
Contents

Preface	xi
Acknowledgments	xvii
A Detailed Guide for the Reader	xxi
Notation and Symbols	xxxv
Chapter 1. Riemannian Geometry	1
§1. Introduction	1
§2. Metrics, connections, curvatures and covariant differentiation	2
§3. Basic formulas and identities in Riemannian geometry	10
§4. Exterior differential calculus and Bochner formulas	14
§5. Integration and Hodge theory	20
§6. Curvature decomposition and locally conformally flat manifolds	25
§7. Moving frames and the Gauss-Bonnet formula	32
§8. Variation of arc length, energy and area	41
§9. Geodesics and the exponential map	52
§10. Second fundamental forms of geodesic spheres	58
§11. Laplacian, volume and Hessian comparison theorems	67
§12. Proof of the comparison theorems	73
§13. Manifolds with nonnegative curvature	80
§14. Lie groups and left-invariant metrics	87
§15. Notes and commentary	89

Chapter 2. Fundamentals of the Ricci Flow Equation	95
§1. Geometric flows and geometrization	96
§2. Ricci flow and the evolution of scalar curvature	98
§3. The maximum principle for heat-type equations	100
§4. The Einstein-Hilbert functional	104
§5. Evolution of geometric quantities	108
§6. DeTurck's trick and short time existence	113
§7. Reaction-diffusion equation for the curvature tensor	119
§8. Notes and commentary	123
Chapter 3. Closed 3-manifolds with Positive Ricci Curvature	127
§1. Hamilton's 3-manifolds with positive Ricci curvature theorem	127
§2. The maximum principle for tensors	128
§3. Curvature pinching estimates	131
§4. Gradient bounds for the scalar curvature	136
§5. Curvature tends to constant	140
§6. Exponential convergence of the normalized flow	142
§7. Notes and commentary	149
Chapter 4. Ricci Solitons and Special Solutions	153
§1. Gradient Ricci solitons	154
§2. Gaussian and cylinder solitons	157
§3. Cigar steady soliton	159
§4. Rosenau solution	162
§5. An expanding soliton	164
§6. Bryant soliton	167
§7. Homogeneous solutions	169
§8. The isometry group	175
§9. Notes and commentary	176
Chapter 5. Isoperimetric Estimates and No Local Collapsing	181
§1. Sobolev and logarithmic Sobolev inequalities	181
§2. Evolution of the length of a geodesic	186
§3. Isoperimetric estimate for surfaces	188
§4. Perelman's no local collapsing theorem	190
§5. Geometric applications of no local collapsing	198
§6. 3-manifolds with positive Ricci curvature revisited	206

§7. Isoperimetric estimate for 3-dimensional Type I solutions	208
§8. Notes and commentary	211
Chapter 6. Preparation for Singularity Analysis	213
§1. Derivative estimates and long time existence	213
§2. Proof of Shi's local first and second derivative estimates	218
§3. Cheeger-Gromov-type compactness theorem for Ricci flow	233
§4. Long time existence of solutions with bounded Ricci curvature	237
§5. The Hamilton-Ivey curvature estimate	240
§6. Strong maximum principles and metric splitting	245
§7. Rigidity of 3-manifolds with nonnegative curvature	248
§8. Notes and commentary	250
Chapter 7. High-dimensional and Noncompact Ricci Flow	253
§1. Spherical space form theorem of Huisken-Margerin-Nishikawa	254
§2. 4-manifolds with positive curvature operator	259
§3. Manifolds with nonnegative curvature operator	263
§4. The maximum principle on noncompact manifolds	272
§5. Complete solutions of the Ricci flow on noncompact manifolds	279
§6. Notes and commentary	286
Chapter 8. Singularity Analysis	291
§1. Singularity dilations and types	292
§2. Point picking and types of singularity models	297
§3. Geometric invariants of ancient solutions	307
§4. Dimension reduction	316
§5. Notes and commentary	326
Chapter 9. Ancient Solutions	327
§1. Classification of ancient solutions on surfaces	328
§2. Properties of ancient solutions that relate to their type	338
§3. Geometry at infinity of gradient Ricci solitons	353
§4. Injectivity radius of steady gradient Ricci solitons	364
§5. Towards a classification of 3-dimensional ancient solutions	368
§6. Classification of 3-dimensional shrinking Ricci solitons	375
§7. Summary and open problems	388
Chapter 10. Differential Harnack Estimates	391

§1. Harnack estimates for the heat and Laplace equations	392
§2. Harnack estimate on surfaces with $\chi > 0$	397
§3. Linear trace and interpolated Harnack estimates on surfaces	401
§4. Hamilton's matrix Harnack estimate for the Ricci flow	405
§5. Proof of the matrix Harnack estimate	410
§6. Harnack and pinching estimates for linearized Ricci flow	415
§7. Notes and commentary	420
Chapter 11. Space-time Geometry	425
§1. Space-time solution to the Ricci flow for degenerate metrics	426
§2. Space-time curvature is the matrix Harnack quadratic	433
§3. Potentially infinite metrics and potentially infinite dimensions	434
§4. Renormalizing the space-time length yields the ℓ -length	452
§5. Space-time DeTurck's trick and fixing the measure	453
§6. Notes and commentary	456
Appendix A. Geometric Analysis Related to Ricci Flow	461
§1. Compendium of inequalities	461
§2. Comparison theory for the heat kernel	463
§3. Green's function	465
§4. The Liouville theorem revisited	466
§5. Eigenvalues and eigenfunctions of the Laplacian	467
§6. The determinant of the Laplacian	476
§7. Parametrix for the heat equation	485
§8. Monotonicity for harmonic functions and maps	492
§9. Bieberbach theorem	494
§10. Notes and commentary	500
Appendix B. Analytic Techniques for Geometric Flows	503
§1. Riemannian surfaces	503
§2. Kazdan-Warner-type identities and solitons	516
§3. Andrews' Poincaré-type inequality	517
§4. The Yamabe flow and Aleksandrov reflection	520
§5. The cross curvature flow	528
§6. Time derivative of the sup function	531
§7. Notes and commentary	532
Appendix S. Solutions to Selected Exercises	535

Bibliography

573

Index

603