Introduction

This monograph is based on a one-semester course on measure theory, which I have taught several times. The prerequisite for the course is an introductory analysis course, covering such matters as metric spaces, uniform convergence of functions, the contraction mapping principle, and aspects of multi-variable calculus, including the inverse function theorem. For the convenience of the reader, some of this material is briefly treated in some appendices.

The core topic for the course treated here is the theory of measure and integration, associated especially with the work of H. Lebesgue, though of course many other mathematicians have contributed to this central subject. We mention particularly E. Borel, M. Riesz, J. Radon, M. Frechet, G. Fubini, C. Caratheodory, F. Hausdorff, and A. Besicovitch, among the classical founders.

We begin with an introductory chapter on the Riemann integral, for functions defined on an interval $[a, b]$ in $\mathbb{R}$. We develop some of the properties of the Riemann integral, including a proof of the Fundamental Theorem of Calculus. We see that while continuous functions are Riemann integrable, some very reasonable-looking functions are not.

In Chapter 2 we construct Lebesgue measure on $\mathbb{R}$. We emphasize that the key difference between Lebesgue measure of $S \subset \mathbb{R}$ and the "content" of S, arising from the Riemann integral, is that the content is approximated by taking finite coverings of S by intervals, while the Lebesgue measure is approximated by taking *countable* (perhaps infinite) coverings of S by intervals.

In Chapter 3 we define the Lebesgue integral and establish some basic properties, such as the Monotone Convergence Theorem and the Lebesgue

Dominated Convergence Theorem. We integrate measurable functions defined on general measure spaces. Though at this point we have only constructed Lebesgue measure on $\mathbb{R}$, the basic theory of integration is not more complicated on general measure spaces, and pursuing it helps clarify what one should do to construct more general measures. In Chapter 4 we introduce L^p spaces, consisting of measurable functions f such that $|f|^p$ is integrable or, more precisely, of equivalence classes of such functions, where we say $f_1 \sim f_2$ provided these functions differ only on a set of measure zero. If μ is a measure on a space X, we study $L^p(X, \mu)$ as a Banach space, for $1 \le p \le \infty$, and in particular we study $L^2(X, \mu)$ as a Hilbert space. We develop some Hilbert space theory and apply it to establish the Radon-Nikodym Theorem, comparing two measures μ and ν when ν is "absolutely continuous" with respect to μ.

Constructing measures other than Lebesgue measure on $\mathbb{R}$ is an important part of measure theory, and we begin this task in Chapter 5, giving some useful general methods, especially due to Caratheodory, for making such constructions, establishing that they are measures, and identifying certain types of sets as measurable. The first concrete application of this is made in Chapter 6, in the construction of the "product measure" on $X \times Y$, when X has a measure μ and Y has the measure ν. The integral with respect to the product measure $\mu \times \nu$ is compared to "iterated integrals," in theorems of Fubini and Tonelli.

In Chapter 7 we construct Lebesgue measure on $\mathbb{R}^n$, for $n > 1$, as a product measure. We study how the Lebesgue integral on $\mathbb{R}^n$ transforms under an invertible linear transformation on $\mathbb{R}^n$ and, more generally, under a C^1 diffeomorphism. We go a bit further, considering transformation via a Lipschitz homeomorphism, and we establish a result under the hypothesis that the transformation is differentiable almost everywhere, a property that will be studied further in Chapter 11. We extend the scope of the n-dimensional integral in another direction, constructing surface measure on an n-dimensional surface M in $\mathbb{R}^m$. This is done in terms of the Riemann metric tensor induced on M. We go further and discuss integration on more general Riemannian manifolds. This central chapter contains a larger number of exercises than the others, divided into several sets of exercises. After the first set, of a nature parallel to exercise sets for other chapters, there is a set relating the Riemann and Lebesgue integrals on $\mathbb{R}^n$, extending the previous discussion of the relationship between the material of Chapter 1 and that of Chapters 2–3, an exercise set on determinants, and an exercise set on row reduction on matrix products, providing linear algebra background for the proof of the change of variable formula. There is also an exercise set on the connectivity of $Gl(n, \mathbb{R})$ and a set on integration on certain matrix groups.

In Chapter 8 we discuss "signed measures," which differ from the measures considered up to that point only in that they can take negative as well as positive values. The key result established there is the Hahn decomposition, $\nu = \nu^+ - \nu^-$, for a signed measure ν, on X, where ν^+ and ν^- are positive measures, with disjoint supports. This allows us to extend the Radon-Nikodym Theorem to the case of a signed measure ν, absolutely continuous with respect to a (positive) measure μ. We consider a further extension, to complex measures, though this is completely routine.

In Chapter 9 we again take up the study of L^p spaces and pursue it a bit further. We identify the *dual* of the Banach space $L^p(X, \mu)$ with $L^q(X, \mu)$, where $1 \le p < \infty$ and $1/p + 1/q = 1$, making use of the Radon-Nikodym Theorem for signed measures as a tool in the demonstration. We study some integral operators, including convolution operators, amongst others, and derive operator bounds on L^p spaces. We consider the Fourier transform $\mathcal{F}$ and prove that $\mathcal{F}$ is an invertible norm-preserving operator on $L^2(\mathbb{R}^n)$. Some mention of Fourier series was made in Chapter 4, particularly in the exercises.

Chapter 10 discusses Sobolev spaces, $H^{k,p}(\mathbb{R}^n)$, consisting of functions whose derivatives of order $\le k$, defined in a suitable weak sense, belong to $L^p(\mathbb{R}^n)$. Certain Sobolev spaces are shown to consist entirely of bounded continuous functions, even Hölder continuous, on $\mathbb{R}^n$. This subject is of great use in the study of partial differential equations, though such applications are not made here. The most significant application we make of Sobolev space theory in these notes appears in the following chapter. We mention that in Chapter 10 certain results on the weak derivative depend on the Fundamental Theorem of Calculus for the Riemann integral; it is for this reason that we included a proof of this result in Chapter 1.

Chapter 11 deals with various results in the area of almost-everywhere convergence. The basic result, called Lebesgue's differentiation theorem, is that a function $f \in L^1(\mathbb{R}^n)$ is equal for almost all $x \in \mathbb{R}^n$ to the limit of its averages over balls of radius r, centered at x, as $r \to 0$. Under a slightly stronger condition on x, we say x is a Lebesgue point for f; more generally there is the notion of an L^p-Lebesgue point, and one shows that if $f \in L^p(\mathbb{R}^n)$, then almost every $x \in \mathbb{R}^n$ is an L^p-Lebesgue point for f, provided $1 \le p < \infty$. We use the Hardy-Littlewood maximal function as a tool to establish these results. This area also requires a "covering lemma," allowing one to select from a collection of sets covering S a subcollection with desirable properties.

Another important result established in Chapter 11 is Rademacher's Theorem that a Lipschitz function on $\mathbb{R}^n$ is differentiable almost everywhere. We get this as a corollary of the stronger result that if $f \in H^{1,p}(\mathbb{R}^n)$ and $p > n$ (or $p = n = 1$), then f is differentiable almost everywhere; in fact, f is differentiable at every L^p-Lebesgue point of the weak derivative ∇f. With this result, we can complete the demonstration of the result from Chapter 7 that the change of variable formula for the integral extends to Lipschitz homeomorphisms. Making use of Rademacher's Theorem, we show that a Lipschitz function can be altered off a small set to yield a C^1 function. These results are important in the study of Lipschitz surfaces in $\mathbb{R}^n$.

The covering lemma we use for the results of Chapter 11 mentioned above is Weiner's Covering Lemma. We also discuss covering lemmas of Vitali and of Besicovitch and show how Besicovitch's result leads to extensions of the Lebesgue differentiation theorem.

In Chapter 12 we construct r-dimensional Hausdorff measure $\mathcal{H}^r$, on a separable metric space X, for any $r \in [0, \infty)$. In case S is a Borel set in $\mathbb{R}^n$, one has $\mathcal{H}^n(S) = \mathcal{L}^n(S)$, Lebesgue measure; this is not a straightforward consequence of the definition (unless $n = 1$), and its proof requires some effort. In particular, the proof requires a covering lemma. We extend this analysis to show that n-dimensional Hausdorff measure on an n-dimensional manifold with a continuous metric tensor coincides with the volume measure constructed in Chapter 7. This applies to C^1 surfaces in Euclidean space. We go further and study Hausdorff measure on Lipschitz surfaces in Euclidean space. Results on Lipschitz functions established in Chapter 11 are invaluable here. These provide basic results in "geometric measure theory." There is a great deal more to geometric measure theory, which has been developed as a tool in the study of minimal surfaces, amongst other applications. A good overview can be found in [**Mor**], a reading of which should prepare one for the treatise [**Fed**].

While the bulk of Chapter 12 deals with $\mathcal{H}^r$ when r is an integer, there are wild sets, some of incredible beauty, called "fractals," for which $\mathcal{H}^r$ is germane for a value $r \notin \mathbb{Z}$. We touch this only briefly; one can consult [**Ed**], [**Mdb**], [**Fal**], and [**PR**] for more material on fractals.

In Chapter 13 we show how a positive linear functional on $C(X)$, the space of continuous functions on X, gives rise to a (positive) measure on X, when X is a compact metric space, and how a bounded linear functional on $C(X)$ gives rise to a signed measure on X. Out of this come compactness results for bounded sets of measures. These results extend to the case where X is a general compact Hausdorff space; treatments of this can be found in several places, including [**Fol**] and [**Ru**]. The argument is somewhat simpler when X is metrizable; in particular, we can appeal to results from Chapter 5 for a lot of the technical work.

Chapters 14–17 explore connections between measure theory and probability theory. The basic connection is that a probability measure is a positive measure of total mass 1. Chapter 14 treats ergodic theory, which deals with statistical properties of iterates of a measure-preserving map φ on a probability space $(X, \mathfrak{F}, \mu)$. In particular, one studies the map $Tf(x) = f(\varphi(x))$ on L^p spaces and the means $A_k f = (1/k) \sum_{j=0}^{k-1} T^j f(x)$. Mean ergodic theorems and Birkhoff's Ergodic Theorem treat L^p-norm behavior and pointwise a.e. behavior of $A_k f(x)$, tending to a limit $Pf(x)$ as $k \to \infty$. Ergodic transformations are those for which such limits are constant. Knowing that certain transformations are ergodic can provide valuable information, as we will see.

Chapter 15 discusses some of the fundamental results of probability theory, dealing with random variables (i.e., measurable functions) on a probability space. These results include weak and strong laws of large numbers, whose basic message is that means of a large number of independent random variables of the same type (i.e., with the same probability distribution) tend to constant limits, with probability one. We approach the strong law via Birkhoff's Ergodic Theorem. We also treat the Central Limit Theorem, giving conditions under which such means have approximately Gaussian probability distributions.

In Chapter 16, we construct Wiener measure on the set of continuous paths in $\mathbb{R}^n$, describing the probabilistic behavior of a particle undergoing Brownian motion. We begin with a probability measure W on a countable product $\mathfrak{P}$ of compactifications of $\mathbb{R}^n$, first defining a positive linear functional on $C(\mathfrak{P})$ and then getting the measure via the results of Chapter 13. The index set for the countable product is $\mathbb{Q}^+$, the set of rational numbers ≥ 0. The space of continuous paths is naturally identified with a subset $\mathfrak{P}_0$, shown to be a Borel subset of $\mathfrak{P}$, of W-measure one. To illustrate how fuzzy Brownian paths are, we show that when $n \geq 2$, almost all paths have Hausdorff dimension 2.

There is a further structure associated with Wiener measure on path space, namely a filtered family of σ-algebras. Certain families of functions $\mathfrak{f}_t$ on $\mathfrak{P}_0$ are "martingales," i.e., $\mathfrak{f}_t$ is obtained from $\mathfrak{f}_s$ by taking the "conditional expectation," when $t < s$. This is discussed in Chapter 17. We define martingales more generally, prove the Martingale Maximal Inequality, and apply this to a number of convergence results for martingales, obtaining both a variant of the Lebesgue differentiation theorem and another proof of the strong law of large numbers. We also produce several martingales associated with Brownian motion and apply the martingale maximal inequality to these.

This book has several appendices, some providing background material, as mentioned above, and others providing supplementary material. Appendix A contains some basic material on metric spaces, topological spaces, and compactness. In particular, we prove the Stone-Weierstrass Theorem, which gives a very useful sufficient condition for a set $\mathcal{A}$ of continuous functions on a compact space X to be dense in the space of all continuous functions $C(X)$. This result is used in a number of places, including Chapters 4, 6, and 16. In Appendix B we give some basic results in multi-variable differential calculus, including the Inverse Function Theorem, the nature of a diffeomorphism, and the concept of a manifold. This material is useful for appreciating change of variable formulas for integrals and also the results on integration on surfaces and more general manifolds in Chapters 7 and 12.

Appendix C is devoted to the Whitney Extension Theorem, needed for the approximation theorem for Lipschitz functions in Chapter 11. Appendix D treats the Marcinkiewicz Interpolation Theorem, giving L^p estimates on an operator satisfying "weak type" estimates on L^q and L^r, with $q < p < r$. Appendix E discusses Sard's Theorem, that the set of critical values of a C^1 map $F : \mathbb{R}^n \to \mathbb{R}^n$ has measure zero. This result is applied in Appendix F, to help prove a change of variable theorem for a C^1 map of $\mathbb{R}^n$ not assumed to be a diffeomorphism.

In Appendix G we discuss the elements of the theory of differential forms and their integration. These results have many applications to problems in differential equations, differential geometry, and topology. A key result is a general Gauss-Green-Stokes formula. To illustrate the power of this formula, we show how it leads to a short proof of the famous Brouwer Fixed-Point Theorem. In Appendix H we apply the differential form results to obtain another approach to the change of variable formula, a modification of an approach put forward by P. Lax [**La**]. Finally in Appendix I we extend the Gauss-Green-Stokes formula from the setting of smoothly bounded domains considered in Appendix G to the setting of Lipschitz domains.

There are several ways to use this monograph in a course. Chapters 1–4 provide a quick introduction to the basics of Lebesgue integration. I have used this material at the end of analysis courses that precede a full-blown course in measure theory. For a one quarter course on measure theory, Chapters 1–9 would provide a solid background in the subject. For a semester course, one could deepen this background with a selection of material from Chapters 10–17 and the appendices.

Each of the seventeen chapters in this monograph ends with a set of exercises. These form an integral part of our presentation, and thinking about them should sharpen the reader's understanding of the material. On occasion, the results of some of the exercises are used in the development of subsequent material.

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