*Chapter II*

# Nonlinear dyadic analysis, microlocal analysis, energy estimates

At first sight it might seem a little ridiculous to present 'applications' of a notion such as that of pseudo-differential operators which is encountered very commonly in nature: nevertheless, without wishing to refer the reader to specialist articles or papers, we describe certain aspects of the applications in an elementary and very limited way here. While Sections A and B can be viewed as natural developments (rich in consequences) of the symbolic calculus of the classical pseudo-differential presented in Chapter I, Section C uses the efficiency of the pseudo-differential *tool* to obtain vital hyperbolic energy estimates.

These estimates can be used to obtain Hörmander's theorem on the propagation of singularities (one of the key results of microlocal analysis); in addition, they are central to the nonlinear perturbation techniques presented in Chapter III.

## A. Nonlinear dyadic analysis

### 1. Littlewood–Paley decomposition: general properties.

1.1. *Littlewood–Paley decomposition.* Suppose $\psi \in C_0^\infty(\mathbb{R}^n)$, $\psi(\xi) = 1$ for $|\xi| \leq 1/2$, $\psi(\xi) = 0$ for $|\psi| \geq 1$. Set $\varphi(\xi) = \psi(\xi/2) - \psi(\xi)$: $\varphi$ is supported in the shell $1/2 \le |\xi| \le 2$, and, for all $\xi$,

$$1 = \psi(\xi) + \sum_{p \ge 0} \varphi(2^{-p}\xi)$$

(there are never more than two non-zero terms in this series). For $u \in \mathcal{S}'(\mathbb{R}^n)$, set

$$u_{-1} = S_0 u = \psi(D)u, \quad u_p = \varphi(2^{-p}D)u$$

so that

$$u = S_0 u + \sum_{p \ge 0} u_p.$$

This is the Littlewood–Paley decomposition of $u$. The partial sums will be denoted by $S_p u = \sum_{q=-1}^{p-1} u_q$. Each term is holomorphic in $\mathbb{C}^n$ (and, in fact, in $H^{+\infty}$ whenever $u$ is in $H^s$) and the 'quality' of $u$ is reflected in the rate of convergence of $\sum u_p$. Two lemmas will be constantly used. Let us recall the notation: $|u|_s$ denotes the norm of $u$ in $H^s$ $(s \in \mathbb{R})$, $\|u\|_0$ the $L^\infty$ norm.

**Lemma 1.1.1** (Almost-orthogonality of the terms)**.** *We have*

$$1/2 \le \psi^2(\xi) + \sum_{p \ge 0} \varphi^2(2^{-p}\xi) \le 1, \tag{1.1.1}$$

*and for all* $u \in L^2$

$$\sum_{p \ge -1} |u_p|_0^2 \le |u|_0^2 \le 2 \sum_{p \ge -1} |u_p|_0^2. \tag{1.1.2}$$

**Proof.**

$$\psi^2(\xi) + \sum \varphi^2(2^{-p}\xi) \le \Big[\psi(\xi) + \sum \varphi(2^{-p}\xi)\Big]^2 = 1,$$

and

$$1 = \Big[\psi(\xi) + \sum \varphi(2^{-p}\xi)\Big]^2 \le \Big(\psi^2(\xi) + \sum \varphi^2(2^{-p}\xi)\Big)$$

by virtue of the inequality $(a+b)^2 \le 2(a^2+b^2)$. Since

$$|u|_0^2 = \text{Const.} \int |\hat{u}(\xi)|^2 \mathrm{d}\xi,$$

in order to obtain (1.1.2), it suffices to multiply the two members of (1.1.1) by $|\hat{u}(\xi)|^2$ and to integrate, remembering that $\hat{u}_p(\xi) = \varphi(2^{-p}\xi)\hat{u}(\xi)$ and $\hat{u}_{-1}(\xi) = \psi(\xi)\hat{u}(\xi)$, by definition. □

This lemma results from the fact that $(u_p, u_q)_{L^2} = 0$ whenever $|p-q| \ge 2$: the terms $u_p$ are not orthogonal, but almost, and Lemma 1.1.1 is just Pythagoras's theorem corresponding to this almost orthogonality.

More generally, if $\{u_p\}$ is a sequence of $L^2$ functions with $\operatorname{supp} \hat{u}_p \subset \{\xi, \frac{1}{C}2^{p-1} \le |\xi| \le C2^{p+1}\}$, we always have the inequality

$$\Big|\sum u_p\Big|_0^2 \le \text{Const.} \sum |u_p|_0^2. \tag{1.1.2'}$$

In what follows, given a function $u$ on $\mathbb{R}^n$, we shall denote its $L^\infty$ norm by $\|u\|_0$.

**Lemma 1.1.2** (Sensitivity of terms to derivations)**.** *There exist constants $C$, independent of $p$ and of $u$, for which:*

i) *For all $\alpha \in \mathbb{N}^n$, $p \ge -1$*

$$|\partial^\alpha u_p|_0 \le C2^{p|\alpha|}|u|_0, \quad |\partial^\alpha S_p u|_0 \le C2^{p|\alpha|}|u|_0, \tag{1.1.3}$$

$$\|\partial^\alpha u_p\|_0 \le C2^{p|\alpha|}\|u\|_0, \quad \|\partial^\alpha S_p u\|_0 \le C2^{p|\alpha|}\|u\|_0. \tag{1.1.4}$$

ii) *For all $s \in \mathbb{R}$ and $p \ge 0$,*

$$\frac{1}{C}2^{ps}|u_p|_0 \le |u_p|_s \le C2^{ps}|u_p|_0. \tag{1.1.5}$$

iii) *For all $k \in \mathbb{N}$ and $p > 0$,*

$$\frac{1}{C}2^{pk}\|u_p\|_0 \le \sum_{|\alpha|=k} \|\partial^\alpha u_p\|_0 \le C2^{pk}\|u_p\|_0. \tag{1.1.6}$$

**Proof.** a) By definition

$$|u_p|_s^2 = (2\pi)^{-n} \int (1+|\xi|^2)^s \varphi^2(2^{-p}\xi)|\hat{u}(\xi)|^2 \mathrm{d}\xi;$$

since $(1+|\xi|^2)^s$ is bounded above and below on the support of $\varphi(2^{-p}\xi)$ by Const. $\times\, 2^{2ps}$, we immediately obtain (1.1.3) and (1.1.5).

b) Denoting the inverse Fourier transform of $\Phi$ by $\check{\Phi}$ (that is, $\hat{\check{\Phi}} = \Phi$), we have $\Phi(D)u = \check{\Phi} * u$, and if $\Phi(\xi) = \Phi_1(\mu\xi)$ ($\mu \in \mathbb{R}$; $\Phi_1$ fixed), $\check{\Phi}(x) = \mu^{-n}\check{\Phi}_1(x/\mu)$, so that

$$\int_{\mathbb{R}^n} |\check{\Phi}(x)|\mathrm{d}x = \int_{\mathbb{R}^n} |\check{\Phi}_1(y)|\mathrm{d}y$$

is independent of $\mu$. We deduce that $\|\check{\Phi} * u\|_0 \le C\|u\|_0$, where $C$ does not depend on $\mu$. For all $\alpha$, we have

$$\partial^\alpha(\Phi(D)u) = (\partial^\alpha)\check{\Phi} * u = \mu^{-|\alpha|}(\mu^{-n}(\partial^\alpha\check{\Phi}_1)(x/\mu)) * u,$$

whence

$$\|\partial^\alpha(\Phi(D)u)\|_0 \le C\mu^{-|\alpha|}\|u\|_0;$$

applying this to $\Phi(\xi) = \varphi(2^{-p}\xi)$ (which defines $u_p$) or to $\Phi(\xi) = \psi(2^{-p}\xi)$ (which defines $S_p u$), we obtain (1.1.4). Similarly,

$$\begin{aligned}\partial^\alpha \hat{u}_p &= \text{Const.}\, \xi^\alpha \varphi(2^{-p}\xi)\hat{u}(\xi) \\ &= \text{Const.}\, 2^{p|\alpha|}(2^{-p}\xi)^\alpha \varphi(2^{-p}\xi)\hat{u}(\xi) \\ &= \text{Const.}\, 2^{p|\alpha|}\Phi_1(2^{-p}\xi)\hat{u}_p(\xi),\end{aligned}$$

where $\Phi_1(\xi) = \xi^\alpha \chi(\xi)$, $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi = 1$ near the support of $\varphi$. We thus obtain the right-hand inequality of (1.1.6).

c) To obtain (1.1.6), we write, for $\chi \in C_0^\infty$ equal to 1 near $\operatorname{supp}\varphi$,

$$\varphi(\xi) = \left(\sum_{|\alpha|=k} \xi^\alpha \chi_\alpha(\xi)\right)\varphi(\xi),$$

where

$$\chi_\alpha(\xi) = \frac{\xi^\alpha \chi(\xi)}{\sum_{|\alpha|=k}(\xi^\alpha)^2} \in C_0^\infty,$$

and we obtain

$$\begin{aligned}\hat{u}_p(\xi) &= \varphi(2^{-p}\xi)\hat{u}(\xi) = \sum (2^{-p}\xi)^\alpha \chi_\alpha(2^{-p}\xi)\hat{u}_p(\xi) \\ &= 2^{-pk}\sum \chi_\alpha(2^{-p}\xi)\widehat{D^\alpha u_p}(\xi),\end{aligned}$$

whence $2^{pk}u_p = \sum (2^{pn}\check{\chi}_\alpha(2^p\cdot)) * D^\alpha u_p$ and the conclusion follows. □

1.2. *Characterization of Sobolev spaces.*

**Proposition 1.2.** i) *If $u \in H^s(\mathbb{R}^n)$, then for all $p \geq -1$,*

$$|u_p|_0 \leq \text{Const.}\, |u|_s c_p 2^{-ps},$$

*where $c_p = c_p(u)$ satisfies $\sum c_p^2 \leq 1$.*

ii) *Conversely, if, for $p \geq -1$, $|u_p|_0 \leq C c_p 2^{-ps}$, with $\sum c_p^2 \leq 1$, then $u \in H^s$ and $|u|_s \leq \text{Const.}\, C$.*

**Proof.** Since $(\langle D\rangle^s u)_p = \langle D\rangle^s u_p$, this is an immediate consequence of (1.1.5) and of the characterization of $L^2$ provided by Lemma 1.1.1. □

1.3. *Characterization of Hölder spaces.* For $\alpha > 0$, $\alpha \notin \mathbb{N}$, we define $C^\alpha = C^\alpha(\mathbb{R}^n)$ to be the space of $u \in C^k(\mathbb{R}^n)$ ($k = E(\alpha) =$ integer part of $\alpha$), which are bounded together with their derivatives up to order $k$, such that

$$\exists C, \forall x, \forall y, \forall \beta, |\beta| = k,$$

$$|\partial^\beta u(x) - \partial^\beta u(y)| \leq \mathrm{C}|x-y|^{\alpha-k}. \tag{1.3.1}$$

The norm in $C^\alpha$ will be denoted by $\|u\|_\alpha$ and defined by $\|u\|_\alpha = \|u\|_0 + \|u\|'_\alpha$, where $\|u\|'_\alpha$ denotes the best constant in (1.3.1).

**Proposition 1.3.** i) *If $u \in C^\alpha(\mathbb{R}^n)$ $(\alpha \notin \mathbb{N})$, then for all $p \geq -1$,*

$$\|u_p\|_0 \leq \text{Const.}\, \|u\|_\alpha 2^{-p\alpha}.$$

ii) *Conversely, if, for $p \geq -1$, $\|u_p\|_0 \leq C2^{-p\alpha}$ $(\alpha \notin \mathbb{N})$, then $u \in C^\alpha$ and $\|u\|_\alpha \leq \text{Const.}\, C$.*

**Proof.** We remark that the inequalities (1.1.6) and the fact that $(\partial^\alpha u)_p = \partial^\alpha u_p$ permit an immediate reduction to the case $0 < \alpha < 1$.

a) Since $\|u_{-1}\|_0 \leq \text{Const.}\, \|u\|_0$, it suffices to consider

$$u_p(x) = \int 2^{pn} \check{\varphi}(2^p(x-y))u(y)\mathrm{d}y \text{ for } p \geq 0.$$

From the fact that $\int \check{\varphi}(z)\mathrm{d}z = \text{Const.}\, \varphi(0) = 0$, we also have

$$u_p(x) = \int 2^{pn} \check{\varphi}(2^p(x-y))(u(y)-u(x))\mathrm{d}y,$$

whence

$$\begin{aligned} |u_p(x)| &\leq \|u\|_\alpha \int 2^{pn} |\check{\varphi}|(2^p(x-y))|x-y|^\alpha \mathrm{d}y \\ &\leq \text{Const.}\, \|u\|y_\alpha 2^{-p\alpha}, \end{aligned}$$

which proves i).

b) Conversely, for some $p$ to be determined, set

$$u = S_p u + R_p u, \quad R_p u = \sum_{q \geq p} u_q;$$

then we have

$$\|R_p u\|_0 \leq \sum_{q \geq p} \|u_p\|_0 \leq C2^{-p\alpha}.$$

Moreover,

$$|S_p u(x) - S_p u(y)| \leq |x-y| \sum_{q=-1}^{p-1} \|\nabla u_q\|_0;$$

following (1.1.6),

$$\|\nabla u_q\|_0 \leq \text{Const.}\, C2^{q(1-\alpha)},$$

and of course $\|\nabla u_{-1}\|_0 \leq \text{Const.}\, C$; hence, if $0 < \alpha < 1$,

$$|S_p u(x) - S_p u(y)| \leq \text{Const.}\, C|x-y|2^{p(1-\alpha)},$$

because the series $\sum \|\nabla u_q\|_0$ is then geometrically divergent.

Regrouping the estimates for $R_p u$ and $S_p u$, we find

$$|u(x) - u(y)| \leq \text{Const.}\, C|x-y|2^{p(1-\alpha)} + 2C2^{-p\alpha}.$$

If we take $p$ to be the largest integer such that $2^p \leq \frac{1}{|x-y|}$, we obtain

$$|u(x) - u(y)| \leq \text{Const.}\, C|x-y|^\alpha,$$

which proves ii). □

Of course, one should be aware that $\|u_p\|_0 \leq C$ does not characterize $L^\infty$ and that $\|u_p\|_0 \leq C2^{-p}$ does not characterize $C^1$ (in the classical sense), but a larger space (see Exercise A.3).

1.4. *Sobolev injections.*

**Proposition 1.4.** *For $s > n/2$ , $s - n/2 \notin \mathbb{N}$, $H^s \subset C^{s-n/2}$ (continuous injection).*

**Proof.** We write $u_p(x) = (2\pi)^{-n} \int e^{\mathrm{i}x\xi} \hat{u}_p(\xi) \mathrm{d}\xi$, whence

$$\begin{aligned}
\|u_p\|_0 &\leq \text{Const.} \int_{|\xi| \leq C2^p} |\hat{u}_p(\xi)| \mathrm{d}\xi \\
&\leq \text{Const.}\, |\hat{u}_p|_0 [\text{Volume} B(0, C2^p)]^{1/2} \\
&\leq \text{Const.}\, 2^{pn/2} |u|_s c_p 2^{-ps},
\end{aligned}$$

following Proposition 1.2.

'Forgetting' $c_p \leq 1$, the result follows using Proposition 1.3. □

1.5. *Convexity inequalities.*

**Proposition 1.5.** i) *If $s = \lambda s_0 + (1-\lambda)s_1$ ($0 \leq \lambda \leq 1, s_0 < s_1, s_0, s_1 \in \mathbb{R}$), then for all $u \in C_0^\infty(\mathbb{R}^n)$, we have the inequality*

(1.5.1) $$|u|_s \leq \text{Const.}\, |u|_{s_0}^\lambda |u|_{s_1}^{1-\lambda}.$$

ii) *If $\alpha = \lambda\alpha_0 + (1-\lambda)\alpha_1$ ($0 \leq \lambda \leq 1$, $\alpha_0 < \alpha_1$, $\alpha_0, \alpha_1$ positive and not in $\mathbb{N}$), then for all $u \in C^{\alpha_1}$ we have*

(1.5.2) $$\|u\|_\alpha \leq \text{Const.}\, \|u\|_{\alpha_0}^\lambda \|u\|_{\alpha_1}^{1-\lambda}.$$

**Proof.** a)

$$\begin{aligned}
|u|_s^2 &= \int (1+|\xi|^2)^s |\hat{u}(\xi)|^2 \mathrm{d}\xi \\
&= \int (1+|\xi|^2)^{\lambda s_0} |\hat{u}(\xi)|^{2\lambda} (1+|\xi|^2)^{(1-\lambda)s_1} |\hat{u}(\xi)|^{2(1-\lambda)} \mathrm{d}\xi,
\end{aligned}$$

and the result follows using Hölder's inequality with '$p = 1/\lambda$' and '$q = 1/1-\lambda$'.

b) We write

$$\begin{aligned}
\|u_p\|_0 \leq \|u_p\|_0^\lambda \|u_p\|_0^{1-\lambda} &\leq \text{Const.}\, \|u\|_{\alpha_0}^\lambda (2^{-p\alpha_0})^\lambda \|u\|_{\alpha_1}^{1-\lambda} (2^{-p\alpha_1})^{1-\lambda} \\
&\leq \text{Const.} \|u\|_{\alpha_0}^\lambda \|u\|_{\alpha_1}^{1-\lambda} 2^{-p\alpha},
\end{aligned}$$

and the result now follows by Proposition 1.3. □

*Remark* It is useful to know that (1.5.2) is still true for $\alpha_0, \alpha_1 \geq 0$, by defining $C^\alpha$ to be $L^\infty$ for $\alpha = 0$, and the space of Lipschitz functions of order $\leq \alpha - 1$ for $\alpha \in \mathbb{N}$, $\alpha \geq 1$ (see Exercise I.2.4 a)).

Here we see very well how the characterization given in Proposition 1.3 works: in its very terms it contains a number of useful properties of Hölder spaces (for example (1.5.2)) which one does not need to know explicitly.

1.6. *Regularization operators.*

**Proposition 1.6.** *There exists a family* $S_\theta$ *(*$\theta \geq 1$*) of operators* $S_\theta : \bigcup_{\alpha \geq 0} C^\alpha \to \bigcap_{\beta \geq 0} C^\beta$ *with the following properties:*

i) $\|S_\theta u\|_\alpha \leq \mathrm{Const.}\|u\|_\beta, \alpha \leq \beta,$

ii) $\|S_\theta u\|_\alpha \leq \mathrm{Const.}\theta^{\alpha-\beta}\|u\|_\beta, \alpha \geq \beta,$

iii) $\|S_\theta u - u\|_\alpha \leq \mathrm{Const.}\theta^{\alpha-\beta}\|u\|_\beta, \alpha \leq \beta,$

iv) $\|\frac{\mathrm{d}}{\mathrm{d}\theta} S_\theta u\|_\alpha \leq \mathrm{Const.}\theta^{\alpha-\beta-1}\|u\|_\beta, \forall \alpha, \beta.$

*When* $\alpha$ *or* $\beta$ *is an integer, here we understand* $C^\alpha$ *to be the space defined by the characterization of Proposition 1.3 with the corresponding norm.*

**Proof.** For $\chi \in C_0^\infty(\mathbb{R})$, $\chi = 1$ in the neighbourhood of the origin, we set

$$S_\theta u = \sum_p \chi(2^p/\theta) u_p,$$

implying that $(S_\theta u_p) = 0$ for $2^p \geq \mathrm{Const.}\theta$ and $\|(S_\theta u)_p\|_0 \leq \mathrm{Const.}\|u\|_\beta 2^{-p\beta}$ otherwise.

In particular, if $\alpha \geq \beta$,

$$\|(S_\theta u)_p\|_0 \leq \mathrm{Const.} 2^{-p\alpha} 2^{p(\alpha-\beta)} \|u\|_\beta$$

implies ii).

Since $S_\theta u - u = \sum(\chi(2^p/\theta) - 1)u_p$, we have $(S_\theta u - u)_p = 0$ for $2^p \leq \mathrm{Const.}\theta$ and $\|(S_\theta u - u)_p\|_0 \leq \mathrm{Const.}\ \|u\|_\beta 2^{-p\beta}$ otherwise; writing

$$\|(S_\theta u - u)_p\|_0 \leq \mathrm{Const.}\|u\|_\beta 2^{-p\alpha} 2^{p(\alpha-\beta)},$$

we obtain iii).

Finally, $\frac{\mathrm{d}}{\mathrm{d}\theta} S_\theta u = \frac{1}{\theta} \sum (\chi_1)(2^p/\theta) u_p$ with $\chi_1(z) = -z\chi'(z)$, and the same arguments as before give iv), because $\mathrm{Const.}\theta \leq 2^p \leq \mathrm{Const.}\theta$ on the support of $\chi_1$. □

## 2. Application to the study of products and composition.

2.1. *Estimates of a product of two functions.*

**Proposition 2.1.1.** i) *If $u, v \in C^\alpha$ $(\alpha \notin \mathbb{N})$,*

$$\|uv\|_\alpha \leq \text{Const.}(\|u\|_0\|v\|_\alpha + \|u\|_\alpha\|v\|_0). \tag{2.1.1}$$

ii) *If $u, v \in L^\infty \cap H^2$ $(s > 0)$, then so also is $uv$, and*

$$|uv|_s \leq \text{Const.}(\|u\|_0|v|_s + |u|_s\|v\|_0). \tag{2.1.2}$$

**Proof.** Let us write

$$u = \sum_p u_p, \quad v = \sum_q v_q,$$

and

$$uv = \sum_{p,q} u_p v_q = \sum_q (S_q u)v_q + \sum_p u_p S_{p+1} v = \Sigma_1 + \Sigma_2.$$

The terms $\Sigma_1$ and $\Sigma_2$ have their spectra (that is to say the supports of their Fourier transforms) in the ball $\{|\xi| \leq \text{Const.}2^p\}$. We shall use the following lemma.

**Lemma 2.1.** *Let $(a_q)_{q\geq -1}$ be a sequence of functions such that*

$$\operatorname{supp} \hat{a}_q \subset \{\xi, |\xi| \leq \text{Const.}2^q\}.$$

*Suppose that $\|a_q\|_0 \leq C2^{-q\alpha}$ for some $\alpha > 0$ (resp. $|a_q|_0 \leq Cc_q 2^{-qs}$ for some $s > 0$, with $\sum c_q^2 \leq 1$).*

*Then $u = \sum_{q\geq -1} a_q$ belongs to $C^\alpha$ (resp. $u \in H^s$), with $\|u\|_\alpha \leq \text{Const.}C$ (resp. $|u|_s \leq \text{Const.}C$).*

**Proof.** It suffices to observe that for some $N$, the dyadic blocks $u_p$ of $u$ satisfy $u_p = \sum_{q\geq p-N}(a_q)_p$, where, denoting the $L^2$ or $L^\infty$ norm by $|\ |$,

$$|u_p| \leq \sum_{q\geq p-N} |(a_q)_p| \leq \text{Const.} \sum_{q\geq p-N} |a_q|,$$

following Lemma 1.1.2 i).

In the case of $L^\infty$, we deduce

$$\|u_p\|_0 \leq \text{Const.}C \sum_{q\geq p-N} 2^{-q\alpha} \leq \text{Const.}C2^{-p\alpha}.$$

In the case of $L^2$,

$$\begin{aligned} |u_p|_0 \quad \leq \quad & \text{Const.}C \sum_{1 \geq p-N} c_q 2^{-qs} \\ & \leq \text{Const.}C 2^{-ps} \left( \sum_{q \geq p-N} c_2^2 2^{-(q-p)s} \right)^{1/2}, \end{aligned}$$

following Schwarz's inequality applied to $c_q 2^{-(q-p)s/2} \times 2^{-(q+p)s/2}$, and it remains to remark that

$$\sum_p \sum_{q \geq p-N} c_q^2 2^{-(q-p)s} \leq \text{Const.}$$

□

We note the crucial role played in these estimates by the geometric nature of the partitioning used.

Taking into account the lemma, it suffices to evaluate $\|(S_q u)v_q\|_0$. Now

$$\|(S_q u)v_q\|_0 \leq \|S_q u\|_0 \|v_q\|_0 \leq \text{Const.}\|u\|_0 \|v\|_\alpha 2^{-q\alpha},$$

whence $\|\Sigma_1\|_\alpha \leq \text{Const.}\|u\|_0\|v\|_\alpha$, and similarly for $\Sigma_2$ with $u$ and $v$ interchanged.

To prove ii), we write

$$|(S_q u)v_q|_0 \leq \|S_q u\|_0 |v_q|_0 \leq \text{Const.}\|u\|_0 |v|_s c_q 2^{-qs},$$

whence $|\Sigma_1|_s \leq \text{Const.}\|u\|_0 |v|_s$ and similarly for $\Sigma_2$.

If, in the above proof, we replace $uv = \sum_{p,q} u_p v_q = \Sigma_1 + \Sigma_2$ by the finer

$$\begin{aligned} uv \quad = \quad & \sum_{p \leq q-3} u_p v_q + \sum_{q \leq p-3} u_p v_q + \sum_{|p-q| \leq 2} u_p v_q \\ = \quad & \sum (S_{q-2}u)v_q + \sum (S_{p-2}v)u_p + \sum_{|p-q| \leq 2} u_p v_q = \Sigma_1' + \Sigma_2' + \Sigma_3', \end{aligned}$$

we remark that the terms of the sums $\Sigma_1'$ and $\Sigma_2'$ have their spectra in *shells* (and not only in balls); for example, for $\Sigma_1'$,

$$\begin{aligned} \text{Spectrum}\, S_{q-2}u \quad & \subset \quad \{\xi, |\xi| \leq 2^{q-2}\}, \\ \text{Spectrum}\, v_q \quad & \subset \quad \left\{\xi, \frac{1}{2} 2^q \leq |\xi| \leq 2 \cdot 2^q \right\}, \end{aligned}$$

whence

$$\text{Spectrum}\,(S_{q-2}u)v_q \subset \text{Sum of spectra} \subset \{\xi, 2^{q-2} \leq |\xi| \leq 9 \cdot 2^{q-2}\}.$$

Moreover, the terms $u_p$ and $v_q$ are both small if $u$ and $v$ are sufficiently regular, so that the last sum, whose terms are small like $u_p v_q$, represents an 'error term' which is more regular than $u$ or $v$. □

This approach is the starting point for J.-M. Bony's introduction of the concept of the 'paraproduct', where the paraproduct (nonsymmetric) of $u$ and $v$ is $T_u v = \sum(S_{q-2}u)v_q$ (see Exercise A.5); we then have $uv = T_u v + T_v u +$ better remainder. This concept and the associated notion of 'paradifferential operators' have turned out to be very fruitful in the study of singularities of solutions of nonlinear equations (cf. Bony [B1]).

We remark that if $\alpha \in \mathbb{N}$, part i) of the proposition is elementary, taking into account the remark of Section 1.5. A convenient version in applications is given by the following proposition.

**Proposition 2.1.2.** *If $u, v \in L^\infty \cap H^s$ ($s > 0$ integer), then for all $\alpha, \beta$, with $|\alpha| + |\beta| = s$, we have*

$$|(\partial^\alpha u)(\partial^\beta v)|_0 \leq \text{Const.}(\|u\|_0 |v|_s + |u|_s \|v\|_0). \tag{2.1.3}$$

**Proof.** The proposition is self-evident if $|\alpha| = 0$ or $|\beta| = 0$. Otherwise (for example, $|\alpha| \geq 1$), we write

$$\partial^\alpha u \partial^\beta v = \sum_j * \partial_{i_j}(\partial^{\alpha_j} u \partial^{\beta_j} v) + *u\partial^{\alpha+\beta} v,$$

where $|\alpha_j| + |\beta_j| = s - 1$ and $*$ denotes coefficients of no importance.

To prove (2.1.3), it suffices to show that

$$|\partial^{\alpha_j} u \partial^{\beta_j} v|_1 \leq \text{Const.}(\|u\|_0 |v|_s + |u|_s \|v\|_0).$$

We then proceed as in the proof of Proposition 2.1.1:

$$\partial^{\alpha_j} u \partial^{\beta_j} v = \sum (S_q \partial^{\alpha_j} u)(\partial^{\beta_j} v)_q + \sum (\partial^{\alpha_j} u)_p (S_{p+1} \partial^{\beta_j} v) = \Sigma_1 + \Sigma_2,$$

and this time we have

$$\begin{aligned} |S_q(\partial^{\alpha_j} u)(\partial^{\beta_j} v)_q|_0 &\leq \|S_q \partial^{\alpha_j} u\|_0 |(\partial^{\beta_j} v)_q|_0 \\ &\leq \text{Const.}\|u\|_0 2^{q|\alpha_j|} |v|_s c_q 2^{-q(s-|\beta_j|)} \\ &\leq \text{Const.}\|u\|_0 |v|_s 2^{-q} c_q, \end{aligned}$$

because of Lemma 1.1.2 iii), which proves the result. □

We remark that the analogue of (2.1.3) in Hölder spaces is a simple result of the convexity inequalities of Section 1.5.

In fact, it is traditional to prove (2.1.3) using the following (so-called Gagliardo–Nirenberg) inequality (see [Au]): if $u \in L^\infty \cap H^s$ (integer $s > 0$), then for all $\alpha$, $0 \leq |\alpha| \leq s$,

$$|\partial^\alpha u|_{L^p} \leq \text{Const.}\|u\|_0^{1-|\alpha|/s}|u|_s^{|\alpha|/s},$$

where $p = 2s/|\alpha|$.

Assuming this inequality, we obtain, via the Hölder inequality ($p = 2s/|\alpha|$, $q = 2s/|\beta|$)

$$\begin{aligned} |(\partial^\alpha u)(\partial^\beta v)|_0 &\leq |\partial^\alpha u|_{L^p}|\partial^\beta v|_{L^q} \\ &\leq \text{Const.}(\|u\|_0|v|_s)^{-1-|\alpha|/s}(|u|_s\|v\|_0)^{|\alpha|/s} \\ &\leq \text{Const.}(\|u\|_0|v|_s + |u|_s\|v\|_0), \end{aligned}$$

by convexity of the exponential $a^\mu b^{1-\mu} \leq \mu a + (1-\mu)b$, where the argument is parallel to that for the spaces $C^\alpha$.

2.2. *Estimation of a composite function.*

**Proposition 2.2.** *Let $F : \mathbb{R} \to \mathbb{C}$ be a $C^\infty$ function with $F(0) = 0$. If $u \in L^\infty \cap H^s$ ($s > 0$), then $F(u) \in L^\infty \cap H^s$ and $|F(u)|_s \leq C|u|_s$, where $C$ depends only on $F$ and on $\|u\|_0$.*

The proof uses the following lemma.

**Lemma 2.2** (Meyer multipliers)**.** *Let $\delta \in \mathbb{R}$, and suppose we have a sequence $m_p \in C^\infty$ with, for all $k \in \mathbb{N}$,*

$$\sum_{|\alpha|=k} \|\partial^\alpha m_p\|_0 \leq C_k 2^{p(k+\delta)}.$$

*The mapping $M : u \mapsto \sum m_p u_p = Mu$ maps $H^s$ to $H^{s-\delta}$ for all $s > \delta$, with operator norm depending only on the $C_k$ for $k \leq E(s-\delta)+1$.*

**Proof.** Since the spectrum of $u_p$ is contained in $\{\xi, 1/2 \leq 2^{-p}|\xi| \leq 2\}$, let us choose $C > 4$ and partition $m_p$ according to the formula

$$\hat{m}_p = \psi(2^{-p}\xi/C)\hat{m}_p + \sum_{k\geq 0}\varphi(2^{-k}2^{-p}\xi/C)\hat{m}_p = \hat{m}_{p,-1} + \sum_{k\geq 0}\hat{m}_{p,k}.$$

We then set $M_k u = \sum m_{p,k}u_p$, $k \geq -1$.

The terms of $M_{-1}u$ have their spectra in balls $\{\xi, |\xi| \leq (C+2)2^p\}$ and $|m_{p,-1}u_p| \leq \text{Const.}|m_{p,-1}|_0 c_p |u|_s 2^{-ps}$. Since, following Lemma 1.1.2 i),

$$\|m_{p,-1}\|_0 \leq \text{Const.}\|m_p\|_0 \leq \text{Const.}2^{p\delta}C_0,$$

Lemma 2.1 shows that $M_{-1}u \in H^{s-\delta}$ if $s > \delta$.

The terms $M_k u$ ($k \geq 0$) have their spectra in the shells

$$\left\{\xi, 2^{p+1}\left(\frac{C}{4}2^k - 1\right) \leq |\xi| \leq 2^{p+1}(1 + C2^k)\right\}$$

and

$$|m_{p,k}u_p|_0 \leq \text{Const.}\|m_{p,k}\|_0 c_p |u|_s 2^{-ps}.$$

Since, following Lemma 1.1.2 i) and iii),

$$\begin{aligned}\|m_{p,k}\|_0 &\leq \text{Const.}\sum_{|\alpha|=\ell}\|\partial^\alpha m_{p,k}\|_0 2^{-(p+k)\ell}\\ &\leq \text{Const.}C_\ell 2^{-k\ell}2^{p\delta},\end{aligned}$$

we obtain, for all $\ell \in \mathbb{N}$,

$$\begin{aligned}|m_{p,k}u_p|_0 &\leq \text{Const.}C_\ell |u|_s 2^{-k\ell}c_p 2^{-p(s-\delta)}\\ &\leq \text{Const.}C_\ell |u|_s 2^{k(s-\ell-\delta)}C_p 2^{-(p+k)(s-\delta)}.\end{aligned}$$

We deduce that $M_k u \in H^{s-\delta}$ with $|M_k u|_{s-\delta} \leq \text{Const.}C_\ell 2^{k(s-\ell-\delta)}|u|_s$, following the remark after Lemma 1.1.1 and (1.1.2′). Finally, choosing $\ell > s-\delta$, $M = \sum_{k\geq -1} M_k$ converges normally in the space of continuous operators from $H^s$ to $H^{s-\delta}$, and $\|M\| \leq \text{Const.}(C_0 + C_\ell)$. □

A typical example of such 'Meyer multipliers' (where $\delta = 0$) is given by $m_p = S_p a$, for some $a \in L^\infty$ (because of Lemma 1.1.2). Of course, in that case, $m_p$ has its spectrum in a ball $\{\xi, |\xi| \leq C2^p\}$, which is not assumed in general; nevertheless, the proof of Lemma 2.2 consists precisely of showing that one can essentially reduce to that situation.

We remark (Exercise A.7) that $M$ is a pseudo-differential operator of order $\delta$, whose symbol $m(x,\xi) = \sum m_p(x)\varphi(2^{-p}\xi)$ does not however satisfy the standard estimates: its action from $H^s$ to $H^{s-\delta}$ is thus not as evident as it seems, and it is therefore subject to the condition $s > \delta$.

**Proof of Proposition 2.2.** We use the so-called 'telescopic series' trick, which involves writing

$$F(u) = F(S_0u) + F(S_1u) - F(S_0u) + \cdots + F(S_{p+1}u) - F(S_pu) + \ldots,$$

then

$$F(S_{p+1}u) - F(S_pu) = m_p u_p, \text{ with } m_p = \int_0^1 F'(S_pu + tu_p)\mathrm{d}t.$$

a) If $u \in L^\infty \cap L^2$, for all $\alpha$, $\partial^\alpha(F(S_0u)) \in L^\infty \cap L^2$: indeed

$$\partial^\alpha(F(S_0u)) = \sum * F^{(q)}(S_0u)(\partial^{\gamma_1}S_0u)\ldots(\partial^{\gamma_q}S_0u),$$

where the $\gamma_j$ are multiple indices, $\gamma_1 + \cdots + \gamma_q = \alpha$, $1 \leq q \leq |\alpha|$, $|\gamma_j| \geq 1$. Each term $\partial^\gamma S_0u$ is in $L^2 \cap L^\infty$, with $\|\partial^\gamma S_0 u\|_0 \leq \text{Const.}\|u\|_0$, $|\partial^\gamma S_0u|_0 \leq \text{Const.}|u|_0$, whence the result follows if $|\alpha| \geq 1$. For $|\alpha| = 0$, $|F(S_0u)(x)| \leq C|S_0u(x)|$ (where $C$ depends only on $\|u\|_0$), whence $|F(S_0u)|_0 \leq C\text{Const.}|u|_0$.

b) Let us verify that $m_p$ is a 'Meyer multiplier' of order $\delta = 0$. It suffices to consider $\tilde{m}_p = G(S_p u)$. We then have

$$\partial^\alpha G(S_p u) = \sum *G^{(q)}(S_p u)(\partial^{\gamma_1} S_p u) \dots (\partial^{\gamma_q} S_p u),$$

as in a), and $\|\partial^\gamma S_p u\|_0 \leq \text{Const.}\|u\|_0 2^{p|\gamma|}$ following Lemma 1.1.2. Hence

$$\|\partial^\alpha G(S_p u)\|_0 \leq \text{Const.}2^{p(|\gamma_1|+\dots+|\gamma_q|)} = \text{Const.}2^{p|\alpha|},$$

where the constant depends only on $G$, $\alpha$ and $\|u\|_0$, which completes the proof. □

Here again, we refer readers to J.-M. Bony for the following so-called 'paralinearization' formula which makes Proposition 2.2 evident: if $u \in H^s$, $s > n/2$, $F(u) = T_{F'(u)}u + R(u)$, with $T$ the paraproduct defined in Section 2.1 (cf. Exercise A.5), and $R(u) \in H^{2s-n/2}$ which is strictly contained in $H^s$. In other words, the estimate for $F(u)$ is 'linear in $u$' since in fact $F(u)$ is equal to $T_{F'(u)}$, up to a residue.

## B. Microlocal analysis: wave front set and pseudo-differential operators

### 1. Wave front set of a distribution.

1.1. *Definition of the wave front set.* The Fourier transform $\hat{u}(\xi)$ of a function $u \in C_0^\infty(\mathbb{R}^n)$ is rapidly decreasing in $\xi$, that is,

$$\forall k, |\hat{u}(\xi)| \leq C_k(1 + |\xi|)^{-k}. \tag{1.1.1}$$

Conversely, if (1.1.1) is satisfied by the Fourier transform of a distribution $u \in \mathcal{E}'(\mathbb{R}^n)$, then $u \in C_0^\infty(\mathbb{R}^n)$, because of the Fourier inversion formula.

**Definition 1.1.1.** For $u \in \mathcal{E}'(\mathbb{R}^n)$, let $\Sigma(u)$ be the complement (in $\mathbb{R}^n \setminus 0$) of the set of directions $\xi \in \mathbb{R}^n \setminus 0$ in the neighbourhood (conical) of which $\hat{u}$ satisfies (1.1.1). By 'conical', we mean the following property of a set $\Gamma$: $\xi \in \Gamma$, $\lambda > 0 \Rightarrow \lambda\xi \in \Gamma$. Just as sing supp $u = \complement\{x, u \text{ is } C^\infty \text{ near } x\}$ is the set of 'bad points' of $u$, so $\Sigma(u)$ is the set of 'bad spectral directions' of $u$, or possibly 'bad frequencies' of $u$.

To combine these two information items in the concept of $WF(u)$, we use the following lemma.

**Lemma 1.1.1.** *If $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $u \in \mathcal{E}'(\mathbb{R}^n)$, $\Sigma(\varphi u) \subset \Sigma(u)$.*

**Proof.** We have $\widehat{\varphi u}(\xi) = (2\pi)^{-n} \int \hat{\varphi}(\eta)\hat{u}(\xi - \eta)\mathrm{d}\eta$. Since $u \in \mathcal{E}'(\mathbb{R}^n)$, $|\hat{u}(\mathcal{S})| \leq C(1+|\mathcal{S}|)^M$ for some $M$, and, on the other hand $\hat{\varphi}$ satisfies (1.1.1). Suppose $\xi_0 \notin \Sigma(u)$: in a conical neighbourhood $\Gamma$ of $\xi_0$, $\hat{u}$ satisfies (1.1.1);

let us split the integral into $\int_{|\eta|\le c|\xi|}$ and $\int_{|\eta|\ge c|\xi|}$, so that for $\xi$ in a neighbourhood $\Gamma_1$ of $\xi_0$, we have $\xi-\eta\in\Gamma$ in the first integral (this is possible for $0<c<1$, $c$ sufficiently small). We then obtain

$$\left|\int_{|\eta|\le c|\xi|}\right| \le c_k(1+|\xi|)^{-k}(1-c)^{-k}|\hat{\varphi}|_{L_1},$$

because $|\eta|\le c|\xi|$ implies $|\xi-\eta|\ge(1-c)|\xi|$. On the other hand

$$\begin{aligned}
&\left|\int_{|\eta|\ge c|\xi|}\right| \\
\le\ & C\int|\hat{\varphi}(\eta)|(1+|\eta|)^{k+M}(1+|\xi-\eta|)^M\frac{\mathrm{d}\eta}{(1+|\eta|)^{k+M}}\\
\le\ & C\int|\hat{\varphi}(\eta)|(1+|\eta|)^{k+M}\frac{(1+(1+1/c)|\eta|)^M}{(1+|\eta|)^M}\mathrm{d}\eta\frac{1}{(1+c|\xi|)^k}\\
\le\ & \frac{C(1+1/c)^M}{c^k}(1+|\xi|)^{-k}\int|\hat{\varphi}(\eta)|(1+|\eta|)^{k_M}\mathrm{d}\eta,
\end{aligned}$$

because $|\xi-\eta|\le(\frac{1}{c}+1)|\eta|$.

The estimate

$$|(1+|\xi|)^k\widehat{\varphi u}(\xi)\mathrm{Const.}_k\int|\hat{\varphi}(\eta)|(1+|\eta|)^{k+M}\mathrm{d}\eta \tag{1.1.2}$$

proves the lemma and at the same time makes precise the dependence upon $\varphi$. □

We can then define, for $u\in\mathcal{D}'(X)$ ($X$ an open subset of $\mathbb{R}^n$), the set $\Sigma_x(u)$ of 'bad spectral directions' of $u$ over $x$ by $\Sigma_x(u)=\bigcap_\varphi\Sigma(\varphi u)$, where $\varphi$ runs over the set of $\varphi\in C_0^\infty(X)$, $\varphi(x)\ne0$.

**Definition 1.1.2.** Let $X$ be an open subset of $\mathbb{R}^n$ and $u\in\mathcal{D}'(X)$. The conical closed subset of $X\times(\mathbb{R}^n\setminus0)$ defined by

$$WF(u)=\{(x,\xi)\in\mathbb{R}^n\setminus0,\xi\in\Sigma_x(u)\}$$

is the wave front set of $u$.

The following proposition shows that $WF(u)$ is the set of bad spectral directions of $u$ over $\operatorname{sing\,supp} u$: this is the synthesis announced in Section I.1.

**Proposition 1.1.2.** *The projection of $WF(u)$ onto $X$ is* sing supp $u$.

**Proof.** In fact, if $x_0\notin\operatorname{sing\,supp}u$, $\Sigma_{x_0}(u)=\emptyset$ by definition. Conversely, let us suppose that for some $x_0$, $\Sigma_{x_0}(u)=\emptyset$: this means that for any direction $\xi\in S^{n-1}$, there exists $\varphi_\xi\in C_0^\infty(X)$, $\varphi_\xi(x_0)\ne0$, such that $\widehat{\varphi_\xi u}$ is rapidly decreasing in a conical neighbourhood of $\xi$, $V_\xi$. By compactness of $S^{n-1}$, we obtain a finite number of functions $\varphi_i(x) = \varphi_{\xi_i}(x)$ such that, for all $i$, $V_{\xi_i} \cap \Sigma(\varphi_i u) = \emptyset$: Lemma 1.1.1 then implies that $\Sigma((\prod_i \varphi_i)u) = \emptyset$, that is $(\prod_i \varphi_i)u \in C_0^\infty$, which implies $x_0 \notin \operatorname{sing\,supp} u$. □

1.2. *Examples. Case of Fourier distributions.*

**Example 1.2.1.** Consider $\delta$, the Dirac mass at the origin. For all $\varphi \in C_0^\infty$, $\varphi(0) \neq 0$,

$$\widehat{\varphi\delta}(\xi) = \langle \delta, \varphi e^{-\mathrm{i}x\xi} \rangle = \varphi(0) \text{ and } \Sigma(\varphi\delta) = \mathbb{R}^n \setminus 0.$$

Thus

$$WF(\delta) = \{(x,\xi), x = 0, \xi \neq 0\}.$$

**Example 1.2.2.** Let

$$u = \begin{cases} -1 & x_1 < 0, \\ +1 & x_1 \geq 0. \end{cases}$$

We have

$$\widehat{\varphi u}(\xi) = -\int_{x_1 \leq 0} e^{-\mathrm{i}x\xi}\varphi(x)\mathrm{d}x + \int_{x_1 \geq 0} e^{-\mathrm{i}x\xi}\varphi(x)\mathrm{d}x.$$

When $|\xi| \to +\infty$ near a direction $\xi_0 = ((\xi_0)_1, \xi_0')$ for which $\xi_0' \neq 0$, each of the integrals is rapidly decreasing, because for example

$$\int_{x_1 \leq 0} = \int_{x_1 \leq 0} \mathrm{d}x_1 e^{-\mathrm{i}x_1\xi_1} \hat{\varphi}'(x_1, \xi')\mathrm{d}\xi'$$

where $\hat{\varphi}'$ is the partial Fourier transform in $\xi'$. Thus, $WF(u) \subset \{(x,\xi), x_1 = 0 \text{ and } \xi' = 0\}$.

Moreover, if a normal at $\{x_1 = 0\}$ were not in $WF(u)$, the others would not be either since $u$ is real and translation invariant in $x'$ and $u$ would be $C^\infty$. Thus $WF(u) = \{(0, x', \xi_1, 0), \xi_1 \neq 0\}$.

An important class of distributions is described in the following theorem.

**Theorem 1.2.** *Suppose $\varphi(x,\xi)$ is real, homogeneous of degree 1 in $\xi$, and $C^\infty$ for $\xi \neq 0$, and suppose $a \in S^m$ is zero for $|x| \geq C$. If $\mathrm{d}\varphi \neq 0$ on $\operatorname{supp} a$, we can define $u = \int e^{\mathrm{i}\varphi(x,\xi)} a(x,\xi)\mathrm{d}\xi$ (oscillating integral) and $WF(u) \subset \{(x,\eta), \eta = \varphi_x'(x,\xi), \varphi_\xi' = 0\}$.*

The definition is an easy consequence of Theorem 1 of the Appendix to Chapter I (cf. Exercise I.4.6), while the control of the wave front set results from the definitions and a non-stationary phase theorem (Exercise B.9).

## 2. Linear operators and wave front set.

2.1. *A general theorem.* Here we shall assume the following theorem, the elementary proof of which uses only Definition 1.1.2 for the wave front set (see [H4]).

**Theorem 2.1.** *Let $X, Y$ be open sets in $\mathbb{R}^n, \mathbb{R}^m$ and let $K \in \mathcal{D}'(X \times Y)$. Suppose that $WF(K)$ does not contain directions parallel to $\mathbb{R}^n$ or $\mathbb{R}^m$ (that is, $(x, y, \xi, \eta) \in WF(K) \Rightarrow \xi \neq 0,\ \eta \neq 0$). The operator $K$ defined by the formula $Ku(x) = \int K(x,y)u(y)\mathrm{d}y$ can then be extended to $\mathcal{E}'(Y)$ (the integral being understood in the sense of distributions), and*

$$WF(Ku) \subset WF'(K) \circ WF(u), \tag{2.1.1}$$

*where*

$$WF'(K) = \{(x, y, \xi, \eta), (x, y, \xi, -\eta) \in WF(K)\}.$$

*Here,*

$$\begin{aligned} & WF'(K) \circ WF(u) \\ & \quad = \{(x, \xi), \exists (y, \eta) \in WF(u), (x, y, \xi, \eta) \in WF'(K)\}. \end{aligned}$$

In other words, $WF'(K)$ describes the displacement of $WF(u)$ under the action of $K$, it being understood that (2.1.1) is only an inclusion. Here are some examples of applications of the theorem.

**Example 2.1.1.** Let $Tu(x) = u(x, 0)$ be the operator 'trace of $u$ on $t = 0$', defined for $u \in C_0^\infty(\mathbb{R}^{n+1})$, where $(x, t) \in \mathbb{R}^n \times \mathbb{R}$. Since $u(x, 0) = \frac{1}{2\pi}\int \hat{u}^2(x, \tau)\mathrm{d}\tau$ ($\hat{}^2$ denotes the partial Fourier transform with respect to $t$), we also have

$$\begin{aligned} Tu(x) &= (2\pi)^{-n} \int e^{\mathrm{i}x\xi}\widehat{Tu}(\xi)\mathrm{d}\xi \\ &= (2\pi)^{-n-1} \int e^{\mathrm{i}x\xi}\hat{u}(\xi, \tau)\mathrm{d}\xi\mathrm{d}\tau \\ &= (2\pi)^{-n-1} \int e^{\mathrm{i}(x-y)\xi}e^{-\mathrm{i}t\tau}u(y, t)\mathrm{d}y\mathrm{d}t\mathrm{d}\xi\mathrm{d}\tau. \end{aligned}$$

The operator $T$ thus has kernel $K(x, y, t)$ (here $X = \mathbb{R}^n$, $Y = \mathbb{R}^{n+1}$) equal to the Fourier distribution

$$K(x, y, t) = (2\pi)^{-n-1} \int e^{\mathrm{i}(x-y)\xi}e^{-\mathrm{i}t\tau}\mathrm{d}\xi\mathrm{d}\tau,$$

with phase

$$\Phi(x, y, t, \xi, \tau) = (x - y)\xi - t\tau.$$

We know (following Theorem 1.2) that $WF(K) \subset \{(x, x, 0, \xi, -\xi, -\tau)\}$. Thus, $WF(K)$ does not contain directions parallel to $\mathbb{R}^n_\xi$ but does contain directions $(\xi, -\xi, -\tau) = (0, 0, -\tau)$ parallel to $\mathbb{R}^n_{\xi,\tau}$; in order to be able to apply (2.1.1), we assume (Exercise B.4) that it is sufficient to suppose that the vertical direction $(0, -\tau)$ is not in the wave front set of $u$. We then find

$$WF(Tu) \subset \{(x, \xi), \exists \tau, (x, 0, \xi, \tau) \in WF(u)\}.$$

If we wish to understand this relation more intuitively, we may observe the following:

i) The singularities of $u$ away from $\{t = 0\}$ clearly cannot play any role in $Tu$.

ii) If $\chi(D_x)$ is a 'tangential' operator, we have $\chi(D_\chi)Tu = T\chi(D_\chi)u$.

If for some $(x_0, \xi_0)$ and all $\tau$, $(x_0, 0, \xi_0, \tau) \notin WF(u)$, we will have $Pu \in C^\infty$ for $P = \tilde{\chi}(D_x, D_t)\chi(D_x)\varphi$ where $\varphi \in C_0^\infty$ is a cut-off function close to $(\chi_0, 0)$ and $\chi$ and $\tilde{\chi}$ are symbols of degree zero, with $\chi$ supported in a conical neighbourhood of $x_0$ and $\tilde{\chi}$ zero in a conical neighbourhood of $(0, \pm 1)$. Since the vertical is not in $WF(u)$, $\tilde{\chi}\varphi u = \varphi u + C^\infty$, whence

$$\chi(D_x)T\varphi u = \chi(D_x)\varphi(x, 0)Tu \in C^\infty,$$

which means that $(x_0, \xi_0) \notin WF(Tu)$.

**Example 2.1.2.** Let $X = \partial_t + \sum_{i=1}^n a_i(x, t)\partial_{x_i}$ be a $C^\infty$ real field in $\mathbb{R}^{n+1}$. Consider the Cauchy problem

$$Xu = 0, \quad u|_{t=0} = u_0,$$

and define $Ku_0 = u(x, T)$ for $T > 0$ sufficiently small. Let $x(t, x_0)$ denote the solution of the differential equation in $\mathbb{R}^n$:

$$\begin{aligned} \frac{\mathrm{d}x_i}{\mathrm{d}t} &= a_i(x, t), \quad 1 \leq i \leq n, \\ x(0) &= x_0. \end{aligned}$$

The curve $t \mapsto (x(t, x_0), t)$ is just the integral curve of $X$ emanating from $(x_0, 0)$; since any solution $u$ is constant on these curves $u(x(t, x_0), t) = u_0(x_0)$, which describes the unique solution $u$ of the Cauchy problem considered (since $(x_0, t) \mapsto (x(t, x_0), t)$ is a local diffeomorphism). Any singularity $x_0$ of $u_0$ is echoed in a singularity of $u$ along the integral curve of $X$ emanating from $x_0$, and also in a singularity of $Ku_0$ at $x(T, x_0)$ (because if the trace of $u$ at $x(T, x_0)$ were $C^\infty$, $u$ itself would be $C^\infty$ near $(x(T, x_0), T)$ following the preceding argument). The operator $K$ thus displaces the singular support of $u$ in the direction of flow of $X$.

Let us now examine the situation at the more detailed level of the wave front set.

We have $Ku_0(x) = u_0(\Phi^{-1}(x))$ (if $\Phi(x_0) = x(T, x_0)$), which can be written as

$$Ku_0 = (2\pi)^{-n} \int e^{\mathrm{i}(\Phi^{-1}(x)-y)\xi} u_0(y) \mathrm{d}y \mathrm{d}\xi.$$

The kernel $K$ is thus the Fourier distribution

$$K(x, y) = (2\pi)^{-n} \int e^{\mathrm{i}(\Phi^{-1}(x)-y)\xi} \mathrm{d}\xi.$$

Following Theorem 1.2, we have (where ${}^tA$ denotes the transpose of $A$)

$$WF(K) \subset \{(x, \Phi^{-1}(x), {}^t(\Phi^{-1})'(x)\xi, -\xi)\},$$

and, by (2.1.1),

$$WF(Ku_0) \subset \{(\Phi(y), \xi), \xi = {}^t\Phi'^{-1}(y)\eta, (y, \eta) \in WF(u_0)\}.$$

We can 'visualize' the mapping $(y, \eta) \mapsto (x, \xi)$ induced by $\Phi$ as follows: if $S$ is a surface passing through $y$ with normal $\eta$ at $y$, $\Phi(S)$ is a surface passing through $x$ with normal $\xi$. At the same time, this interpretation renders intuitive the result on $WF(Ku_0)$ obtained (proceeding by analogy with Example 1.2.2). The flow of $X$ thus induces a mapping $(y, \eta) \mapsto (x, \eta)$ which describes the displacement of the wave front set under the action of $K$.

**Example 2.1.3.** Let us consider the solution of the Cauchy problem for the wave equation

$$\square u = (\partial_t^2 - \Delta_x)u = 0, \quad u(x, 0) = 0, \quad \partial_t u(x, 0) = u_0 \tag{2.1.2}$$

where $\Delta_x = \partial_1^2 + \cdots + \partial_n^2$ is the Laplacian.

Using a partial Fourier transformation in $x$ (with abuse of notation), it is easy to calculate $u$; indeed

$$\partial_t^2 \hat{u} + |\xi|^2 \hat{u}(\xi, t) = 0, \quad \hat{u}(\xi, 0) = 0, \quad \partial_t \hat{u}(\xi, 0) = \hat{u}_0,$$

whence

$$\hat{u}(\xi, t) = A(\xi) e^{\mathrm{i}t|\xi|} + B(\xi) e^{-\mathrm{i}t|\xi|}$$

with

$$A + B = 0, \ \mathrm{i}|\xi|(A - B) = \hat{u}_0,$$

or finally,

$$\begin{aligned} u(x, t) \quad = \quad & \frac{1}{2\mathrm{i}(2\pi)^n} \left\{ \int e^{\mathrm{i}(x-y)\xi + \mathrm{i}t|\xi|} \frac{1}{|\xi|} u_0(y) \mathrm{d}y \mathrm{d}\xi \right. \\ & \left. - \int e^{\mathrm{i}(x-y)\xi - \mathrm{i}t|\xi|} \frac{1}{|\xi|} u_0(y) \mathrm{d}y \mathrm{d}\xi \right\}. \end{aligned}$$

We shall assume that $u$ is the only solution to the problem in hand and that replacing $1/|\xi|$ by $(1-\chi(\xi))/|\xi|$ in the above integrals ($\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi = 1$ near 0) defines a new function $\tilde{u}$ which differs from $u$ only in a $C^\infty$ function.

We remark that $\tilde{u} = (K_+ + K_-)u_0$, where the kernel $K$ of the operator $K_\pm$ is the Fourier distribution

$$K_\pm = \frac{1}{2\mathrm{i}(2\pi)^n} \int e^{\mathrm{i}(x-y)\xi \pm \mathrm{i}t|\xi|} \frac{1-\chi(\xi)}{|\xi|} \mathrm{d}\xi.$$

Following Theorem 1.2 we have

$$WF(K_\pm) \subset \left\{ (x,t,y,\xi,\pm|\xi|,-\xi), x - y \pm t\frac{\xi}{|\xi|} = 0 \right\},$$

whence

$$WF(K_\pm u_0) \subset \left\{ (x,t,\xi,\pm|\xi|), (y,\xi) \in WF(u_0), \text{ where } y = x \pm t\frac{\xi}{|\xi|} \right\}.$$

Geometrically, let us draw $\Gamma_y$, the 'light cone' emanating from $y$, with equation $t^2 = |x-y|^2$: the singularity $(y,\xi)$ of $u_0$ is echoed in the singularity of $K_\pm u_0$ consisting of the single direction $(\xi, \pm|\xi|)$ along the generatrix of $\Gamma_y$ which is perpendicular to it.

In the particular case in which $n = 1$, $\Box = (\partial_t + \partial_x)(\partial_t - \partial_x)$, and any function $u = \Phi(x+t)$ (a solution of $(\partial_t - \partial_x)u = 0$) or $u = \Psi(x-t)$ (a solution of $(\partial_t + \partial_x)u = 0$) is a solution of $\Box u = 0$. The solution of (2.1.2) is then $u = \Phi(x+t) - \Phi(x-t)$ for $2\Phi'(x) = u_0(x)$, or $u(x,t) = 1/2 \int_{x-t}^{x+t} u_0(s)\mathrm{d}s$. As in Example 2.1.2, a singularity of $u_0$ will be echoed in two (since there are now two fields $X$) singularities of $u$; this analysis can be refined at the level of the wave front set.

The fundamental difference of the case $n \geq 2$ from the case $n = 1$ is that the actual position of the singular support of $u$ depends on the wave front set of $u_0$ and not on its own singular support. If, for example, $u_0$ is $C^\infty$ away from 0, we know that $\operatorname{sing\,supp} u \subset \Gamma_0$; but in order to know exactly which generatrices constitute sing supp $u$, we need to know what the directions of $WFu_0$ are over 0. We see in this example that the introduction of the wave front set is not a simple refinement of the analysis of the singular support, it is a necessity.

2.2. *Pseudo-differentials and wave front set.* The application of the general Theorem 2.1 to pseudo-differentials is of particular interest.

**Proposition 2.2.1.** *Let $A$ be a pseudo-differential operator, with kernel $K$. Then $WF(K) \subset \{(x,x,\xi,-\xi), \xi \neq 0\}$.*

**Proof.** In fact $K(x,y) = \int e^{i(x-y)\xi} a(x,\xi) d\xi$ is a Fourier distribution and the proposition follows from Theorem 1.2. □

**Corollary 2.2.1.** *Let $A$ be a pseudo-differential operator. For all $u \in \mathcal{S}'(\mathbb{R}^n)$, $WF(Au) \subset WF(u)$.*

**Proof.** If $u \in \mathcal{E}'(\mathbb{R}^n)$, the result follows from the general Theorem 2.1 and Proposition 2.2.1, because $WF'(K) \subset \{(x,x,\xi,\xi)\}$. If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $(x_0,\xi_0) \notin WF(u)$, $(x_0,\xi_0) \notin WF(\chi u)$ for $\chi \in C_0^\infty$ having value 1 near $x_0$. Thus $(x_0,\xi_0) \notin WF(A\chi u)$ and also $(x_0,\xi_0) \notin WF(\tilde{\chi} A\chi u)$ for $\tilde{\chi} \in C_0^\infty$ having value 1 near $x_0$; but $\tilde{\chi} A\chi u = \tilde{\chi} Au + C^\infty$ if $\chi = 1$ in the neighbourhood of $\operatorname{supp}\tilde{\chi}$; hence $(x_0,\xi_0) \notin WF(\tilde{\chi} Au)$ for such a choice of $\chi$ and $\tilde{\chi}$, which implies $(x_0,\xi_0) \notin WF(Au)$.

The property of Corollary 2.2.1 is called the 'pseudo-local property': it means that the action of a pseudo-differential operator does not displace the wave front set (compare with Examples 2.1.2 and 2.1.3), but, if need be, diminishes it: for example, $\partial_t \Phi(x) = 0$, while $\Phi$ can be taken to be as singular as one wishes.

We shall give more precise details of this action.

**Proposition 2.2.2.** *Let $\Gamma$ be a conical open set in $\mathbb{R}^n \times \mathbb{R}^n$, and let $a \in S^m$, such that $a \in S^{-\infty}$ in $\Gamma$ (which means that for all $(x_0,\xi_0) \in \Gamma$, there exists a conical neighbourhood of $(x_0,\xi_0)$ on which the estimates which characterize $S^{-\infty}$ hold). Then, for $u \in \mathcal{E}'(\mathbb{R}^n)$ and $A = \operatorname{Op}(a)$, $WF(Au) \cap \Gamma = \emptyset$.*

**Proof.** Let $(x_0,\xi_0) \in \Gamma$ and $q(x,\xi) \in S^0$ have value 1 near $(x_0,\xi_0)$ and be such that $aq \in S^{-\infty}$; the operator $A_1 = \operatorname{Op}(aq)$ has a $C^\infty$ kernel, and the kernel of $A_2 = \operatorname{Op}(a(1-q))$ is $C^\infty$ near $(x_0,x_0,\xi_0,-\xi_0)$ following Theorem 1.2. Thus $(x_0,\xi_0) \notin WF(A_2 u)$ following the general Theorem 2.1, and $Au = A_2 u + C^\infty$, which completes the proof. □

In other words, the action of an operator $A$ destroys the wave front set where its symbol is of order $-\infty$.

The 'inverse' phenomenon is the subject of the following corollary.

**Corollary 2.2.2.** *If $a \in S^m$, $Au \in C^\infty$ near $(x_0,\xi_0)$ and $|a(x,\xi)| \geq c|\xi|^m$ for $|\xi| \geq C$ in a conical neighbourhood of $(x_0,\xi_0)$, then $(x_0,\xi_0) \notin WF(u)$.*

**Proof.** By definition, there exists $\tilde{b} \in S^{-m}$, $a\tilde{b} - 1 \in S^{-1}$ in a neighbourhood of $(x_0,\xi_0)$. As in Section I.5.4, we see that there exists $b \in S^{-m}$ such that $b \# a - 1 \in S^{-\infty}$ near $(x_0,\xi_0)$. Since $u = BAu - (BA - \mathrm{id})u$, the conclusion follows from Proposition 2.2.2 and from the pseudo-local property. □

In other words, at non-characteristic points of the symbol for $A$ (those which, by definition, satisfy the hypothesis of Corollary 2.2.2), the wave front set is conserved by the action of $A$.

Corollary 2.2.2 can be used as a characterization of the wave front set: $WF(u) = \bigcap \text{char}(A)$, where $A$ describes the properly supported operators such that $Au \in C^\infty$. It is nice to have a more direct proof of Corollary 2.2.2 using only the initial definition of the wave front set. Here it is, in the case of a differential operator $P(x, D)$ in an open set $X$ in $\mathbb{R}^n$: let $u \in \mathcal{D}'(X)$; suppose that $p_m(x_0, \xi_0) \neq 0$ and $(x_0, \xi_0) \notin WF(Pu)$.

To prove that $(x_0, \xi_0) \notin WF(u)$, we consider $\widehat{\varphi u}(\xi)$, that is to say $\langle u, \varphi e^{-\mathrm{i}x\xi}\rangle$: the idea of the proof is to write $\varphi e^{-\mathrm{i}x\xi}$ in the form ${}^tP(\psi e^{-\mathrm{i}x\xi})$, such that then

$$\langle u, \varphi e^{-\mathrm{i}x\xi}\rangle = \langle u, {}^tP(\psi e^{-\mathrm{i}x\xi})\rangle = \langle Pu, \psi e^{-\mathrm{i}x\xi}\rangle$$

decreases as desired according to the hypothesis on $Pu$. We denote $Q = {}^tP$ (with $q_m = p_m$) and observe the formula

$$e^{\mathrm{i}x\xi} Q(\psi e^{-\mathrm{i}x\xi}) = q_m(x, \xi)\psi + Q_{m-1}\psi + \cdots + Q_0\psi,$$

where $Q_j\psi$ is the homogeneous part in $\xi$ of the term on the left of degree $j$, which is a polynomial whose coefficients are differential operators of order $m - j$ applied to $\psi$.

To obtain $q_m(x, \xi)\psi + \cdots + Q_0\psi = \varphi$ for $(x, \xi)$ near $(x_0, \xi_0)$, we simply take

$$\psi = \psi_N = \frac{1}{q_m(x, \xi)}(\varphi + a_1(x, \xi) + \cdots + a_N(x, \xi)),$$

where the $a_j(x, \xi)$ are homogeneous of degree $-j$ in $\xi$ and are determined by the relations

$$\begin{aligned} a_1 + Q_{m-1}(\varphi/q_m) &= 0, \\ a_2 + Q_{m-1}(a_1/q_m) + Q_{m-2}(\varphi/q_m) &= 0, \text{ etc.} \end{aligned}$$

In that fashion, $e^{\mathrm{i}x\xi} Q(\psi e^{-\mathrm{i}x\xi}) = \varphi + r_{N+1}$, where $r_{N+1}$ is a symbol in $\xi$ of order $-N - 1$, $\operatorname{supp} r_{N+1} \subset \operatorname{supp} \varphi$.

What can we now say about $\langle Pu, \psi_N e^{-\mathrm{i}x\xi}\rangle$? We know that for some $\chi \in C_0^\infty$, $\chi = 1$ near $x_0, \xi_0 \notin \Sigma(\chi Pu)$; taking $\text{supp}\varphi \supset \text{supp}\psi_N$ sufficiently small so that $\chi = 1$ in the neighbourhood of $\text{supp}\varphi$, we obtain $\langle Pu, \psi_N e^{-\mathrm{i}x\xi}\rangle = (\psi_N \chi Pu)(\xi)$, whence

$$\begin{aligned} (1 + |\xi|)^k |\widehat{\varphi u}(\xi)| \quad \leq \quad & (1 + |\xi|)^k |(\widehat{r_{N+1} u})|(\xi) \\ & + C_k \int |\widehat{\psi_N}(\eta, \xi)| (1 + |\eta|)^{k+M} \mathrm{d}\eta, \end{aligned}$$

following (1.1.2). For $N$ sufficiently large, the second term is bounded in $\xi$, which completes the proof. □