

vector fields as holomorphic sections in the complex tangent bundle. A *holomorphic one parameter group* is a holomorphic map $g : \mathbb{C} \times M \rightarrow M$ with the obvious properties. In particular, for each $t \in \mathbb{C}$ the map $g_t : M \rightarrow M$, $a \mapsto g(t, a)$, is a biholomorphic transformation of M . Correspondingly one also defines the concept of a *holomorphic local one parameter group*: This is a holomorphic map $g : I \times U \rightarrow M$, $U \subset M$ open, with the corresponding properties. Then, as in the differentiable case, holomorphic vector fields and holomorphic local one parameter groups correspond. The proofs are identical and will not be presented here.

Definition. A *complex Lie group* is a set G with the following properties:

- (i) G is a group.
- (ii) G is a complex manifold.
- (iii) The map $G \times G \rightarrow G$, $(a, b) \mapsto ab^{-1}$, is holomorphic.

Example 3.6. The Lie group $\mathrm{GL}(n, \mathbb{C})$ is also a complex Lie group of complex dimension n^2 .

Definition. Let M be a complex manifold and G a complex Lie group. An *action of G on M* is a holomorphic map $M \times G \rightarrow M$, $(p, g) \mapsto pg$ with the following properties:

- (i) $(pg_1)g_2 = p(g_1g_2)$ for all $p \in M$ and $g_1, g_2 \in G$,
- (ii) $pe = p$ for all $p \in M$.

Remark 3.14. All the results of §3.4 hold analogously for complex Lie groups and actions of complex Lie groups on complex manifolds. The proofs can be taken over almost word-for-word to this situation, replacing the concept differentiable by complex or holomorphic throughout.

3.6. Isolated critical points

We now come to the main theme of this chapter, namely the study of isolated singularities of holomorphic functions.

Let M be an $(n + 1)$ -dimensional complex manifold and $f : M \rightarrow \mathbb{C}$ a holomorphic function.

Definition. A point $p \in M$ is called a *critical point* or *singularity* of f if its differential $T_p f$ is the null map. If a is a critical point of f , then $f(a)$ is called a *critical value* of f .

Remark 3.15. If (z_1, \dots, z_{n+1}) is a local coordinate system around a (with $z_j(a) = 0$), then a is a critical point of f if and only if

$$\frac{\partial f}{\partial z_1}(0) = \dots = \frac{\partial f}{\partial z_{n+1}}(0) = 0.$$

Definition. A point $p \in M$ is called an *isolated critical point* or an *isolated singularity* of f if there is a neighborhood U of p in M such that no point of $U \setminus \{p\}$ is critical.

Remark 3.16. One should note that this definition also includes the case when p is not a critical point at all. This may appear illogical but is, however, the usual definition.

Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be the germ of a holomorphic function with an isolated critical point at 0. If $M \subset \mathbb{C}^{n+1}$ is a suitable open neighborhood of $0 \in \mathbb{C}^{n+1}$, if $\tilde{f} : M \rightarrow \mathbb{C}$ is a representative of f , and $X := \tilde{f}^{-1}(0)$, then X has a regular or an isolated singular point at 0. We also then say simply that f has an isolated singularity at 0. One often calls f or the analytic set germ $(X, 0)$ an isolated singularity. In the sequel we shall not distinguish notationally between a germ f and a representative $f : M \rightarrow \mathbb{C}$, where M is an open neighborhood of $0 \in \mathbb{C}^{n+1}$.

By the implicit function theorem the level surface $f^{-1}(w)$ for $w \in \mathbb{C}$, $w \neq 0$, $|w|$ suitably small, is a complex submanifold of \mathbb{C}^{n+1} in a neighborhood of $0 \in \mathbb{C}^{n+1}$. The zero set $f^{-1}(0)$ has a singularity at $0 \in \mathbb{C}^{n+1}$, but it is a complex submanifold of \mathbb{C}^{n+1} in a neighborhood of 0, apart from 0.

Lemma 3.5. *There exists an $\varepsilon > 0$ such that the sphere $S_\rho \subset \mathbb{C}^{n+1}$, center 0, of radius $\rho \leq \varepsilon$, intersects the zero set $f^{-1}(0)$ transversally.*

For the proof of this auxiliary result we need the following lemma, which we cite without proof.

Lemma 3.6 (Curve selection lemma). *Let $V \subset \mathbb{R}^m$ be an open neighborhood of $p \in \mathbb{R}^m$, let $f_1, \dots, f_k, g_1, \dots, g_l : V \rightarrow \mathbb{R}$ be real analytic functions, and let*

$$Z := \{x \in V \mid f_1(x) = \dots = f_k(x) = 0, g_1(x) > 0, \dots, g_l(x) > 0\}.$$

If $p \in \bar{Z}$, there exists a real analytic curve $\gamma : [0, \delta) \rightarrow V$, $0 < \delta$, with $\gamma(0) = p$ and $\gamma(t) \in Z$ for all $t \in (0, \delta)$.

For the proof see [Mil68, §3]. It is assumed there that $V = \mathbb{R}^m$ and that the $f_1, \dots, f_k, g_1, \dots, g_l$ are polynomials. The proof can be adapted immediately to the analytic case.

Proof of Lemma 3.5. We consider the function $r|_{f^{-1}(0)} : f^{-1}(0) \rightarrow \mathbb{R}$, $r(z) = |z|^2$. The critical points of r on $f^{-1}(0)$ are exactly those points of $f^{-1}(0)$ at which $S_{\sqrt{r}}$ and $f^{-1}(0)$ do not intersect transversally. Put

$$Z = \{z \in f^{-1}(0) \mid z \text{ a critical point of } r|_{f^{-1}(0) \setminus \{0\}}\}.$$

Then Z is specified by real analytic equations and inequalities, so it satisfies the hypothesis of the curve selection lemma. We must show that 0 is not an accumulation point of Z .

Suppose $0 \in \bar{Z}$. Then, by the curve selection lemma, there are a $\delta > 0$ and a curve $\gamma : [0, \delta) \rightarrow f^{-1}(0)$ with $\gamma(0) = 0$ and $\gamma(t) \in Z$ for $t \in (0, \delta)$. Thus

$$(r \circ \gamma)'(t) = \langle \text{grad } r(\gamma(t)), \gamma'(t) \rangle = 0$$

for all $t \in (0, \delta)$. Hence $r \circ \gamma$ is constant, and, since $r \circ \gamma(0) = 0$, we have $r \circ \gamma \equiv 0$ on $[0, \delta)$. Since $r^{-1}(0) = 0$, it follows that $\gamma(t) = 0$ for all $t \in [0, \delta)$. This, however, contradicts $\gamma(t) \in Z$ for $t \in (0, \delta)$. \square

Now let M be an open neighborhood of $0 \in \mathbb{C}^{n+1}$ and let $f : M \rightarrow \mathbb{C}$ be a representative of f . Let $\varepsilon > 0$ be as in Lemma 3.5, $B_\varepsilon \subset M$ the open ball around $0 \in \mathbb{C}^{n+1}$ of radius ε . This is a complex submanifold of M . It follows from Lemma 3.5 that there is an $\eta_0 > 0$, $\eta_0 \ll \varepsilon$, such that $f^{-1}(w)$ intersects the sphere S_ε transversally for $w \in \mathbb{C}$, $|w| \leq \eta_0$.

Now again let M be an $(n+1)$ -dimensional complex manifold and $f : M \rightarrow \mathbb{C}$ a holomorphic function.

Definition. A critical point $p \in M$ of f is called *nondegenerate* if there is a local coordinate system (z_1, \dots, z_{n+1}) around p (with $p = 0$) such that $\det((\partial^2 f / \partial z_i \partial z_j)(0)) \neq 0$. Here $\det((\partial^2 f / \partial z_i \partial z_j)(0))$ is the determinant of the Hessian matrix of f in the coordinates (z_1, \dots, z_{n+1}) at 0.

Remark 3.17. One can show that the condition for a nondegenerate critical point is independent of the choice of the coordinate system.

There is a complex Morse lemma:

Proposition 3.15 (Complex Morse lemma). *Let M be an $(n+1)$ -dimensional complex manifold, $f : M \rightarrow \mathbb{C}$ a holomorphic function and p a nondegenerate critical point of f . Then there is a local coordinate system (z_1, \dots, z_{n+1}) in a neighborhood V of p with $z_i(p) = 0$, $i = 1, \dots, n+1$, such that on V ,*

$$f(z_1, \dots, z_{n+1}) = f(p) + z_1^2 + \dots + z_{n+1}^2.$$

To prove Proposition 3.15, we need the following lemma.

Lemma 3.7. *Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be the germ of a holomorphic function with a nondegenerate critical point in 0. Then the partial derivatives $\partial f / \partial z_1, \dots, \partial f / \partial z_{n+1}$ generate the maximal ideal \mathfrak{m}_{n+1} of \mathcal{O}_{n+1} .*

Proof. Let $U \subset \mathbb{C}^{n+1}$ be an open neighborhood of 0 and $f : U \rightarrow \mathbb{C}$ a representative of f . We consider the map

$$\begin{aligned} \text{grad } f : U &\longrightarrow \mathbb{C}^{n+1} \\ z &\longmapsto \left(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_{n+1}}(z) \right). \end{aligned}$$

Since 0 is a nondegenerate critical point of f , we have

$$\text{rank}(J_{\text{grad } f}(0)) = n + 1.$$

From the rank theorem (Proposition 2.34) follows: There are open neighborhoods V, V' of 0 in U and a biholomorphic map $\varphi : V \rightarrow V'$ such that for all $z \in V$,

$$\text{grad } f \circ \varphi(z_1, \dots, z_{n+1}) = (z_1, \dots, z_{n+1}).$$

Since z_1, \dots, z_{n+1} generate the maximal ideal \mathfrak{m}_{n+1} of \mathcal{O}_{n+1} , the result follows. \square

Proof of Proposition 3.15. Since we are dealing with a local assertion, we can assume that M is an open neighborhood of 0 in \mathbb{C}^{n+1} , $p = 0$ and $f(p) = 0$.

By a linear change of coordinates we can ensure that in suitable coordinates f has the form

$$f(u) = u_1^2 + \dots + u_{n+1}^2 + \varphi(u)$$

with $\varphi \in \mathfrak{m}_{n+1}^3$. We consider the function

$$F(u, t) := u_1^2 + \dots + u_{n+1}^2 + t\varphi(u)$$

for u in a neighborhood U of 0 in \mathbb{C}^{n+1} and $t \in [0, 1]$. We now search for a holomorphic one parameter group $g : [0, 1] \times U' \rightarrow U'$ for a suitable neighborhood U' of 0 in \mathbb{C}^{n+1} such that

$$(3.2) \quad F(g_t(u), t) = u_1^2 + \dots + u_{n+1}^2, \quad g_0(u) = u, \quad g_t(0) = 0 \text{ for } u \in U', t \in [0, 1].$$

A holomorphic vector field

$$X = \frac{\partial}{\partial t} + \sum_{j=1}^{n+1} a_{t,j} \frac{\partial}{\partial u_j}$$

is induced on $[0, 1] \times U'$ by g . From the equation (3.2) it follows that $XF = 0$ for this vector field. Hence one obtains the following equation for the functions $a_{t,j}$ sought:

$$\frac{\partial F}{\partial t} + \sum_{j=1}^{n+1} a_{t,j} \frac{\partial F}{\partial u_j} = \varphi + \sum_{j=1}^{n+1} a_{t,j} \frac{\partial F}{\partial u_j} = 0.$$

We can rewrite this equation as

$$(3.3) \quad \sum_{j=1}^{n+1} a_{t,j} \frac{\partial F}{\partial u_j} = -\varphi.$$

However, the function F_t with $F_t(u) = F(u, t)$ has a nondegenerate critical point in 0 for each $t \in [0, 1]$. By Lemma 3.7 the partial derivatives $\partial F_t / \partial u_1, \dots, \partial F_t / \partial u_{n+1}$ generate the maximal ideal \mathfrak{m}_{n+1} of \mathcal{O}_{n+1} . Hence in a suitable neighborhood W of $\{0\} \times [0, 1]$ in $U \times \mathbb{C}$ we can take $w_1 := \partial F / \partial u_1, \dots, w_{n+1} := \partial F / \partial u_{n+1}$ and t as new coordinates.

Since by hypothesis φ lies in \mathfrak{m}_{n+1}^3 , we can write φ as

$$\varphi = \sum_{j=1}^{n+1} w_j \psi_j,$$

where the ψ_j are holomorphic functions on W with $\psi_j(0, t) = 0$ for $t \in [0, 1]$. We can thus solve the equation (3.3) as follows:

$$a_{t,j} := -\psi_j.$$

By Proposition 3.3 and Remark 3.13 there is now a local holomorphic one parameter group $g' : I \times U'' \rightarrow \mathbb{C}^{n+1}$, where $U'' \subset U$ is a suitable neighborhood of $0 \in \mathbb{C}^{n+1}$, that induces the vector field X . The vector field X in this case also depends on t , but Proposition 3.3 can be generalized immediately to time dependent vector fields too. Since $X_{(0,t)} = 0$ for $t \in [0, 1]$, we can extend g' to a holomorphic one parameter group $g : [0, 1] \times U' \rightarrow U'$, where, if necessary, we shrink U'' to a suitable neighborhood U' of $0 \in \mathbb{C}^{n+1}$. Then $g_t(0) = 0$ for $t \in [0, 1]$. The coordinate transformation sought is thus g_1 . \square

Corollary 3.3. *Nondegenerate critical points are isolated.*

Definition. A holomorphic function $f : M \rightarrow \mathbb{C}$ is called a *Morse function* if all its critical points are nondegenerate and all its critical values are distinct.

3.7. The universal unfolding

In this section we introduce the concept of an unfolding of a holomorphic function germ.

Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ.

Definition. An *unfolding* of f is a holomorphic function germ $F : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$ with $F(z, 0) = f(z)$.

Definition. Two unfoldings $F : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$, $G : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$ of f are called *equivalent* if there is a holomorphic map germ

$$\psi : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}^{n+1}, 0) \text{ with } \psi(z, 0) = z$$

such that

$$G(z, u) = F(\psi(z, u), u).$$

Definition. Let $F : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$ be an unfolding of f and let $\varphi : (\mathbb{C}^l, 0) \rightarrow (\mathbb{C}^k, 0)$ be a holomorphic map germ. The unfolding $G : (\mathbb{C}^{n+1} \times \mathbb{C}^l, 0) \rightarrow (\mathbb{C}, 0)$ with

$$G(z, t) = F(z, \varphi(t))$$

is called the *unfolding of f induced from F* by φ .

Definition. An unfolding $F : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$ of f is called *versal* if every unfolding of f is equivalent to an unfolding induced from F .

A versal unfolding $F : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$ of f is called *universal* (or *miniversal*) if k is minimal.

Proposition 3.16. *Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ with an isolated singularity at 0. Then*

$$\mathcal{O}_{n+1}/\mathcal{O}_{n+1} \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{n+1}} \right)$$

is a finite-dimensional \mathbb{C} -vector space.

Proof. Let $f_j := \partial f / \partial z_j$, $j = 1, \dots, n+1$, be the partial derivatives of f and let J_f be the ideal spanned by the f_1, \dots, f_{n+1} . Now f has an isolated singularity at 0, so 0 is the unique solution of

$$f_1(z) = \dots = f_{n+1}(z) = 0$$

in an open neighborhood M of 0. Then

$$V(J_f) = V(f_1, \dots, f_{n+1}) = (\{0\}, 0).$$

It follows that

$$I(V(J_f)) = \mathfrak{m},$$

where \mathfrak{m} is the maximal ideal of \mathcal{O}_{n+1} . By Rückert's Nullstellensatz (Proposition 2.27)

$$I(V(J_f)) = \text{rad } J_f.$$

Hence there is a natural number $q > 0$ such that $\mathfrak{m}^q \subset J_f$. But $\mathcal{O}_{n+1}/\mathfrak{m}^q$ is a finite-dimensional \mathbb{C} -vector space (cf. the proof of Proposition 2.42). Hence \mathcal{O}_{n+1}/J_f is also a finite-dimensional \mathbb{C} -vector space. This completes the proof of Proposition 3.16. \square

If f has an isolated singularity at 0, one can construct a universal unfolding of f as follows.

Proposition 3.17. *Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ with an isolated singularity at 0. Then one obtains a universal unfolding F of f as follows: Let $g_0 = -1, g_1, \dots, g_{p-1}$ be representatives of a basis of the \mathbb{C} -vector space*

$$\mathcal{O}_{n+1}/\mathcal{O}_{n+1} \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{n+1}} \right).$$

Then put

$$\begin{aligned} F : (\mathbb{C}^{n+1} \times \mathbb{C}^p, 0) &\longrightarrow (\mathbb{C}, 0) \\ (z, u) &\longmapsto f(z) + \sum_{j=0}^{p-1} g_j(z)u_j. \end{aligned}$$

We present a proof of this proposition modelled on [AGV85, Mar82]. We need a few preparations.

Let $F : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$, $(z, u) \mapsto F(z, u)$ be an unfolding of f . We put

$$\dot{F}_j(z) = \left. \frac{\partial F(z, u)}{\partial u_j} \right|_{u=0}, \quad j = 1, \dots, k.$$

This defines function germs $\dot{F}_j \in \mathcal{O}_{n+1}$. We denote the set of all linear combinations of $\dot{F}_1, \dots, \dot{F}_k$ with complex coefficients by

$$\mathbb{C}(\dot{F}_1, \dots, \dot{F}_k).$$

Furthermore, for brevity, we put

$$\mathcal{O}_{n+1} \left(\frac{\partial f}{\partial z_i} \right) = \mathcal{O}_{n+1} \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{n+1}} \right)$$

for the ideal of \mathcal{O}_{n+1} spanned by the partial derivatives $\partial f/\partial z_1, \dots, \partial f/\partial z_{n+1}$.

Definition. An unfolding $F : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$ of f is called *infinitesimally versal* if

$$\mathcal{O}_{n+1} = \mathcal{O}_{n+1} \left(\frac{\partial f}{\partial z_i} \right) + \mathbb{C}(\dot{F}_1, \dots, \dot{F}_k).$$

Example 3.7. The unfolding F as in Proposition 3.17 is infinitesimally versal.

Now let $F : (\mathbb{C}^{n+1} \times \mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$, $(z, u) \mapsto F(z, u)$, be an infinitesimally versal unfolding of f . Let

$$\begin{aligned} G : (\mathbb{C}^{n+1} \times \mathbb{C}^k \times \mathbb{C}, 0) &\longrightarrow (\mathbb{C}, 0) \\ (z, u, v) &\longmapsto G(z, u, v) \end{aligned}$$

be a one-parameter unfolding of F , i.e., $G(z, u, 0) = F(z, u)$. Since $G(z, 0, 0) = F(z, 0) = f(z)$, we can also regard G as an unfolding of f .

Lemma 3.8. *The unfolding G of f is equivalent to an unfolding of f induced from F .*

Proof. Since F is infinitesimally versal,

$$\mathcal{O}_{n+1} = \mathcal{O}_{n+1} \left(\frac{\partial f}{\partial z_i} \right) + \mathbb{C}(\dot{F}_1, \dots, \dot{F}_k).$$

Since

$$\begin{aligned} \frac{\partial G}{\partial z_i}(z, 0, 0) &= \frac{\partial f}{\partial z_i}, \quad i = 1, \dots, n+1, \\ \frac{\partial G}{\partial u_j}(z, 0, 0) &= \dot{F}_j, \quad j = 1, \dots, k, \end{aligned}$$

it follows that

$$(3.4) \quad \mathcal{O}_{n+1+k+1} = \mathcal{O}_{n+1+k+1} \left(\frac{\partial G}{\partial z_i} \right) + \mathcal{O}_{n+1+k+1} \mathfrak{m}_{k+1} + \mathbb{C} \left(\frac{\partial G}{\partial u_1}, \dots, \frac{\partial G}{\partial u_k} \right),$$

where \mathfrak{m}_{k+1} is the maximal ideal of $\mathcal{O}_{k+1} = \mathbb{C}\{u_1, \dots, u_k, v\}$. Now let

$$\begin{aligned} \pi : \quad \mathbb{C}^{n+1} \times \mathbb{C}^k \times \mathbb{C} &\longrightarrow \mathbb{C}^k \times \mathbb{C} \\ (z, u, v) &\longmapsto (u, v) \end{aligned}$$

be the canonical projection and $\pi^* : \mathcal{O}_{k+1} \rightarrow \mathcal{O}_{n+1+k+1}$ the algebra homomorphism induced by it. Let

$$M := \mathcal{O}_{n+1+k+1} / \mathcal{O}_{n+1+k+1} \left(\frac{\partial G}{\partial z_i} \right).$$

Then M is obviously finite over $\mathcal{O}_{n+1+k+1}$. We apply the Weierstrass preparation theorem for modules (Corollary 2.5) to π^* , M and $\partial G / \partial u_1, \dots, \partial G / \partial u_k$. From this proposition and the equation (3.4) it follows that

$$\mathcal{O}_{n+1+k+1} = \mathcal{O}_{n+1+k+1} \left(\frac{\partial G}{\partial z_i} \right) + \mathcal{O}_{k+1} \left(\frac{\partial G}{\partial u_1}, \dots, \frac{\partial G}{\partial u_k} \right).$$

Now we consider the element $\partial G / \partial v \in \mathcal{O}_{n+1+k+1}$. Then $\partial G / \partial v$ has a representation

$$\frac{\partial G}{\partial v} = \sum_{i=1}^{n+1} \xi_i(z, u, v) \frac{\partial G}{\partial z_i} + \sum_{j=1}^k \eta_j(u, v) \frac{\partial G}{\partial u_j}$$

with $\xi_i \in \mathcal{O}_{n+1+k+1}$ and $\eta_j \in \mathcal{O}_{k+1}$. This equation can be formulated as follows: For the germ of the holomorphic vector field

$$X = \frac{\partial}{\partial v} - \sum_{j=1}^k \eta_j(u, v) \frac{\partial}{\partial u_j} - \sum_{i=1}^{n+1} \xi_i(z, u, v) \frac{\partial}{\partial z_i}$$

we have

$$X(G) = 0.$$

This vector field defines, by Proposition 3.3 and Remark 3.13, a holomorphic local one parameter group and a system of phase curves. For a point (z, u, v) near 0 let $(\psi(z, u, v), \varphi(u, v), 0)$ be the point of intersection of the phase curve through (z, u, v) with the hyperplane $\mathbb{C}^{n+1} \times \mathbb{C}^k \times \{0\}$. This defines holomorphic map germs

$$\begin{aligned} \psi : (\mathbb{C}^{n+1} \times \mathbb{C}^k \times \mathbb{C}, 0) &\longrightarrow (\mathbb{C}^{n+1}, 0), \\ \varphi : (\mathbb{C}^k \times \mathbb{C}, 0) &\longrightarrow (\mathbb{C}^k, 0) \end{aligned}$$

with $\psi(z, 0, 0) = z$. From the condition $X(G) = 0$ it follows that G is constant along the phase curves. Thus

$$G(z, u, v) = G(\psi(z, u, v), \varphi(u, v), 0) = F(\psi(z, u, v), \varphi(u, v)).$$

This means that ψ is an equivalence of G with the unfolding F induced by φ . This proves Lemma 3.8. \square

Proof of Proposition 3.17. Let $F : (\mathbb{C}^{n+1} \times \mathbb{C}^p, 0) \rightarrow (\mathbb{C}, 0)$ be the unfolding of f from Proposition 3.17 and let $F' : (\mathbb{C}^{n+1} \times \mathbb{C}^l, 0) \rightarrow (\mathbb{C}, 0)$, $(z, u') \mapsto F'(z, u')$, be an arbitrary unfolding of f . We form the “sum” of F and F' :

$$\begin{aligned} H : (\mathbb{C}^{n+1} \times \mathbb{C}^p \times \mathbb{C}^l, 0) &\longrightarrow (\mathbb{C}, 0) \\ (z, u, u') &\longmapsto F(z, u) + F'(z, u') - f(z). \end{aligned}$$

We denote by H_0, H_1, \dots, H_l the restrictions of H to the subspaces

$$\mathbb{C}^{n+1} \times \mathbb{C}^p \subset \mathbb{C}^{n+1} \times \mathbb{C}^p \times \mathbb{C} \subset \dots \subset \mathbb{C}^{n+1} \times \mathbb{C}^p \times \mathbb{C}^l.$$

In particular $H_0 = F$ and $H_l = H$. Since F is infinitesimally versal, so too the unfoldings H_0, H_1, \dots, H_l are infinitesimally versal, as is easy to see. We can therefore apply Lemma 3.8 successively to show that the unfolding H is equivalent to an unfolding of f induced from F . Since the unfolding F' of f is induced from H , it follows that the unfolding F' is equivalent to an unfolding of f induced from F . Thus the unfolding F is universal and Proposition 3.17 is proved. \square

We record yet another corollary of Proposition 3.17.

Corollary 3.4. *Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ with a nondegenerate critical point at 0. Then*

$$\begin{aligned} F : (\mathbb{C}^{n+1} \times \mathbb{C}, 0) &\longrightarrow (\mathbb{C}, 0) \\ (z, t) &\longmapsto f(z) - t \end{aligned}$$

is a universal unfolding of f .

Proof. This follows from Proposition 3.17 and Lemma 3.7. \square

3.8. Morsifications

Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ.

Definition. A *morsification* of $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ is a representative $F : M \times U \rightarrow \mathbb{C}$ of an unfolding

$$\begin{aligned} F : (\mathbb{C}^{n+1} \times \mathbb{C}, 0) &\longrightarrow (\mathbb{C}, 0) \\ (z, \lambda) &\longmapsto f_\lambda(z) \end{aligned}$$

of f such that for almost all $\lambda \in U \setminus \{0\}$ (everywhere except on a Lebesgue null set) the function $f_\lambda : M \rightarrow \mathbb{C}$ is a Morse function. The Morse function f_λ is itself often called a morsification of f .

We next want to show that every isolated singularity of f has a morsification.

Proposition 3.18. *Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ with an isolated singularity at 0. Then f has a morsification.*

Proof. Let $f : M \rightarrow \mathbb{C}$ be a representative of f , where M is an open neighborhood of 0 in \mathbb{C}^{n+1} . We consider the map

$$\begin{aligned} \text{grad } f : M &\longrightarrow \mathbb{C}^{n+1} \\ z &\longmapsto \text{grad } f(z) = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{n+1}} \right). \end{aligned}$$

By Proposition 3.7 the critical values of $\text{grad } f$ form a Lebesgue null set in \mathbb{C}^{n+1} . Let $\lambda(\tilde{a}_1, \dots, \tilde{a}_{n+1})$ for $\lambda \in \mathbb{C}$ be a regular value near $0 \in \mathbb{C}^{n+1}$. Put

$$\tilde{f}_\lambda(z) := f(z) - \lambda \sum_{i=1}^{n+1} \tilde{a}_i z_i.$$

A point $p \in \mathbb{C}^{n+1}$ is then a critical point of \tilde{f}_λ if and only if $\text{grad } f(p) = \lambda(\tilde{a}_1, \dots, \tilde{a}_{n+1})$. But since $\lambda(\tilde{a}_1, \dots, \tilde{a}_{n+1})$ is not a critical value of $\text{grad } f$, we see that $\text{grad } f$ is biholomorphic at p , whence

$$\det \left(\frac{\partial^2 \tilde{f}_\lambda}{\partial z_i \partial z_j}(p) \right) \neq 0,$$

i.e., p is a nondegenerate critical point of \tilde{f}_λ . Therefore \tilde{f}_λ has only nondegenerate critical points. Since the set of regular values of the map $\text{grad } f$ is open, we may replace $\lambda(\tilde{a}_1, \dots, \tilde{a}_{n+1})$ by a neighboring regular value $\lambda(a_1, \dots, a_{n+1})$ so that the function

$$f_\lambda(z) := f(z) - \lambda \sum_{i=1}^{n+1} a_i z_i$$

has only nondegenerate points with distinct critical values. \square

Definition. The number $\mu = \mu(f)$ of nondegenerate critical points of a morsification f_λ of f is called the *Milnor number* (or *multiplicity*) of the singularity f .

In the following proposition we give another description of this number. It follows from this proposition that the Milnor number is independent of the morsification chosen. Our proof follows [Orl76, I.5].

Proposition 3.19. *Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ with an isolated singularity at 0. Then*

$$\mu(f) = \dim_{\mathbb{C}} \mathcal{O}_{n+1} / \mathcal{O}_{n+1} \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{n+1}} \right).$$

Proof. By Proposition 3.16 the number on the right side is finite. We put

$$d := \dim_{\mathbb{C}} \mathcal{O}_{n+1} / \mathcal{O}_{n+1} \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{n+1}} \right).$$

By definition, $\mu(f)$ is equal to the number of nondegenerate critical points of a morsification f_λ of f . By the proof of Proposition 3.18 this last number is equal to the number of preimage points (in a neighborhood of 0) of a regular value near 0 of the finite holomorphic map

$$\text{grad } f : M \longrightarrow \mathbb{C}^{n+1},$$

where M is a suitable open neighborhood of 0 in \mathbb{C}^{n+1} . We now show that this number is equal to d .

We denote the component functions of the map $\text{grad } f$ by $f_1 := \partial f / \partial z_1, \dots, f_{n+1} := \partial f / \partial z_{n+1}$. Since f has an isolated singularity at 0, by hypothesis, 0 is an isolated point of $(\text{grad } f)^{-1}(0)$. By Proposition 2.42 the map $(\text{grad } f)^* : \mathcal{O}_{n+1} \rightarrow \mathcal{O}_{n+1}$ is finite. Since \mathcal{O}_{n+1} has dimension $n + 1$ (Corollary 2.6), it follows from the proof of Proposition 2.44 that the homomorphism $(\text{grad } f)^*$ is injective. To distinguish, we denote the coordinates of the image space of $\text{grad } f$ by y_1, \dots, y_{n+1} . Furthermore, we put $A = \mathbb{C}\{z_1, \dots, z_{n+1}\}$, $R = \mathbb{C}\{f_1, \dots, f_{n+1}\} \subset A$. Since $(\text{grad } f)^* : \mathbb{C}\{y_1, \dots, y_{n+1}\} \rightarrow A$ is injective, R is isomorphic to $\mathbb{C}\{y_1, \dots, y_{n+1}\}$.

Now $d < \infty$ so (f_1, \dots, f_{n+1}) is a parameter system for A and by Corollary 2.9 it is also a prime sequence in A . From Proposition 2.52 it follows that A is a free R -module. By the Weierstrass preparation theorem for modules (Corollary 2.5), A is generated as an R -module by d elements and d is the minimal number of generators. Thus A has an R -basis of d elements. Let K_R and K_A be the respective quotient fields of R and A . Then K_A is a field extension of K_R of degree d . Since all elements of K_A are algebraic

over K_R , we have $K_A = K_R(z_1, \dots, z_{n+1})$. By the primitive element theorem (see e.g. [Lan02, V, §4]) there is a $\xi \in K_A$, $\xi = \sum_{i=1}^{n+1} c_i z_i$ with $c_i \in K_R$, with $K_A = K_R(\xi)$. By multiplying the minimal polynomial of ξ over K_R by the common denominator of the coefficients, we can assume that ξ satisfies an equation

$$b_0 \xi^d + b_1 \xi^{d-1} + \dots + b_d = 0$$

with $b_j \in R$, $j = 0, \dots, d$, $b_0 \neq 0$. On multiplying this equation by b_0^{d-1} , one obtains the equation

$$(b_0 \xi)^d + b_1 (b_0 \xi)^{d-1} + \dots + b_0^{d-1} b_d = 0.$$

It follows from this equation that $b_0 \xi$ is integral over R . By changing coordinates we can assume that $b_0 \xi = z_1$. Then z_1 satisfies an equation $p(f_1, \dots, f_{n+1})(z_1) = 0$, where

$$p(f_1, \dots, f_{n+1})(t) = t^d + a_1(f_1, \dots, f_{n+1})t^{d-1} + \dots + a_d(f_1, \dots, f_{n+1})$$

and $a_i(f_1, \dots, f_{n+1})$, for $i = 1, \dots, d$, is a power series in f_1, \dots, f_{n+1} that converges in a neighborhood $U \subset M$ of 0. By Proposition 2.43, $a_i(0) = 0$ for $i = 1, \dots, d$.

Next we consider the hypersurface

$$V = \{(f_1, \dots, f_{n+1}, z_1) \in U \times \mathbb{C} \mid p(f_1, \dots, f_{n+1})(z_1) = 0\} \subset U \times \mathbb{C}$$

with the projection

$$\begin{aligned} \pi : \quad V &\longrightarrow U \\ (f_1, \dots, f_{n+1}, z_1) &\longmapsto (f_1, \dots, f_{n+1}). \end{aligned}$$

Let $\Delta(f_1, \dots, f_{n+1})$ be the discriminant of the polynomial $p(f_1, \dots, f_{n+1})(t)$ (cf. §1.6). This is a polynomial in the coefficients $a_i(f_1, \dots, f_{n+1})$ of the polynomial $p(f_1, \dots, f_{n+1})(t)$. Let

$$D = \{(f_1, \dots, f_{n+1}) \in U \mid \Delta(f_1, \dots, f_{n+1}) = 0\} \subset U.$$

The set D is a hypersurface in U . Outside D the equation

$$p(f_1, \dots, f_{n+1})(z_1) = 0$$

has exactly d distinct solutions in the variable z_1 . Let $C = \pi^{-1}(D)$.

We consider further the map

$$\begin{aligned} \tilde{\sigma} : \quad M &\longrightarrow M \times \mathbb{C} \\ (z_1, \dots, z_{n+1}) &\longmapsto (f_1(z), \dots, f_{n+1}(z), z_1). \end{aligned}$$

Let $W = \tilde{\sigma}^{-1}(V)$. Then W is a neighborhood of 0 in M . We put $\sigma = \tilde{\sigma}|_W$, $B = \sigma^{-1}(C)$. By shrinking U and V , if necessary, we can ensure that σ maps the set $W \setminus B$ homeomorphically onto $V \setminus C$. The composition $\sigma \circ \pi : W \rightarrow U$ agrees with the map $\text{grad } f : W \rightarrow U$. Since $\sigma \circ \pi$ has exactly d preimage

points outside D , this is true for $\text{grad } f$ too. By the remark at the beginning of the proof it follows that

$$\mu(f) = d.$$

This completes the proof of Proposition 3.19. \square

Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ with an isolated singularity at 0, $\text{grad } f(0) = 0$. To be able to survey all possible morsifications, we consider a universal unfolding F of f . By Proposition 3.16 one obtains a universal unfolding F of f as follows: Let $g_0 = -1, g_1, \dots, g_{\mu-1}$ be representatives of a basis of the \mathbb{C} -vector space

$$\mathcal{O}_{n+1} / \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right) \mathcal{O}_{n+1},$$

which has dimension μ ; cf. Proposition 3.19. Then put

$$\begin{aligned} F : (\mathbb{C}^{n+1} \times \mathbb{C}^\mu, 0) &\longrightarrow (\mathbb{C}, 0) \\ (z, u) &\longmapsto f(z) + \sum_{j=0}^{\mu-1} g_j(z) u_j. \end{aligned}$$

Let

$$F : M \times U \rightarrow \mathbb{C}$$

be a representative of the unfolding F , where M is an open neighborhood of 0 in \mathbb{C}^{n+1} and U is an open neighborhood of 0 in \mathbb{C}^μ . We put

$$\begin{aligned} \mathcal{Y} &:= \{(z, u) \in M \times U \mid F(z, u) = 0\}, \\ \mathcal{Y}_u &:= \{z \in M \mid F(z, u) = 0\}. \end{aligned}$$

Since $F(z, 0) = f(z)$, there is, by Lemma 3.5, an $\varepsilon > 0$ such that each sphere $S_\rho \subset M$ around 0 of radius $\rho \leq \varepsilon$ intersects the set \mathcal{Y}_0 transversally. Let $\varepsilon > 0$ be so chosen. Then there is also an $\eta > 0$ such that for $|u| \leq \eta$ the set $\{u \in \mathbb{C}^\mu \mid |u| \leq \eta\}$ lies entirely in U and \mathcal{Y}_u intersects the sphere S_ε transversally. Let η be so chosen. We put

$$\begin{aligned} \mathcal{X} &:= \{(z, u) \in \mathcal{Y} \mid |z| < \varepsilon, |u| < \eta\}, \\ \bar{\mathcal{X}} &:= \{(z, u) \in \mathcal{Y} \mid |z| \leq \varepsilon, |u| < \eta\}, \\ \partial \bar{\mathcal{X}} &:= \{(z, u) \in \mathcal{Y} \mid |z| = \varepsilon, |u| < \eta\}, \\ S &:= \{u \in U \mid |u| < \eta\}, \end{aligned}$$

$$\begin{aligned} p : \bar{\mathcal{X}} &\longrightarrow S \\ (z, u) &\longmapsto u. \end{aligned}$$

Let C be the set of critical points of p . If $p_0, \dots, p_{\mu-1}$ are the component functions of p , i.e., $p_i(z, u) = u_i$ for $(z, u) \in \bar{\mathcal{X}}$, then the critical points of p

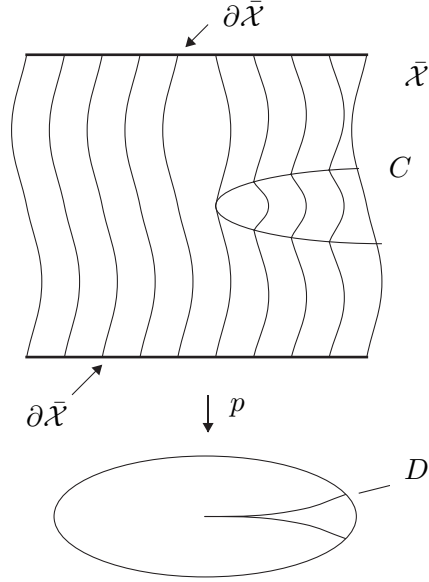


Figure 3.7. Critical set C and discriminant D

are determined by the equations and inequalities

$$\begin{aligned} F(z, u) &= 0, \\ \text{grad } F &= \lambda_0 \text{grad } p_0 + \dots + \lambda_{\mu-1} \text{grad } p_{\mu-1}, \\ |z| &\leq \varepsilon, \quad |u| < \eta, \end{aligned}$$

in the unknowns $z_1, \dots, z_{n+1}, u_0, \dots, u_{\mu-1}, \lambda_0, \dots, \lambda_{\mu-1} \in \mathbb{C}$. It follows that

$$\begin{aligned} C &= \left\{ (z, u) \in M \times U \left| \begin{array}{l} |z| \leq \varepsilon, \quad |u| < \eta, \\ F(z, u) = 0, \\ \frac{\partial F}{\partial z_1}(z, u) = \dots = \frac{\partial F}{\partial z_{n+1}}(z, u) = 0 \end{array} \right. \right\} \\ &= \{(z, u) \in \bar{\mathcal{X}} \mid z \text{ is a critical point of } F(\cdot, u)\}. \end{aligned}$$

Let $D = p(C) \subset S$ be the discriminant of p (see Figure 3.7). The map

$$p' := p|_{\bar{\mathcal{X}} - p^{-1}(D)} : \bar{\mathcal{X}} - p^{-1}(D) \longrightarrow S - D$$

is then a submersion.

For the proof of the following proposition we need a generalization of the Morse lemma (Proposition 3.15).

Proposition 3.20 (Generalized Morse lemma). *Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ with*

$$\text{grad } f(0) = 0 \quad \text{and} \quad \text{rank}((\partial^2 f / \partial z_i \partial z_j)(0)) = k.$$

Then there are coordinates (z_1, \dots, z_{n+1}) of \mathbb{C}^{n+1} around 0 such that f is represented in a neighborhood of 0 as

$$(z_1, \dots, z_{n+1}) \longmapsto z_1^2 + \dots + z_k^2 + g(z_{k+1}, \dots, z_{n+1}),$$

where $g \in \mathfrak{m}_{n+1}^3$.

Proof. By a linear coordinate transformation one can ensure that f is of the form

$$f(y_1, \dots, y_{n+1}) = y_1^2 + \dots + y_k^2 + h(y_1, \dots, y_{n+1})$$

with $h \in \mathfrak{m}^3$. Let $f_0 : (\mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$ be defined by

$$f_0(y_1, \dots, y_k) = f(y_1, \dots, y_k, 0, \dots, 0).$$

Then one can regard f as an unfolding of f_0 . The function germ f_0 has a nondegenerate critical point at 0. By Corollary 3.4, f is equivalent as an unfolding to an unfolding of f_0 induced from the universal unfolding

$$\begin{aligned} F : (\mathbb{C}^k \times \mathbb{C}, 0) &\longrightarrow (\mathbb{C}, 0) \\ (y, t) &\longmapsto f_0(y) - t. \end{aligned}$$

Thus there are holomorphic map germs

$$\begin{aligned} g : (\mathbb{C}^{n+1-k}, 0) &\longrightarrow (\mathbb{C}, 0), \\ \psi : (\mathbb{C}^k \times \mathbb{C}^{n+1-k}, 0) &\longrightarrow (\mathbb{C}^k, 0) \end{aligned}$$

with $\psi(y_1, \dots, y_k, 0, \dots, 0) = (y_1, \dots, y_k)$ such that

$$f(y_1, \dots, y_{n+1}) = f_0(\psi(y_1, \dots, y_{n+1})) + g(y_{k+1}, \dots, y_{n+1}).$$

By the hypothesis on ψ the map germ

$$\begin{aligned} \tilde{\psi} : (\mathbb{C}^{n+1}, 0) &\longrightarrow (\mathbb{C}^{n+1}, 0) \\ y = (y_1, \dots, y_{n+1}) &\longmapsto (\psi_1(y), \dots, \psi_k(y), y_{k+1}, \dots, y_{n+1}) \end{aligned}$$

is biholomorphic. We put

$$(z_1, \dots, z_{n+1}) := (\psi_1(y), \dots, \psi_k(y), y_{k+1}, \dots, y_{n+1}).$$

It follows that

$$(f \circ \tilde{\psi}^{-1})(z_1, \dots, z_{n+1}) = f_0(z_1, \dots, z_k) + g(z_{k+1}, \dots, z_{n+1}).$$

We now apply the Morse lemma (Proposition 3.15) to f_0 , proving the result. \square

Proposition 3.21. *For a suitable $\eta > 0$ we have*

- (i) *the map $p : \bar{\mathcal{X}} \rightarrow S$ is proper,*
- (ii) *C is a nonsingular analytic subset of \mathcal{X} and is closed in $\bar{\mathcal{X}}$,*
- (iii) *the restriction $p|_C : C \rightarrow S$ is finite (i.e., proper with finite fibers),*
- (iv) *the discriminant D is an irreducible hypersurface in S .*

Proof. (i) If $K \subset S$ is compact, so too is $\bar{\mathcal{X}} \cap p^{-1}(K)$ compact. Thus $p : \bar{\mathcal{X}} \rightarrow S$ is proper.

(ii) By the choice of ε and η the map $p|_{\partial\bar{\mathcal{X}}}$ is a submersion. It follows that $C \cap \partial\mathcal{X} = \emptyset$, i.e., $C \subset \mathcal{X}$. From the description above, C is an analytic subset of \mathcal{X} and so is closed in $\bar{\mathcal{X}}$.

We must next show that C is nonsingular for some suitable $\eta > 0$. By Proposition 3.20 we can write f in suitable coordinates as

$$f(z_1, \dots, z_{n+1}) = h(z_1, \dots, z_r) + z_{r+1}^2 + \dots + z_{n+1}^2$$

where $h \in \mathfrak{m}_{n+1}^3$. Then we can choose the coordinate functions z_1, \dots, z_r as representatives g_1, \dots, g_r of elements of

$$\mathcal{O}_{n+1} / \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{n+1}} \right) \mathcal{O}_{n+1} .$$

Then a representative of the universal unfolding $F : (\mathbb{C}^{n+1} \times \mathbb{C}^\mu, 0) \rightarrow (\mathbb{C}, 0)$ of f has the form

$$(3.5) \quad F(z, u) = -u_0 + u_1 z_1 + u_2 z_2 + \dots + u_r z_r + z_{r+1}^2 + \dots + z_{n+1}^2 + a(z, u),$$

where $a(z, u)$ has terms only of third or higher order.

The critical set C can now be specified as

$$C = \{(z, u) \in M \times U \mid |z| < \varepsilon, |u| < \eta, \sigma(z, u) = 0\},$$

where

$$\sigma : M \times U \rightarrow \mathbb{C}^{n+2}$$

is the map given by $\sigma = (F, \partial F / \partial z_1, \dots, \partial F / \partial z_{n+1})$. In view of the above form for F we have

$$\begin{aligned} \frac{\partial F}{\partial z_j}(0) &= 0, \quad \frac{\partial^2 F}{\partial z_i \partial z_j}(0) = 2\delta_{ij} \text{ for } 1 \leq i \leq n+1, r+1 \leq j \leq n+1, \\ \frac{\partial F}{\partial u_0}(0) &= -1, \quad \frac{\partial F}{\partial u_i}(0) = 0 \text{ for } 1 \leq i \leq r, \\ \frac{\partial^2 F}{\partial z_i \partial u_j}(0) &= \delta_{ij} \text{ for } 1 \leq i \leq n+1, 0 \leq j \leq r. \end{aligned}$$

The matrix

$$\left(\begin{array}{ccc|ccc} \frac{\partial F}{\partial z_{r+1}}(0) & \cdots & \frac{\partial F}{\partial z_{n+1}}(0) & \frac{\partial F}{\partial u_0}(0) & \cdots & \frac{\partial F}{\partial u_r}(0) \\ \frac{\partial^2 F}{\partial z_1 \partial z_{r+1}}(0) & \cdots & \frac{\partial^2 F}{\partial z_1 \partial z_{n+1}}(0) & \frac{\partial^2 F}{\partial z_1 \partial u_0}(0) & \cdots & \frac{\partial^2 F}{\partial z_1 \partial u_r}(0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial z_{n+1} \partial z_{r+1}}(0) & \cdots & \frac{\partial^2 F}{\partial z_{n+1} \partial z_{n+1}}(0) & \frac{\partial^2 F}{\partial z_{n+1} \partial u_0}(0) & \cdots & \frac{\partial^2 F}{\partial z_{n+1} \partial u_r}(0) \end{array} \right)$$

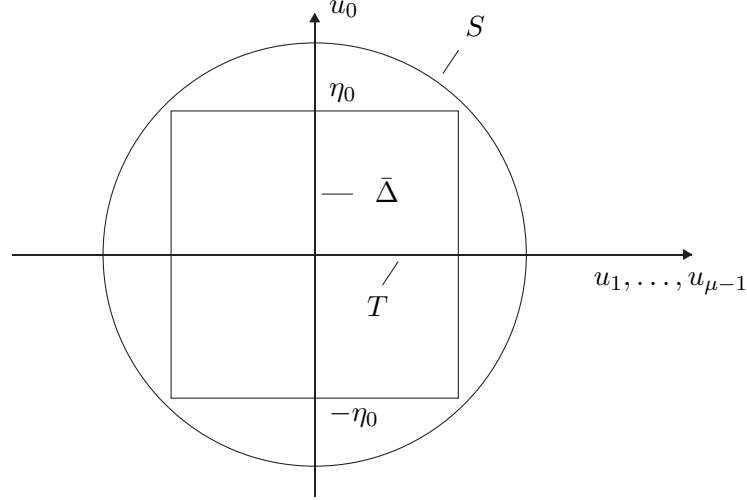


Figure 3.8. $\bar{\Delta} \times T \subset S$

is therefore an $(n+2) \times (n+2)$ -submatrix of rank $n+2$ of the functional matrix of σ at 0. By the implicit function theorem C is therefore a $(\mu-1)$ -dimensional submanifold of \mathcal{X} in a neighborhood of 0. Therefore C is not singular for $\eta > 0$ sufficiently small.

(iii), (iv) By (ii), C is an irreducible analytic subset of \mathcal{X} . The map $p : \bar{\mathcal{X}} \rightarrow S$ is the restriction of a linear projection and, by (i), is proper. Since, by (ii), C is closed in $\bar{\mathcal{X}}$, so too $p|_C : C \rightarrow S$ is proper. By Proposition 2.53 the image $D = p(C)$ is thus an analytic subset of S and the map $p|_C : C \rightarrow S$ is finite. By Proposition 2.54, D is also irreducible. Since the dimension of the analytic set germ (C, x) is equal to $\mu-1$ (Corollary 2.7) for each point $x \in C$, and since $p|_C : C \rightarrow D$ is finite, we have $\dim_u D = \mu-1$ for all $u \in D$. It follows from Proposition 2.47 that D is a hypersurface in S . \square

Let $T := \{t = (u_1, \dots, u_{\mu-1}) \in \mathbb{C}^{\mu-1} \mid |t| < \eta_1\}$ be an open ball of radius $\eta_1 > 0$ around $0 \in \mathbb{C}^{\mu-1}$ and let $\bar{\Delta} \subset \mathbb{C}$ be a disc of radius $\eta_0 > 0$ around 0 such that

$$\bar{\Delta} \times T = \{(u_0, u_1, \dots, u_{\mu-1}) \in \mathbb{C}^\mu \mid u_0 \in \bar{\Delta}, (u_1, \dots, u_{\mu-1}) \in T\} \subset S$$

(see Figure 3.8). We replace S by $\bar{\Delta} \times T$ and $\bar{\mathcal{X}}$ by $\bar{\mathcal{X}} \cap p^{-1}(\bar{\Delta} \times T)$, retaining, however, the same symbols. For $t \in T$ we put

$$\begin{aligned} S_t &:= \bar{\Delta} \times \{t\} \\ &= \{(u_0, u_1, \dots, u_{\mu-1}) \in \bar{\Delta} \times T \mid (u_1, \dots, u_{\mu-1}) = t\}. \end{aligned}$$

We consider a suitable representative $\tilde{F} : \tilde{\mathcal{X}} \rightarrow S = \bar{\Delta} \times T$ of

$$\begin{aligned} \tilde{F} : (\mathbb{C}^{n+1} \times \mathbb{C}^\mu, 0) &\longrightarrow (\mathbb{C} \times \mathbb{C}^{\mu-1}, 0) \\ (z, u) &\longmapsto (F(z, u), u_1, \dots, u_{\mu-1}) \end{aligned}$$

where F is of the form

$$F(z, u) = f(z) - u_0 + \sum_{j=1}^{\mu-1} g_j(z)u_j.$$

Lemma 3.9. *Let $t \in T$, $t \neq 0$. The function*

$$f_{\lambda t} : M \longrightarrow \Delta \times \{\lambda t\}$$

with $f_{\lambda t}(z) = \tilde{F}(z, 0, \lambda t)$ represents a morsification of f for $\lambda \neq 0$.

Proof. We again consider a representative of the universal unfolding of f having the form (3.5) as in the proof of Proposition 3.21. Then it follows as in the proof of Proposition 3.18 that $F(\cdot, u)$, for $(u_1, \dots, u_{\mu-1}) = \lambda t$ outside a Lebesgue null set in a neighborhood of 0, has only nondegenerate critical points with pairwise distinct critical values. Hence $f_{\lambda t}$ for $\lambda \neq 0$ is a morsification of f . \square

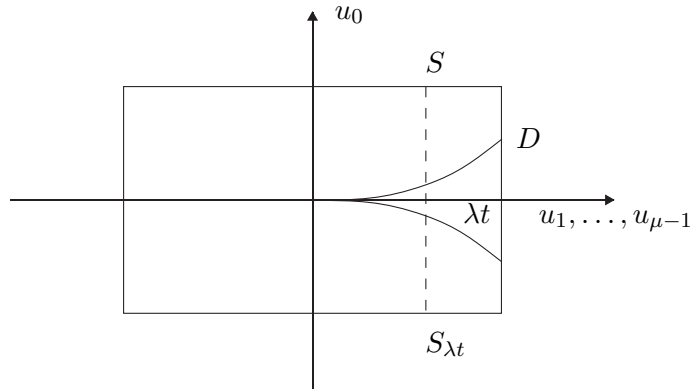


Figure 3.9. The line $\mathbb{C} \times \{\lambda t\}$ intersects the discriminant D transversally

Remark 3.18. That $f_{\lambda t}$ is a Morse function is tantamount to the line $\mathbb{C} \times \{\lambda t\}$ being in general position in the discriminant D , i.e., intersecting the discriminant D transversally at regular points (see Figure 3.9). This follows from the description

$$C = \{(z, u) \in \tilde{\mathcal{X}} \mid z \text{ is a critical point of } F(\cdot, u)\}.$$

If the representative F of the universal unfolding of f has the form (3.5), it follows that the line $\mathbb{C} \times \{0\}$ does not lie in the tangent space (tangent

cone) of D at 0. It follows that for λt outside a Lebesgue null set the line $\mathbb{C} \times \{\lambda t\}$ is in general position to the discriminant D , whence $f_{\lambda t}$ is a Morse function.

3.9. Finitely determined function germs

Next we consider an equivalence relation on the set of all holomorphic function germs $f : (\mathbb{C}^{n+1}, p) \rightarrow (\mathbb{C}, s)$.

Definition. Two holomorphic function germs $f_1 : (\mathbb{C}^{n+1}, p_1) \rightarrow (\mathbb{C}, s_1)$, $f_2 : (\mathbb{C}^{n+1}, p_2) \rightarrow (\mathbb{C}, s_2)$ are called *right equivalent* if there are representatives $\tilde{f}_1 : U_1 \rightarrow \mathbb{C}$ of f_1 and $\tilde{f}_2 : U_2 \rightarrow \mathbb{C}$ of f_2 and a biholomorphic map $\varphi : U_1 \rightarrow U_2$ with $\varphi(p_1) = p_2$ such that $\tilde{f}_2 \circ \varphi = \tilde{f}_1$, where $\psi : \mathbb{C} \rightarrow \mathbb{C}$ is the translation $z \mapsto z + (s_2 - s_1)$, i.e., if the following diagram commutes:

$$\begin{array}{ccc} U_1 & \xrightarrow{\varphi} & U_2 \\ \tilde{f}_1 \downarrow & & \downarrow \tilde{f}_2 \\ \mathbb{C} & \xrightarrow{\psi} & \mathbb{C} \end{array}$$

The right equivalence class of a holomorphic function germ $f : (\mathbb{C}^{n+1}, p) \rightarrow (\mathbb{C}, s)$ is denoted by $[f]$. If f is an isolated singularity, we also call $[f]$ an isolated singularity.

Definition. Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ with an isolated singularity at 0. The *modality* (or *module number*) of f , symbolically, $\text{mod}(f)$, is the smallest number m for which there exists a representative $p : \mathcal{X} \rightarrow S$ of the universal unfolding $F : (\mathbb{C}^{n+1} \times \mathbb{C}^m, 0) \rightarrow (\mathbb{C}, 0)$ of f such that for all $(z, u) \in \mathcal{X}$ the function germs $F_u : (\mathbb{C}^{n+1}, z) \rightarrow (\mathbb{C}, F(z, u))$ given by $F_u(z') = F(z', u)$ fall into finitely many families of right equivalence classes depending on at most m (complex) parameters. The modality depends only on the right equivalence class $[f]$ of f , so we may write $\text{mod}[f]$. If $\text{mod}[f] = m$, we call $[f]$ *m-modal* (or *m-modular*). For $m = 0, 1, 2$ one says *simple*, *unimodal* (or *unimodular*), and *bimodal* (*bimodular*), respectively.

The fact that f is simple means that there are finitely many function germs $f_i : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ ($i = 1, \dots, k$) such that for each unfolding $F : (\mathbb{C}^{n+1} \times \mathbb{C}^r, 0) \rightarrow (\mathbb{C}, 0)$ of f there exists a representative $p : \mathcal{X} \rightarrow S$ such that for all $(z, u) \in \mathcal{X}$ the function germ $F_u : (\mathbb{C}^{n+1}, z) \rightarrow (\mathbb{C}, F(z, u))$ is right equivalent to an f_i .

Example 3.8. If $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ is a holomorphic function germ with a nondegenerate critical point at 0, then f is simple. This follows from Corollary 3.4.

Example 3.9. Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be given by $f(z_1, z_2) = z_1 z_2 (z_1 + z_2)(z_1 - z_2)$. We show that f is not simple. To do this, we consider the unfolding

$$F(z_1, z_2, u) := (z_1 + uz_2)z_2(z_1 + z_2)(z_1 - z_2).$$

Then the germs of the functions

$$f_u : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0), \quad f_u(z_1, z_2) = F(z_1, z_2, u),$$

are in general not right equivalent to one another, for, if this were the case, for two values $u_1, u_2 \in \mathbb{C}$ there would have to be a biholomorphic map $\varphi : U_1 \rightarrow U_2$, $U_1, U_2 \subset \mathbb{C}^2$ open, with $\varphi(0) = 0$, such that $f_{u_1} = f_{u_2} \circ \varphi$. Since f_{u_1} and f_{u_2} are homogeneous polynomials of fourth degree, one can assume that φ is linear. The space of all lines in \mathbb{C}^2 through the origin is the 1-dimensional complex projective space $\mathbb{P}_1\mathbb{C}$. This space is isomorphic to the Riemann sphere $\hat{\mathbb{C}}$. The four lines

$$z_1 + uz_2 = 0, \quad z_2 = 0, \quad z_1 + z_2 = 0, \quad z_1 - z_2 = 0$$

determine four points on the projective line $\mathbb{P}_1\mathbb{C}$. The map φ induces a fractional linear transformation $\hat{\varphi}$ of $\hat{\mathbb{C}}$ onto itself. By Proposition 1.26 a fractional linear transformation leaves the cross ratio of these four points invariant. The cross ratio of these four points depends, however, on u .

In what follows we shall classify the simple function germs. To do this, we need a means of deciding whether two function germs are right equivalent. We now address this matter.

Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ.

Definition. The Taylor polynomial of f around 0 of degree r is called the r -jet of f and is denoted by $j^r f$.

Definition. The function germ f is called r -determined if every holomorphic function germ $g : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ with $j^r f = j^r g$ is right equivalent to f .

In particular an r -determined function germ f is right equivalent to the polynomial $j^r f$.

Let us again write \mathfrak{m} for the maximal ideal of \mathcal{O}_{n+1} . Then

$$\mathfrak{m}^r = \{f \in \mathcal{O}_{n+1} \mid j^{r-1} f = 0\}.$$

Theorem 3.1 (Mather). *Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ. If*

$$\mathfrak{m}^{r+1} \subset \mathfrak{m}^2 \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{n+1}} \right) + \mathfrak{m}^{r+2},$$

then f is r -determined.

Proof. We must show that for each element $h \in \mathfrak{m}^{r+1}$ the function germ $f + h$ is right equivalent to f . We proceed in a way similar to that of the proof of the complex Morse lemma (Proposition 3.15).

So we consider the function

$$F(z, t) = f(z) + th(z)$$

for z in a neighborhood U of 0 in \mathbb{C}^{n+1} and $t \in [0, 1]$. We seek a holomorphic local one parameter group $g : [0, 1] \times U' \rightarrow U'$ for a suitable neighborhood U' of 0 in \mathbb{C}^{n+1} such that

$$(3.6) \quad F(g_t(z), t) = f(z), \quad g_0(z) = z, \quad g_t(0) = 0 \text{ for } z \in U', t \in [0, 1].$$

A holomorphic vector field

$$X = \frac{\partial}{\partial t} + \sum_{j=1}^{n+1} a_{t,j} \frac{\partial}{\partial z_j}$$

is induced on $[0, 1] \times U'$ by g . From the equation (3.6) it follows that $XF = 0$ for this vector field. Hence we obtain the following equation for the functions $a_{t,j}$ that we seek:

$$\frac{\partial F}{\partial t} + \sum_{j=1}^{n+1} a_{t,j} \frac{\partial F}{\partial z_j} = h + \sum_{j=1}^{n+1} a_{t,j} \frac{\partial F}{\partial z_j} = 0.$$

We can rewrite this equation as follows:

$$(3.7) \quad \sum_{j=1}^{n+1} a_{t,j} \frac{\partial F}{\partial z_j} = -h.$$

Claim 3.1. Under the hypotheses of Theorem 3.1 the equation (3.7) is solvable for the coefficients $a_{t,j}$.

Proof. Let $t_0 \in [0, 1]$, let \mathcal{E} be the ring of holomorphic function germs

$$(\mathbb{R} \times \mathbb{C}^{n+1}, (t_0, 0)) \rightarrow (\mathbb{C}, 0)$$

and let \mathfrak{n} be the corresponding maximal ideal. Let

$$J = \mathcal{E} \left(\frac{\partial F}{\partial z_1}, \dots, \frac{\partial F}{\partial z_{n+1}} \right).$$

By hypothesis, $h \in \mathfrak{m}^{r+1}$. We can solve the equation (3.7) for $a_{t,j}$ if

$$\mathfrak{m}^{r+1} \subset J\mathcal{E}\mathfrak{m}.$$

From the definition of F

$$\frac{\partial F}{\partial z_j} = \frac{\partial f}{\partial z_j} + t \frac{\partial h}{\partial z_j},$$

and so

$$\frac{\partial f}{\partial z_j} = \frac{\partial F}{\partial z_j} - t \frac{\partial h}{\partial z_j}.$$

Hence

$$\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{n+1}} \right) \subset J + \mathfrak{m}^{r-1}.$$

Since $\mathfrak{m}^{r+1} \subset \mathfrak{m}^2 (\partial f / \partial z_1, \dots, \partial f / \partial z_{n+1}) + \mathfrak{m}^{r+2}$, by hypothesis, we obtain

$$\begin{aligned} \mathcal{E}\mathfrak{m}^{r+1} &\subset \mathcal{E}\mathfrak{m}^2 \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{n+1}} \right) + \mathcal{E}\mathfrak{m}^{r+2} \\ &\subset \mathfrak{m}^2 J + \mathfrak{n}\mathcal{E}\mathfrak{m}^{r+1}. \end{aligned}$$

The claim follows from Nakayama's lemma (Corollary 2.3). \square

As in the proof of the complex Morse lemma it now follows that there exists a holomorphic local one parameter group $g : [0, 1] \times U' \rightarrow U'$ satisfying the equation (3.6). Then the map germ $g_1 : (U', 0) \rightarrow (U', 0)$ satisfies

$$(f + h) \circ g_1 = f$$

so f is right equivalent to $f + h$. \square

Corollary 3.5. *Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ. If*

$$\mathfrak{m}^{r-1} \subset \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{n+1}} \right),$$

for $r \geq 1$, then f is r -determined.

Proof. This follows from Theorem 3.1 since $\mathfrak{m}^{r+1} = \mathfrak{m}^2 \cdot \mathfrak{m}^{r-1}$ for $r \geq 1$. \square

Corollary 3.6. *Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ with an isolated singularity at 0. Then f is right equivalent to a polynomial.*

Proof. Again let $f_j := \partial f / \partial z_j$, $j = 1, \dots, n+1$, be the partial derivatives of f , and let J_f of f_1, \dots, f_{n+1} be the ideal they span. From the proof of Proposition 3.16 it follows that $\mathfrak{m}^{r-1} \subset J_f$ for some $r \geq 1$. By Corollary 3.5 f is right equivalent to $j^r f$, so to a polynomial. \square

We next apply Proposition 3.12 and Remark 3.14 to the study of holomorphic function germs.

Proposition 3.22. *Suppose $k \geq 3$ and let $f(z_1, \dots, z_k)$ be a homogeneous polynomial of degree 3 having an isolated singularity at 0. Then the corresponding holomorphic function germ $f : (\mathbb{C}^k, 0) \rightarrow (\mathbb{C}, 0)$ is not simple.*

Proof. The homogeneous polynomials in the variables z_1, \dots, z_k of degree 3 form a complex vector space V . The complex dimension of this vector space

is the number of distinct monomials in the variables z_1, \dots, z_k , i.e.,

$$\binom{k+3-1}{3} = \frac{1}{6}k(k+1)(k+2).$$

We consider the unfolding

$$F(z, u) = f(z) + \sum_{|\nu|=3} u_\nu z^\nu$$

of f , where ν runs through all multiindices $\nu = (\nu_1, \dots, \nu_k)$ with $|\nu| = \nu_1 + \dots + \nu_k = 3$. All the function germs $f_u = F(\cdot, u)$ are homogeneous polynomials of degree 3. Two homogeneous polynomials g and \tilde{g} of degree 3 are right equivalent if and only if there is a $\varphi \in \text{GL}(k, \mathbb{C})$ with $\tilde{g} = g \circ \varphi$.

Now $\text{GL}(k, \mathbb{C})$ is a complex Lie group of complex dimension k^2 . The orbit of the natural action of this Lie group on V under $g \in V$ corresponds to the right equivalence class of g . By Proposition 3.12 and Remark 3.14 the orbit is locally a complex submanifold of V of complex dimension $\leq k^2$. But

$$k^2 < \frac{1}{6}k(k+1)(k+2)$$

for $k \geq 3$. On applying this to the unfolding F of f , we see that in every neighborhood of f we can find infinitely many function germs not right equivalent to one another. Therefore f is not simple. \square

Corollary 3.7. *Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a simple holomorphic function germ with an isolated singularity at 0. Then*

$$\text{rank} \left(\frac{\partial^2 f}{\partial z_i \partial z_j}(0) \right) \geq n - 1.$$

Proof. Suppose that

$$\text{rank} \left(\frac{\partial^2 f}{\partial z_i \partial z_j}(0) \right) = k < n - 1.$$

By the generalized Morse lemma (Proposition 3.20) there are coordinates (z_1, \dots, z_{n+1}) such that in these coordinates f can be written as

$$f(z_1, \dots, z_{n+1}) = z_1^2 + \dots + z_k^2 + g(z_{k+1}, \dots, z_{n+1}),$$

where $g \in \mathfrak{m}_{n+1}^3$. But the 3-jet of g is a homogeneous polynomial of degree 3 in $n+1-k \geq 3$ variables having an isolated singularity at 0. By Proposition 3.22, $j^3(g)$, hence g also, is not simple. By the following proposition f is then also not simple. \square

Proposition 3.23. *Let $f_i : (\mathbb{C}^k \times \mathbb{C}^l, 0) \rightarrow (\mathbb{C}, 0)$, $(x, y) \mapsto f_i(x, y)$, $i = 1, 2$, be two function germs of the following form:*

$$\begin{aligned} f_1(x, y) &= x_1^2 + \dots + x_k^2 + g_1(y) && \text{with } g_1 \in \mathfrak{m}^3, \\ f_2(x, y) &= x_1^2 + \dots + x_k^2 + g_2(y) && \text{with } g_2 \in \mathfrak{m}^3. \end{aligned}$$

If f_1 and f_2 are right equivalent, then so too are g_1 and g_2 right equivalent.

Proof. Let

$$\begin{aligned} \varphi : (\mathbb{C}^k \times \mathbb{C}^l, 0) &\longrightarrow (\mathbb{C}^k \times \mathbb{C}^l, 0) \\ (x, y) &\longmapsto (x', y') \end{aligned}$$

be the germ of a biholomorphic map with

$$(3.8) \quad f_1(x', y') = f_1 \circ \varphi(x, y) = f_2(x, y).$$

We consider the partial derivative of f_2 with respect to the coordinate x_i at a point $(0, y)$. Let $\varphi(0, y) = (x', y')$. Then

$$0 = \frac{\partial f_2}{\partial x_i}(0, y) = \sum_{j=1}^k \frac{\partial f_1}{\partial x'_j}(x', y') \frac{\partial x'_j}{\partial x_i}(0, y) + \sum_{j=1}^l \frac{\partial f_1}{\partial y'_j}(x', y') \frac{\partial y'_j}{\partial x_i}(0, y).$$

Let $A(0, y)$ be the matrix

$$A(0, y) = \left(\left(\frac{\partial x'_j}{\partial x_i}(0, y) \right) \right)_{\substack{i=1, \dots, k \\ j=1, \dots, k}}.$$

Since

$$\frac{\partial f_1}{\partial x'_j}(x', y') = 2x'_j$$

and

$$\frac{\partial f_1}{\partial y'_j}(x', y') = \frac{\partial g_1}{\partial y'_j}(x', y'),$$

we obtain following system of equations

$$-2A(0, y)x' = \left(\sum_{j=1}^l \frac{\partial g_1}{\partial y'_j}(x', y') \frac{\partial y'_j}{\partial x_1}(0, y), \dots, \sum_{j=1}^l \frac{\partial g_1}{\partial y'_j}(x', y') \frac{\partial y'_j}{\partial x_k}(0, y) \right)^t.$$

The matrix $A(0, y)$ is invertible for y in a neighborhood of 0 in \mathbb{C}^l . Hence we can solve the system of equations for x' in a neighborhood of $(0, 0)$ in $\varphi(\{0\} \times \mathbb{C}^l)$, i.e., write the coordinate x'_i as a function $x'_i = \xi_i(y)$ of y , where

$$\xi_i \in \mathcal{O}_l \left(\frac{\partial g_1}{\partial y'_1}, \dots, \frac{\partial g_1}{\partial y'_l} \right).$$

It then follows from equation (3.8) that

$$\begin{aligned} g_2(y) &= f_2(0, y) = f_1 \circ \varphi(0, y) = f_1(\xi_1(y), \dots, \xi_k(y), y') \\ &= g_1(y') + \sum_{i=1}^k \xi_i(y)^2. \end{aligned}$$

Thus g_1 is right equivalent to $g_2 - \sum_{i=1}^k \xi_i^2$. The proposition now follows from the following lemma. \square

Lemma 3.10. *Let $g : (\mathbb{C}^l, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ with $g \in \mathfrak{m}^3$. Let*

$$J_g := \mathcal{O}_l \left(\frac{\partial g}{\partial z_1}, \dots, \frac{\partial g}{\partial z_l} \right).$$

If $h \in (J_g)^2$, then $g + h$ is right equivalent to g .

Proof. The proof proceeds similarly to that of Theorem 3.1.

We consider, as there,

$$G(z, t) = g(z) + th(z), \quad t \in [0, 1].$$

Let $t_0 \in [0, 1]$, let \mathcal{E} be the ring of holomorphic function germs

$$(\mathbb{R} \times \mathbb{C}^l, (t_0, 0)) \rightarrow (\mathbb{C}, 0)$$

and let \mathfrak{n} be the corresponding maximal ideal. Let

$$J_G = \mathcal{E} \left(\frac{\partial G}{\partial z_1}, \dots, \frac{\partial G}{\partial z_l} \right).$$

We need to show that

$$h \in \mathfrak{m}J_G.$$

Now

$$\begin{aligned} \mathfrak{m}J_g &\subset \mathfrak{m}J_G + t\mathcal{E} \left(\frac{\partial h}{\partial z_1}, \dots, \frac{\partial h}{\partial z_l} \right) \\ &\subset \mathfrak{m}J_G + \mathfrak{n}\mathfrak{m}J_g. \end{aligned}$$

From Nakayama's lemma (Corollary 2.3) it follows that

$$\mathfrak{m}J_g \subset \mathfrak{m}J_G.$$

Since

$$h \in (J_g)^2 \subset \mathfrak{m}J_g,$$

it follows that $h \in \mathfrak{m}J_G$, as was to be shown. \square

By Corollary 3.7 the simple function germs $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ play a special role. For this reason we shall classify them first of all in the next section. The general classification will then follow with the help of Proposition 3.23.

Proposition 3.24. *Let $f(z_1, z_2)$ be a homogeneous polynomial of degree 4 having an isolated singularity at 0. Then the corresponding function germ $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ is not simple.*

Proof. The proof proceeds as that of Proposition 3.22, where now we apply the fact that the vector space of homogeneous polynomials of degree 4 in the two variable z_1, z_2 has complex dimension

$$\binom{2+4-1}{4} = 5$$

while

$$\dim \mathrm{GL}(2, \mathbb{C}) = 4.$$

This proves Proposition 3.24. \square

Immediately from this:

Corollary 3.8. *Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be a simple holomorphic function germ with an isolated singularity at 0. Then $j^3 f \neq 0$.*

3.10. Classification of simple singularities

Our aim is now, first of all, to classify the simple singularities in \mathbb{C}^2 . We wish to prove the following proposition:

Proposition 3.25. *Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be a simple holomorphic function germ with an isolated singularity at 0 and $\mathrm{grad} f(0) = 0$. Then f is right equivalent to one of the following simple function germs:*

- (a) $x^{k+1} + y^2$ with $k \geq 1$ (A_k),
- (b) $x^2y + y^{k-1}$ with $k \geq 4$ (D_k),
- (c) $x^3 + y^4$ (E_6),
- (d) $x^3 + xy^3$ (E_7),
- (e) $x^3 + y^5$ (E_8).

Our proof of this proposition follows the representation in [BK91] which in turn rests on the original work [Arn73]. We need three lemmas for the proof:

Lemma 3.11. *Let $f(x, y)$ be a homogeneous polynomial of degree 3. Then f can be brought by a \mathbb{C} -linear transformation into one of the forms*

$$0, \quad x^2y + y^3, \quad x^2y, \quad x^3.$$

Proof. We have

$$f(x, y) = (a_1x + b_1y)(a_2x + b_2y)(a_3x + b_3y)$$

for suitable complex numbers a_i, b_i , $i = 1, 2, 3$. Now the equation

$$a_ix + b_iy = 0$$

is the equation of a line in \mathbb{C}^2 , which can be regarded as a point of the complex projective line $\mathbb{P}_1\mathbb{C}$. Then Lemma 3.11 follows from the fact that three distinct points in $\mathbb{P}_1\mathbb{C}$ can be mapped to three arbitrary other points by a fractional linear transformation. \square

Lemma 3.12. *The holomorphic function germs*

$$\begin{aligned} f_1 & : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0), \quad (x, y) \mapsto x^2, \\ f_2 & : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0), \quad (x, y) \mapsto x^2y, \end{aligned}$$

are not simple.

Proof. The unfoldings $F_k(x, y, t) := x^2 + ty^k$ of f_1 are not right equivalent to one another for fixed t and $k = 1, 2, \dots$. This follows from the fact that y^k is not right equivalent to y^l for $k \neq l$ (Exercise). Hence f_1 is not simple.

For f_2 one considers correspondingly the unfoldings $F_k(x, y, t) := x^2y + ty^k$. \square

Lemma 3.13. *Let $f_t : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$, $(x, y) \mapsto x^3 + xy^4 + ty^6$ with $t \in \mathbb{C}$. Then in general f_t and $f_{t'}$ are not right equivalent for $t \neq t'$.*

Proof. We write f_t in the form

$$f_t(x, y) = (x - \lambda_1 y^2)(x - \lambda_2 y^2)(x - \lambda_3 y^2),$$

where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the equation $\lambda^3 + \lambda + t = 0$. The zero set

$$V_t := \{(x, y) \in \mathbb{C}^2 \mid f_t(x, y) = 0\}$$

of f_t consists of the three parabolas $x = \lambda_i y^2$, $i = 1, 2, 3$, with the same tangents at 0. We show that two systems V_t and $V_{t'}$ of three such parabolas for $t \neq t'$ cannot in general be mapped into one another by a biholomorphic map.

Not all three roots of the equation $\lambda^3 + \lambda + t = 0$ can be equal to 0. Without loss of generality let $\lambda_2 \neq 0$. We assume in the sequel that all three roots are pairwise distinct. By using the coordinate transformation

$$\begin{aligned} x' & := x - \lambda_1 y^2, \\ y' & := \sqrt{\lambda_2 - \lambda_1} y, \end{aligned}$$

we can ensure that for V_t ,

$$V_t = \{(x', y') \mid x' = 0, x' = y'^2, x' = \theta y'^2\}.$$

Correspondingly we can find coordinates (x'', y'') for $V_{t'}$ such that

$$V_{t'} = \{(x'', y'') \mid x'' = 0, x'' = y''^2, x'' = \theta' y''^2\}.$$

We assume that $\theta \neq 0, 1$ and $\theta' \neq 0, 1$. Let $t \neq t'$. Then $\theta \neq \theta'$ too. Suppose V_t can be transported to $V_{t'}$ by a biholomorphic map. Then, since the common tangent $x' = 0$ of the parabolas V_t must be transported to the common tangent $x'' = 0$ of the parabolas $V_{t'}$, this map must have the form

$$\begin{aligned} x'' & = x'(a_{11} + u(x', y')), \quad u \in \mathfrak{m}, \\ y'' & = a_{21}x' + a_{22}y' + v(x', y'), \quad v \in \mathfrak{m}^2. \end{aligned}$$

Since the parabola $x' = y'^2$ must be mapped to the parabola $x'' = y''^2$, it follows that $a_{11} = a_{22}^2$. But the image of the parabola $x' = \theta y'^2$ is the curve $x'' = \theta y''^2 + w(y'')$ with $w \in \mathfrak{m}^3$. But this is not a parabola of the form $x'' = \theta' y''^2$ for any $\theta' \neq \theta$. \square

Proof of Proposition 3.25. We denote the coordinates of \mathbb{C}^2 by (x, y) . Let now $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$, $(x, y) \mapsto f(x, y)$, be simple, and let $\text{grad } f(0) = 0$. Then $j^1 f = 0$. Consider the rank of the Hessian matrix

$$r := \text{rank} \left(\frac{\partial^2 f}{\partial z_i \partial z_j}(0) \right).$$

Then $r = 0, 1$ or 2 .

(1) If $r = 2$, then, by the Morse lemma, f is right equivalent to $x^2 + y^2$ and we are in case (a) ($k = 1, A_1$).

(2) If $r = 1$, then, by the generalized Morse lemma (Proposition 3.20), f is right equivalent to

$$x^2 + g(y) \quad \text{with } g \in \mathfrak{m}^3.$$

We have two cases to distinguish:

(2.1) $g = 0$: By Lemma 3.12, f would not be simple.

(2.2) $g \neq 0$: Then $g(y) = ay^{k+1} + h(y)y^{k+1}$ with $a \neq 0, h \in \mathfrak{m}$ and $k \geq 2$. The transformation $y \mapsto y \sqrt[k+1]{a + h(y)}$ shows that g is right equivalent to y^{k+1} . Hence f is right equivalent to $x^2 + y^{k+1}$, $k \geq 2$, and we are in case (a) ($A_k, k \geq 2$).

(3) The case $r = 0$ remains. In this case we examine the 3-jet $j^3 f$. By Lemma 3.11 this is right equivalent to one of the following polynomials:

$$0, \quad x^2 y + y^3, \quad x^2 y, \quad x^3.$$

(3.1) $j^3 f = 0$: By Corollary 3.8, f is not simple.

(3.2) $j^3 f = x^2 y + y^3$: Then $x^2 y + y^3$ is 3-determined, by Theorem 3.1 (Exercise). Hence f is right equivalent to $x^2 y + y^3$ and we are in case (b) ($k = 4, D_4$).

(3.3) $j^3 f = x^2 y$: Then there are two possibilities:

(3.3.1) $f \sim x^2 y$: By Lemma 3.12, f would not be simple.

(3.3.2) $f \not\sim x^2 y$: Then

$$j^s f = x^2 y + ay^s + 2bxy^{s-1} + x^2 g(x, y) \quad \text{with } g \in \mathfrak{m}^{s-2}, \quad s \geq 4.$$

We now put

$$\begin{aligned} x_1 &:= x + by^{s-2}, \\ y_1 &:= y + g(x, y). \end{aligned}$$

Then

$$j^s f = x_1^2 y_1 + a y_1^s.$$

We again distinguish cases.

(3.3.2.1) $a = 0$ for all $s \geq 4$: Then f would again be right equivalent to $x^2 y$ and so not simple.

(3.3.2.2) $a \neq 0$: By Theorem 3.1, $x^2 y + a y^s$ is s -determined (Exercise) and thus f is right equivalent to $x^2 y + y^{k-1}$, $k \geq 5$. We have then case (b) (D_k , $k \geq 5$).

(3.4) $j^3 f = x^3$: Next we consider the 4-jet $j^4 f$ of f . It has the form

$$j^4 f = x^3 + a y^4 + b x y^3 + 3x^2 g(x, y) \quad \text{with } g \in \mathfrak{m}^2.$$

After the substitution $x_1 := x + g(x, y)$ one obtains

$$j^4 f = x_1^3 + a y^4 + b x_1 y^3.$$

We again distinguish cases.

(3.4.1) $a \neq 0$: After the substitution

$$y_1 := \sqrt[4]{a} y + \frac{b}{4(\sqrt[4]{a})^3} x_1$$

$j^4 f$ becomes

$$j^4 f = x_1^3 + y_1^4 + 3x_1^2 h(x, y).$$

If we now put $x_2 := x_1 + h(x, y)$, we obtain

$$j^4 f = x_2^3 + y_1^4.$$

But now $x^3 + y^4$ is, by Theorem 3.1, 4-determined (Exercise). Hence f is right equivalent to $x^3 + y^4$ and we find ourselves in case (c) (E_6).

(3.4.2) $a = 0$, $b \neq 0$: By the substitution $y_1 := \sqrt[3]{b} y$ we obtain

$$j^4 f = x_1^3 + x_1 y_1^3.$$

By Theorem 3.1 the function germ $x^3 + x y^3$ is only 5-determined. Therefore we consider the 5-jet $j^5 f$ of f :

$$j^5 f = x_1^3 + x_1 y_1^3 + \alpha y_1^5 + \beta x_1 y_1^4 + 3x_1^2 h_1(x_1, y_1) \quad \text{with } h_1 \in \mathfrak{m}^3.$$

On substituting $x_2 := x_1 + \alpha y_1^2$, we obtain

$$j^5 f = x_2^3 + x_2 y_1^3 - 3\alpha x_2^2 y_1^2 + \beta' x_2 y_1^4 + 3x_2^2 h_2(x_2, y_1), \quad h_2 \in \mathfrak{m}^3.$$

The substitution $y_2 := y_1 - \alpha x_2$ yields

$$j^5 f = x_2^3 (1 - 2\alpha^3 x_2 - 3\alpha^2 y_2) + x_2 y_2^3 + \beta'' x_2 y_2^4 + 3x_2^2 h_3(x_2, y_2),$$

where $h_3 \in \mathfrak{m}^3$. Finally we put

$$\begin{aligned} x_3 &:= x_2 \sqrt[3]{1 - 2\alpha^3 x_2 - 3\alpha^2 y_2}, \\ y_3 &:= y_2 \frac{1}{\sqrt[9]{1 - 2\alpha^3 x_2 - 3\alpha^2 y_2}}. \end{aligned}$$

We then obtain

$$j^5 f = x_3^3 + x_3 y_3^3 + \beta''' x_3 y_3^4 + 3x_3^2 h_4(x_3, y_3) \text{ with } h_4 \in \mathfrak{m}^3.$$

Next we replace x_3 by $x_4 := x_3 + h_4(x_3, y_3)$ and then y_3 by $y_4 := y_3 \sqrt[3]{1 + \beta''' y_3}$ to obtain

$$j^5 f = x_4^3 + x_4 y_4^3.$$

Since $x^3 + xy^3$ is 5-determined, f is right equivalent to $x^3 + xy^3$ and we find ourselves in case (d) (E_7).

(3.4.3) $a = b = 0$: We consider again the 5-jet $j^5 f$ of f :

$$j^5 f = x^3 + \alpha y^5 + \beta x y^4 + 3x^2 h(x, y) \text{ with } h \in \mathfrak{m}^3.$$

The substitution $x_1 := x + h(x, y)$ yields

$$j^5 f = x_1^3 + \alpha y^5 + \beta x_1 y^4.$$

We again distinguish cases.

(3.4.3.1) $\alpha \neq 0$: Putting $y_1 := \sqrt[5]{\alpha} y$, one obtains

$$j^5 f = x_1^3 + y_1^5 + \beta' x_1 y_1^4.$$

If one now substitutes $y_2 := y_1 + (1/5)\beta' x_1$, one obtains

$$j^5 f = x_1^3 + y_2^5 + 3x_1^2 h_1(x_1, y_2) \text{ with } h_1 \in \mathfrak{m}^3.$$

The substitution $x_2 := x_1 + h_1(x_1, y_2)$ finally yields $j^5 f = x_2^3 + y_2^5$. It again follows from Theorem 3.1 that $x^3 + y^5$ is 5-determined. Hence f is right equivalent to $x^3 + y^5$ and we are in case (e) (E_8).

(3.4.3.2) $\alpha = 0$: Next we consider the 6-jet $j^6 f$ of f . With the same coordinate transformations as in case (3.4.2) we can achieve

$$j^6 f = x^3 + xy^4 + \lambda y^6, \text{ with } \lambda \in \mathbb{C}.$$

It follows then from Lemma 3.13 that f is not simple.

It remains to show that the function germs (a) – (e) are simple. To do this, we consider the universal unfoldings of these function germs, as listed in the following proposition. Let $F : (\mathbb{C}^2 \times \mathbb{C}^k, (0, 0)) \rightarrow (\mathbb{C}, 0)$, $(x, y, u) \mapsto F(x, y, u)$, be such an unfolding and $p : \mathcal{X} \rightarrow S$ a suitable representative. If one works with the function germ $F(\cdot, \cdot, u) : (\mathbb{C}^2, (x, y)) \rightarrow (\mathbb{C}, F(x, y, u))$ in the above proof, one can show that this function germ is right equivalent to one of the finitely many function germs of type A_l , D_l or E_l with $l \leq k$. We leave the details to the reader as an exercise. \square

Proposition 3.26. *The following functions represent universal unfoldings of the holomorphic function germs of type A_k , D_k , E_6 , E_7 and E_8 :*

$$\begin{aligned}
A_k & : F(x, y, u) := x^{k+1} + y^2 - u_0 + u_1x + u_2x^2 + \dots + u_{k-1}x^{k-1}, \\
D_k & : F(x, y, u) := x^2y + y^{k-1} - u_0 + u_1y + \dots + u_{k-2}y^{k-2} + u_{k-1}x, \\
E_6 & : F(x, y, u) := x^3 + y^4 - u_0 + u_1x + u_2y + u_3y^2 + u_4xy + u_5xy^2, \\
E_7 & : F(x, y, u) := x^3 + xy^3 - u_0 + u_1x + u_2y + u_3y^2 \\
& \quad + u_4y^3 + u_5y^4 + u_6xy, \\
E_8 & : F(x, y, u) := x^3 + y^5 - u_0 + u_1x + u_2y + u_3y^2 + u_4y^3 \\
& \quad + u_5xy + u_6xy^2 + u_7xy^3.
\end{aligned}$$

Proof. This follows from Proposition 3.17 and the fact that the monomials by which the corresponding equations are perturbed form a basis of the vector space

$$\mathcal{O}_2/\mathcal{O}_2 \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right).$$

□

From the proof of Proposition 3.25 one can develop an algorithm to decide whether a given holomorphic function germ is simple, and, if so, to which of the function germs of Proposition 3.25 (a) – (e) it is right equivalent. Such an algorithm is presented in [BK91, p. 17].

We finally obtain the general classification of simple holomorphic function germs that goes back to V. I. Arnold [Arn73].

Theorem 3.2 (Arnold). *Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a simple holomorphic function germ with $\text{grad } f(0) = 0$. Then f is right equivalent to one of the following simple function germs:*

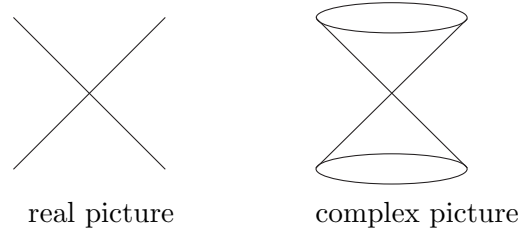
$$\begin{aligned}
A_k & : z_1^{k+1} + z_2^2 + \dots + z_{n+1}^2 \quad \text{for } n \geq 0, k \geq 1, \\
D_k & : z_1^2 z_2 + z_2^{k-1} + z_3^2 + \dots + z_{n+1}^2 \quad \text{for } n \geq 1, k \geq 4, \\
E_6 & : z_1^3 + z_2^4 + z_3^2 + \dots + z_{n+1}^2, \\
E_7 & : z_1^3 + z_1 z_2^3 + z_3^2 + \dots + z_{n+1}^2, \\
E_8 & : z_1^3 + z_2^5 + z_3^2 + \dots + z_{n+1}^2.
\end{aligned}$$

Proof. Let $n \geq 1$. By Corollary 3.7

$$k := \text{rank} \left(\frac{\partial^2 f}{\partial z_i \partial z_j}(0) \right) \geq n - 1.$$

By the generalized Morse lemma f is then right equivalent to

$$\tilde{f}(z) := z_1^2 + \dots + z_{n-1}^2 + g(z_n, z_{n+1}) \quad \text{with } g \in \mathfrak{m}_2^2.$$

Figure 3.10. X_0

From the generalized Morse lemma it follows, besides, that each unfolding of \tilde{f} is right equivalent to an unfolding of the form

$$z_1^2 + \dots + z_{n-1}^2 + G(z_n, z_{n+1}, u)$$

where $G(z_n, z_{n+1}, u)$ represents an unfolding of g . If, now, g is simple, then \tilde{f} is simple too. If on the other hand, g is not simple, it follows from Proposition 3.23 that \tilde{f} is also not simple. Thus Theorem 3.2 follows from Proposition 3.25. \square

3.11. Real morsifications of the simple curve singularities

We shall now motivate the notation of the simple singularities in the previous section. For this purpose we present a short introduction to the topology of singularities in the context of a singularity of type A_1 .

We consider the function

$$f: \mathbb{C}^2 \longrightarrow \mathbb{C} \\ (z_1, z_2) \longmapsto z_1^2 + z_2^2.$$

The unique critical point of this function is the origin $(z_1, z_2) = (0, 0)$. This is also the unique singularity on the hypersurface

$$X_0 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1^2 + z_2^2 = 0\}$$

that consists of two complex lines that intersect at the origin (see Figure 3.10). The space germ $(X_0, 0)$ thus has an isolated singularity at 0. We also called more briefly $(X_0, 0)$ an isolated singularity.

We now want to study the topology of the map f .

For $\lambda \in \mathbb{C}$, $\lambda \neq 0$ we put

$$X_\lambda = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1^2 + z_2^2 = \lambda\},$$

the fiber of f over the value λ . Since X_λ contains no singularities, X_λ is a Riemann surface. It coincides with the Riemann surface of the algebraic function

$$z_2 = \sqrt{\lambda - z_1^2}.$$

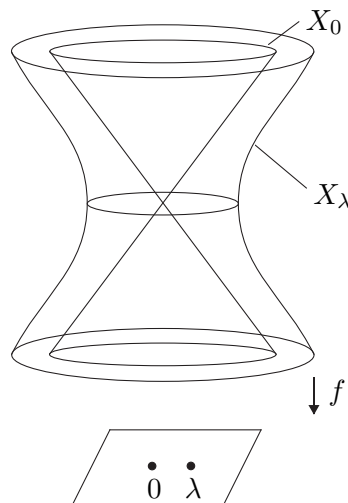


Figure 3.11. Fibers of the map f

We recall how one obtains this surface topologically: We take two copies of the complex z_1 -plane and paste the edges of the cuts $(-\sqrt{\lambda}, \sqrt{\lambda})$ crosswise (see Figure 1.8). In this way one sees that X_λ is homeomorphic to a cylinder $S^1 \times \mathbb{R}$. The real 4-dimensional preimage space \mathbb{C}^2 thus decomposes as the singular fiber X_0 over 0 and the nonsingular fibers X_λ over $\lambda \neq 0$ which are homeomorphic to cylinders (see Figure 3.11).

We wish to describe the level surface X_λ in terms of coordinates. To do this, we put

$$\begin{aligned} z_1 &= x_1 + iy_1, \\ z_2 &= x_2 + iy_2. \end{aligned}$$

Let $\lambda \in \mathbb{R}$. Then X_λ is given by

$$z_1^2 + z_2^2 = \lambda \iff \begin{cases} x_1^2 + x_2^2 - y_1^2 - y_2^2 = \lambda, \\ x_1y_1 + x_2y_2 = 0. \end{cases}$$

With the help of the coordinate transformation

$$u = x_1, \quad v = x_2, \quad w = \pm \sqrt{y_1^2 + y_2^2}$$

(the sign preceding w is determined by the x_1, x_2, y_1, y_2), we can combine both these equations together in the following equation:

$$u^2 + v^2 - w^2 = \lambda.$$

This equation describes a hyperboloid in \mathbb{R}^3 (see Figure 3.12). As $\lambda \rightarrow 0$, its waist shrinks to a point.

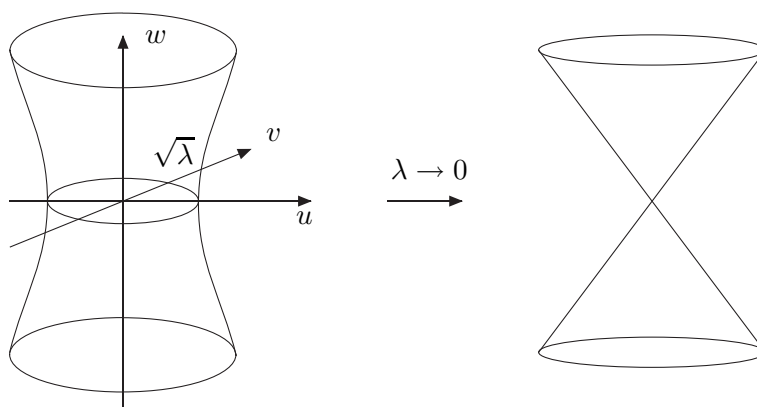


Figure 3.12. The level surface X_λ

We next want to examine how the fiber X_λ varies as λ goes once around the critical value 0. This is the idea of *monodromy* that will play a central role in Chapter 5. We therefore consider the path

$$\lambda(t) = \eta \exp(2\pi it), \quad 0 \leq t \leq 1, \quad \eta > 0,$$

in the image plane. It goes once around the origin in the positive direction (counterclockwise) on the boundary of a circle of radius $\eta > 0$ in the λ -plane (see Figure 3.13). We look at how the fiber $X_{\lambda(t)}$ varies as we let t go from 0

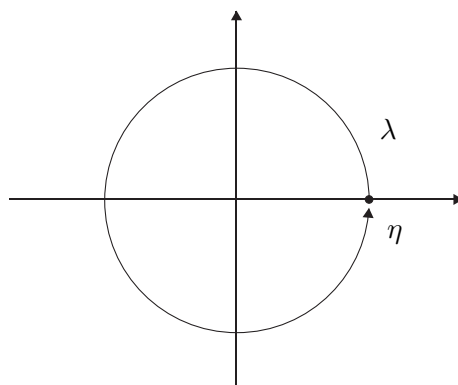


Figure 3.13. The path λ

to 1. For this we again use the description of the fiber $X_{\lambda(t)}$ as the Riemann surface of the function

$$z_2 = \sqrt{\lambda(t) - z_1^2}.$$

As t increases, the branch points $\pm\sqrt{\lambda(t)} = \pm\sqrt{\eta}\exp(\pi it)$ wind round the point 0 in the positive direction. We thus obtain the sequence of Riemann surfaces represented in Figure 3.14. At $t = 1$ the cut has turned itself

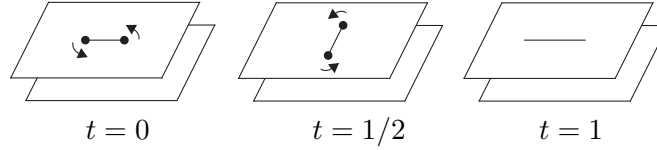


Figure 3.14. $X_{\lambda(t)}$ for $t = 0, 1/2, 1$

through 180° and we come back again to the surface $X_\eta = X_{\lambda(0)}$.

We can now construct a family of diffeomorphisms

$$h_t : X_\eta \longrightarrow X_{\lambda(t)}$$

with $h_0 = \text{id}$ that is continuous in t . We first consider a differentiable bell function

$$\chi : \mathbb{R} \longrightarrow \mathbb{R}$$

with

$$\begin{aligned} \chi(\tau) &= 1 \text{ for } 0 \leq |\tau| \leq 2\sqrt{\eta}, \\ \chi(\tau) &= 0 \text{ for } 3\sqrt{\eta} \leq |\tau| \end{aligned}$$

(see Figure 3.15). Then we put

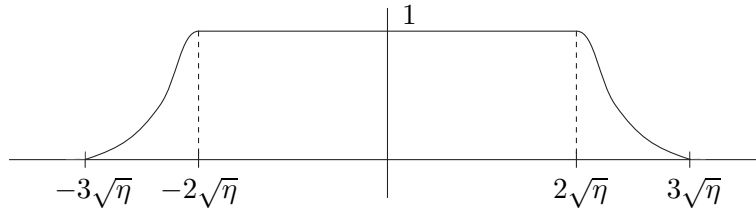


Figure 3.15. Graph of the bell function χ

$$\begin{aligned} g_t : \mathbb{C} &\longrightarrow \mathbb{C} \\ z_1 &\longmapsto z_1 \exp[\pi it \cdot \chi(|z_1|)]. \end{aligned}$$

Let

$$h_t : X_\eta \longrightarrow X_{\lambda(t)}$$

be a lifting of g_t to the covering

$$\begin{array}{ccc} X_\eta & \xrightarrow{h_t} & X_{\lambda(t)} \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{g_t} & \mathbb{C} \end{array}$$

The map

$$h := h_1 : X_\eta \longrightarrow X_\eta$$

is called the *geometric monodromy* of f . The diffeomorphisms h_t , and h_1 with them, are of course uniquely determined only up to homotopy.

We now want to study the diffeomorphism h more closely. We consider the circle

$$\delta := \{(u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 = \eta, w = 0\}$$

in the coordinates (u, v, w) (cf. Figure 3.16). The cycle δ shrinks to the point 0 as $\eta \rightarrow 0$. Following Picard-Lefschetz, δ is called a *vanishing cycle* (cf. §5.2).

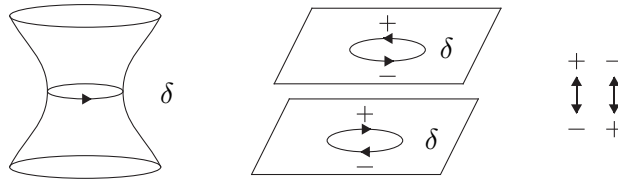


Figure 3.16. Vanishing cycle δ

On the other hand we consider the curve

$$\delta^* := \{(u, v, w) \in \mathbb{R}^3 \mid u = 0, v^2 - w^2 = \eta, v < 0\}$$

(cf. Figure 3.17). If instead of X_η we consider the corresponding compact Riemann surface \bar{X}_η of $\sqrt{\eta - z_1^2}$ over $\mathbb{P}_1(\mathbb{C})$, then the boundary of the corresponding compactified δ^* consists of the two points over ∞ . For this reason we speak of a relative cycle. The cycle δ^* will be called a “*covanishing cycle*”. It intersects the cycle δ transversally in exactly one point. We orient δ^* so that the tangent vectors to δ^* and δ at the point of intersection, taken in this order, define the orientation of the complex z_1 -plane. This means that the intersection number $\langle \delta^*, \delta \rangle$ between the cycles δ^* and δ is equal to $+1$.

We next look at what the diffeomorphisms h_t do to the cycles δ and δ^* (see Figure 3.18). Outside the compact subset of X_η determined by $|z_1| \leq 3\sqrt{\eta}$ the map $h : X_\eta \rightarrow X_\eta$ is the identity. Inside this subset it is as in Figure 3.18. Transported to the cylinder the action h looks as in Figure 3.19.

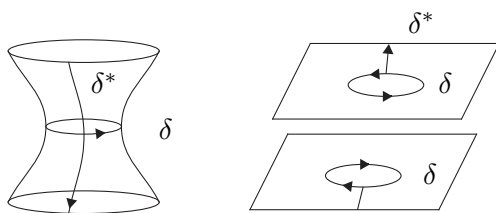


Figure 3.17. Covanishing cycle δ^*

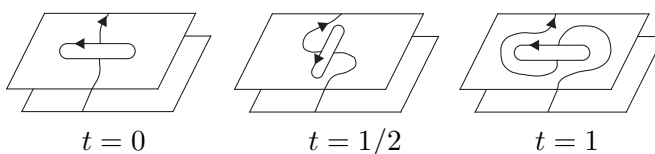


Figure 3.18. Image of δ and δ^* under h_t

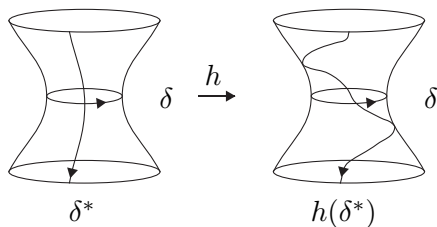


Figure 3.19. Effect of the monodromy h

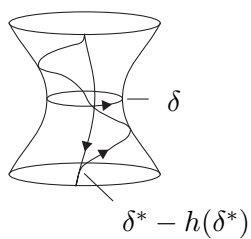


Figure 3.20. The cycle $\delta^* - h(\delta^*)$

The monodromy h is the identity outside a certain ring, while inside this ring the individual circles are twisted, through 0 at one end to 2π at the other end.

We see in this figure that this cycle is homologous to δ , i.e.,

$$\delta^* - h(\delta^*) \sim \delta.$$

The diffeomorphism h is the identity outside a compact set. The cycle $\delta^* - h(\delta^*)$ is represented in Figure 3.20. This is the simplest form of the *Picard-Lefschetz formula* (cf. §5.3).

Next we consider particular morsifications of the simple holomorphic function germs $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$. The representatives described in Proposition 3.25 have the property that they are real, i.e., the restriction to $\mathbb{R}^2 \subset \mathbb{C}^2$ maps \mathbb{R}^2 to \mathbb{R} .

Definition. Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ with $f(M \cap \mathbb{R}^2) \subset \mathbb{R}$ for a suitable representative $f : M \rightarrow \mathbb{C}$. A morsification f_λ of f is called *real* if $f_\lambda(M \cap \mathbb{R}^2) \subset \mathbb{R}$.

Now we consider first of all a simple holomorphic function germ $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ of type A_k given by $f(x, y) = x^{k+1} - y^2$. One obtains a real morsification f_λ of f as follows: In the case when k is even we put

$$f_\lambda(x, y) = \lambda^{2k+2} T_{k+1} \left(\frac{x}{\lambda^2} \right) - y^2 + \frac{1}{2^k} \lambda^{2k+2}.$$

Here $T_n(x) = (1/2^{n-1}) \cos(n \cdot \arccos x)$ is the n -th Chebyshev polynomial. When k is odd, we consider

$$f_\lambda(x, y) = (\lambda^{k+1} T_{(k+1)/2} \left(\frac{x}{\lambda^2} \right) - y)(\lambda^{k+1} T_{(k+1)/2} \left(\frac{x}{\lambda^2} \right) + y).$$

The real curve

$$X_{\mathbb{R},0} := \{(x, y) \in \mathbb{R}^2 \mid f_\lambda(x, y) = 0\}$$

appears as in Figure 3.21 (k even), resp. Figure 3.22 (k odd).

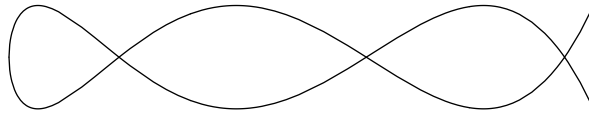


Figure 3.21. The curve $X_{\mathbb{R},0}$ for $k = 6$

Now let $\eta \in \mathbb{R}$, $\eta < 0$, with $|\eta|$ sufficiently small. Then the real level set

$$X_{\mathbb{R},\eta} := \{(x, y) \in \mathbb{R}^2 \mid f_\lambda(x, y) = \eta\}$$



Figure 3.22. The curve $X_{\mathbb{R},0}$ for $k = 7$

consists of cycles. As we let η tend to the function value of the minima of f_λ , these cycles vanish. We can treat these cycles as vanishing cycles corresponding to the minima.

Next we consider the situation of a double point of $X_{\mathbb{R},\eta}$ that lies between two minima. In a neighborhood of such a double point we can introduce real coordinates (\tilde{x}, \tilde{y}) so that f_λ has the form

$$f_\lambda(\tilde{x}, \tilde{y}) = \tilde{x}^2 - \tilde{y}^2.$$

In order to be able to apply our results on the local topology of the function $f(z_1, z_2) = z_1^2 + z_2^2$, we consider the complex coordinates

$$\begin{aligned} z_1 &= x_1 + iy_1, \\ z_2 &= x_2 + iy_2 \end{aligned}$$

with $x_1 = \tilde{x}$ and $y_2 = \tilde{y}$. The vanishing cycles corresponding to the two neighboring minima are the two branches of the hyperbola

$$\tilde{x}^2 - \tilde{y}^2 = \eta \quad \Leftrightarrow \quad x_1^2 - y_2^2 = \eta.$$

In the space $y_1 = 0$, with the coordinates (u, v, w) , these are the curves

$$\begin{aligned} \delta_1^* &:= \{(u, v, w) \in \mathbb{R}^3 \mid v = 0, u^2 - w^2 = \eta, u < 0\}, \\ \delta_2^* &:= \{(u, v, w) \in \mathbb{R}^3 \mid v = 0, u^2 - w^2 = \eta, u > 0\}. \end{aligned}$$

These curves correspond exactly to the covanishing cycles. We orient the vanishing cycle δ of the double points and the covanishing cycles so that

$$\langle \delta_1^*, \delta \rangle = \langle \delta_2^*, \delta \rangle = 1.$$

In this way we obtain a vanishing cycle for each minimum and each double point. We enumerate the cycles beginning with the cycles for the minima. Let $\delta_1, \dots, \delta_p$ be the cycles of the minima and $\delta_{p+1}, \dots, \delta_k$ the cycles of the double points. Then we assign a graph to the system $(\delta_1, \dots, \delta_k)$ of vanishing cycles as follows: The vertices of this graph correspond to the vanishing cycles. We connect the vertices δ_i and δ_j for $i < j$ by an edge if

and only if $\langle \delta_i, \delta_j \rangle = 1$. This graph is called a *Coxeter-Dynkin diagram* of f (cf. §5.5).

In our case we obtain the graph represented in Figure 3.23. This is a

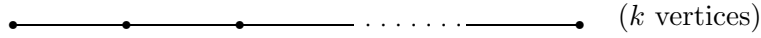


Figure 3.23. The Coxeter-Dynkin diagram of type A_k

Coxeter-Dynkin diagram of type A_k , as known from the theory of Lie groups.

We can also carry out an analogous construction for the simple holomorphic function germs $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ of type D_k . Let f be given by

$$f(x, y) = x^2y - y^{k-1} = y(x^2 - y^{k-2}).$$

Then we consider the real morsification

$$f_\lambda(x, y) = (y - 2\lambda^2)\tilde{f}_\lambda(x, y),$$

where \tilde{f}_λ is the above real morsification determined through the given singularity of type A_{k-3} . For the remaining three types E_6 , E_7 and E_8 it is not so simple to describe real morsifications whose only critical points are minima and saddle points. But it is possible to do so in these cases too. In

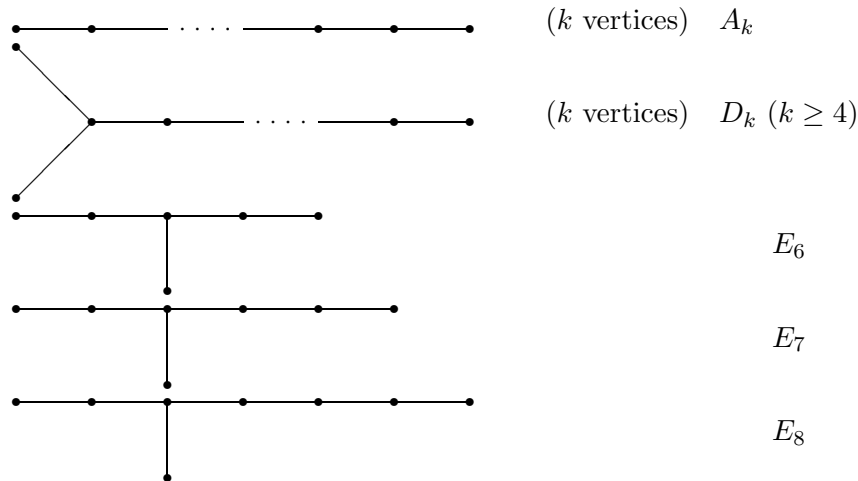


Figure 3.24. Coxeter-Dynkin diagrams of simple curve singularities

fact the simple holomorphic function germs $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ are the only

holomorphic function germs for which there are real morsifications with only one or two critical values (cf. [Dur79, Characterization B6]).

In all one obtains the list of Coxeter-Dynkin diagrams presented in Figure 3.24. These diagrams are also known in the theory of Lie groups and take their name from there.

A definition and a thorough discussion of the concepts of monodromy, vanishing cycles and Coxeter-Dynkin diagrams for general isolated singularities of holomorphic functions form the subject matter of Chapter 5.