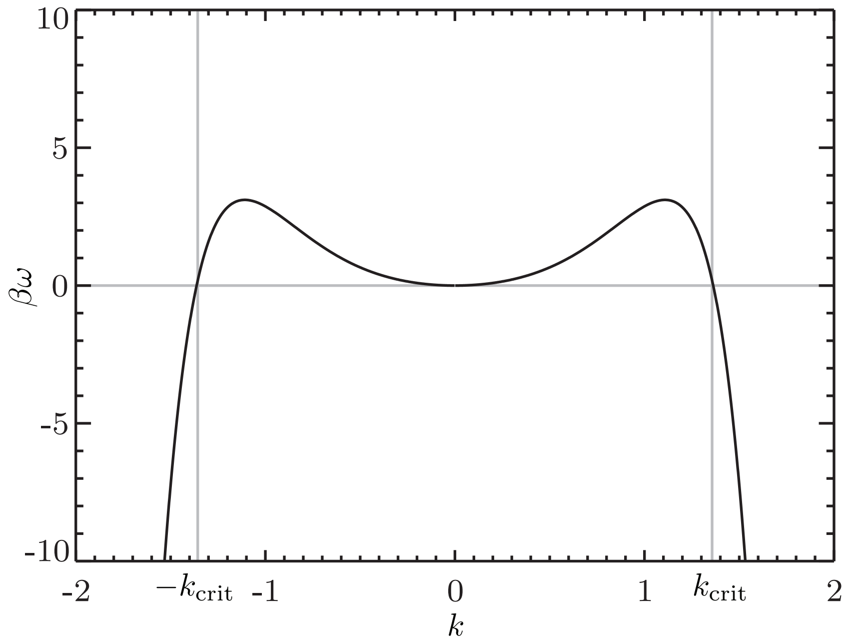


**Figure 10.6.** *The nonlinear coefficient for water waves.*

We begin with the nondimensionalized water wave problem consisting of Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0\,,\qquad -1<y<h(x,t)\,,$$

the boundary condition at the bottom:

$$\frac{\partial\phi}{\partial y}=0\,,\quad \text{at } y=-1\,,$$

and the boundary conditions at the free surface:

$$\frac{\partial h}{\partial t}+\frac{\partial\phi}{\partial x}\frac{\partial h}{\partial x}=\frac{\partial\phi}{\partial y}\,,\quad \text{at } y=h(x,t)\,,$$

and

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial\phi}{\partial y}\right)^2+h=0\,,\quad \text{at } y=h(x,t)\,.$$

One way to treat the partial differential equation for the potential $\phi$ and the boundary condition at the bottom is to solve the problem using power series about $y=-1$ (this is the Cauchy-Kovaleskaya method). Thus, we seek a solution in the form of a convergent power series

$$\phi(x,y,t)=\phi_0(x,t)+\sum_{n=2}^{\infty}\phi_n(x,t)(y+1)^n\,, \tag{10.86}$$

where the linear term in $y+1$ is missing to satisfy the boundary condition at $y=-1$. Substitution into Laplace's equation determines all coefficients $\phi_n(x,t)$ in terms of the leading coefficient $\phi_0(x,t)$:

$$\phi_n(x,t) \equiv 0\,, \quad \text{for } n \text{ odd}\,,$$

and

$$\phi_{2k}(x,t) = \frac{(-1)^k}{(2k)!}\frac{\partial^{2k}\phi_0}{\partial x^{2k}}(x,t)\,, \quad \text{for } k = 0,1,2,\ldots\,.$$

The power series (10.86) is convergent if $\phi_0(x,t)$ is an analytic function of $x$.

The power series solution (10.86) for the potential $\phi(x,y,t)$ becomes an asymptotic series in a small dimensionless parameter $\epsilon$ if $\phi_0(x,t)$ is a function $w(X,T)$ of the combinations $X=\epsilon^{1/2}x$ and $T=\epsilon^{1/2}t$ which represent, respectively, slow horizontal and temporal scales for this problem. Thus, we have

$$\phi(x,y,t) = w - \frac{\epsilon}{2}(y+1)^2\frac{\partial^2 w}{\partial X^2} + \frac{\epsilon^2}{24}(y+1)^4\frac{\partial^4 w}{\partial X^4} + O(\epsilon^3)\,. \qquad (10.87)$$

Next, we pass to the weakly nonlinear context by scaling the dependent variables:

$$h(x,t) = \epsilon G(X,T)\,, \quad w(X,T) = \epsilon^{1/2}N(X,T)\,,$$

where $G$ and $N$ are now supposed to be, along with all derivatives with respect to $X$ and $T$, uniformly bounded functions. Then, it remains to satisfy the boundary conditions at $y=h=\epsilon G$, which after substituting (10.87) take the form

$$\frac{\partial G}{\partial T} + (y+1)\frac{\partial^2 N}{\partial X^2} + \epsilon\left[\frac{\partial N}{\partial X}\frac{\partial G}{\partial X} - (y+1)^3\frac{1}{6}\frac{\partial^4 N}{\partial X^4}\right] = O(\epsilon^2)\,, \quad \text{at } y = \epsilon G(X,T)\,,$$

and

$$\frac{\partial N}{\partial T} + G - \frac{\epsilon}{2}(y+1)^2\frac{\partial^3 N}{\partial X^2 \partial T} = O(\epsilon^2)\,, \quad \text{at } y = \epsilon G(X,T)\,.$$

Expanding about $y=0$ the free boundary conditions become

$$\frac{\partial G}{\partial T} + \frac{\partial^2 N}{\partial X^2} + \epsilon\left[\frac{\partial N}{\partial X}\frac{\partial G}{\partial X} + G\frac{\partial^2 N}{\partial X^2} - \frac{1}{6}\frac{\partial^4 N}{\partial X^4}\right] = O(\epsilon^2) \qquad (10.88)$$

and

$$\frac{\partial N}{\partial T} + G - \frac{\epsilon}{2}\frac{\partial^3 N}{\partial X^2 \partial T} = O(\epsilon^2)\,. \qquad (10.89)$$

Eliminating $G$ by substituting (10.89) into (10.88), assuming that derivatives remain bounded, gives a closed equation for $N$:

$$\frac{\partial^2 N}{\partial T^2} - \frac{\partial^2 N}{\partial X^2} + \frac{\epsilon}{6}\frac{\partial^4 N}{\partial X^4} - \frac{\epsilon}{2}\frac{\partial^4 N}{\partial X^2 \partial T^2} + \epsilon\frac{\partial N}{\partial T}\frac{\partial^2 N}{\partial X^2} + \epsilon\frac{\partial N}{\partial X}\frac{\partial^2 N}{\partial X \partial T} = O(\epsilon^2)\,. \tag{10.90}$$

This equation also says that $\partial^2 N/\partial T^2 = \partial^2 N/\partial X^2 + O(\epsilon)$, so we may combine the fourth derivative terms to obtain

$$\frac{\partial^2 N}{\partial T^2} - \frac{\partial^2 N}{\partial X^2} - \frac{\epsilon}{3}\frac{\partial^4 N}{\partial X^4} + \epsilon\frac{\partial N}{\partial T}\frac{\partial^2 N}{\partial X^2} + \epsilon\frac{\partial N}{\partial X}\frac{\partial^2 N}{\partial X \partial T} = O(\epsilon^2)\,. \tag{10.91}$$

(Note that the $O(\epsilon^2)$ correction term represents a different function in (10.91) than in (10.90).) This is nearly a Boussinesq equation, differing only in the mixed derivative terms, which would be pure $X$ derivatives in the Boussinesq equation.

Now the leading terms suggest motion to the left or right with speed one. Without loss of generality, we go into a right-moving frame with this speed. Thus, we introduce the coordinates

$$\xi = X - T\,, \quad \tau = \epsilon T\,,$$

so that

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial \xi} \quad \text{and} \quad \frac{\partial}{\partial T} = \epsilon\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \xi}\,.$$

The equation (10.91) for $N$ thus becomes, after cancelling an $\epsilon$,

$$2\frac{\partial^2 N}{\partial \xi \partial \tau} + \frac{1}{3}\frac{\partial^4 N}{\partial \xi^4} + 2\frac{\partial N}{\partial \xi}\frac{\partial^2 N}{\partial \xi^2} = O(\epsilon)\,.$$

Dropping the $O(\epsilon)$ term and letting $F := \partial N/\partial \xi$, we arrive at the Korteweg-de Vries equation:

$$\frac{\partial F}{\partial \tau} + F\frac{\partial F}{\partial \xi} + \frac{1}{6}\frac{\partial^3 F}{\partial \xi^3} = 0\,.$$

## 10.4. Notes and References

Aside from its physical applications, the sine-Gordon equation (10.1) is also important mathematically for at least two reasons. First of all, its solutions intrinsically describe surfaces of constant negative curvature. The sine-Gordon equation is therefore of great importance in differential geometry. Secondly, it is an example of an *integrable system*. This means that the sine-Gordon equation (10.1) is a rather special nonlinear partial differential equation for which there is a mathematical framework for solving certain initial-value problems *exactly*. There exists a tool called the *inverse-scattering transform*, which should be viewed as a nonlinear analogue of the Fourier transform that is precisely adapted to the sine-Gordon equation

(10.1). Some information on the inverse-scattering transform can be found in Whitham [**37**], and more introductory details about integrable systems can be found in Drazin and Johnson [**8**].

If in the sine-Gordon equation (10.1) the term $\sin(\varphi)$ is replaced by $m\varphi$, where $m$ is a positive constant, then this equation becomes the Klein-Gordon equation[8] describing the relativistic quantum mechanics of a spin-zero particle with mass $m$. More generally, equations of the form (10.1) with the term $\sin(\varphi)$ replaced by an arbitrary, generally nonlinear, function of $\varphi$ are also frequently called *Klein-Gordon equations*. Thus the sine-Gordon equation (10.1) is a special case of a Klein-Gordon equation involving the sine function.

Like the sine-Gordon equation, the cubic nonlinear Schrödinger equation is another integrable nonlinear partial differential equation. However, it must be stressed that this property is not inherited from that of the sine-Gordon equation, from which it can be obtained by a multiple-scales analysis. Indeed, as we have seen, the cubic nonlinear Schrödinger equation may be derived from other equations (in particular from Klein-Gordon equations of a general form) that are not themselves integrable. This means that many kinds of dispersive nonlinear waves become "more integrable" in the weakly nonlinear limit. See Drazin and Johnson [**8**] for more information about the integrability of the cubic nonlinear Schrödinger equation. Another treatment can be found in Faddeev and Takhtajan [**10**].

Like the sine-Gordon equation and the cubic nonlinear Schrödinger equation, the 3-wave interaction equations also comprise an integrable system for which there exist analytical methods for extracting and analyzing exact solutions. See Ablowitz and Clarkson [**1**]. The Manley-Rowe relations constitute a physically important subset of the infinitely many constants of motion in the integrable case. The Manley-Rowe relations originated in the paper [**26**] of Manley and Rowe.

Boussinesq was a French mathematician and physicist who made important contributions to the field of hydrodynamics. The Boussinesq equation is another one of the nonlinear partial differential equations that is, like the sine-Gordon equation, the cubic nonlinear Schrödinger equation, and the 3-wave interaction equations, an integrable system. Note that the parameter $\epsilon$ in (10.49) need not be small for the equation to be integrable.

As it happens, the Korteweg-de Vries equation (10.51) is also an integrable nonlinear partial differential equation. The Korteweg-de Vries equation is distinguished as being the first dispersive nonlinear wave equation ever recognized to have this remarkable mathematical property. The initial observation (at least the earliest sufficiently precise one) was made in 1965

[8] Oskar Klein, 1894–1977, and Walter Gordon, 1893–1939.

by Zabusky and Kruskal [**38**], who were conducting numerical experiments using the Korteweg-de Vries equation as a long-wave asymptotic model for the Fermi-Pasta-Ulam chain. The inverse-scattering transform adapted to the Korteweg-de Vries equation first appeared in 1967 in a paper of Gardner, Greene, Kruskal, and Miura [**14**]. It turns out that the *modified* Korteweg-de Vries equation introduced in Exercise 10.5 is also completely integrable, and although it may seem strange, in many ways the theory of its solution by an inverse-scattering transform is closer to that of the nonlinear Schrödinger equation than to that of the Korteweg-de Vries equation. Also, note that equations of the form $\varphi_t + \varphi^p \varphi_x + \varphi_{xxx} = 0$ are not completely integrable unless we are in one of the two special cases $p = 1$ or $p = 2$.

The original paper containing the numerical experiments of Fermi, Pasta, and Ulam is [**11**]. For more information about water waves and the various partial differential equation models for them, see Whitham [**37**]. Similar asymptotic derivations of model equations in the case of slowly-varying topography of the ocean floor can be found in Newell's text [**29**], which also contains lots of information about the theory of integrable systems.

# Subject Index