Preface

Ordered vector spaces made their debut at the beginning of the twentieth century. They were developed in parallel (but from a different perspective) with functional analysis and operator theory. Before the 1950s ordered vector spaces appeared in the literature in a fragmented way. Their systematic study began in various schools around the world after the 1950s. We mention the Russian school (headed by L. V. Kantorovich and the Krein brothers), the Japanese school (headed by H. Nakano), the German school (headed by H. H. Schaefer), and the Dutch school (headed by A. C. Zaanen). At the same time several monographs dealing exclusively with ordered vector spaces appeared in the literature; see for instance [**55, 56, 71, 75, 89, 91**]. The special class of ordered vector spaces known as *Riesz spaces* or *vector lattices* has been studied more extensively; see the monographs [**14, 15, 66, 68, 86, 88, 93**].

The theory of ordered vector spaces plays a prominent role in functional analysis. It also contributes to a wide variety of applications and is an indispensable tool for studying a variety of problems in engineering and economics; see for instance [**29, 31, 35, 36, 38, 42, 47, 49, 54, 64, 65, 76**]. The introduction of Riesz spaces and more broadly ordered vector spaces to economic theory has proved tremendously successful and has allowed researchers to answer difficult questions in general price equilibrium theory, economies with differential information, the theory of perfect competition, and incomplete assets economies.

The goal of this monograph is to present the theory of ordered vector spaces from a contemporary perspective that has been influenced by the study of ordered vector spaces in economics as well as other recent applications. We try to imbue the narrative with geometric intuition, which is in keeping with a long tradition in mathematical economics. We also approach the subject with our own personal presentiment that the special class of Riesz spaces is somehow "perfect" and thus loosely conceive of general ordered vector spaces as "deviations" from this "perfection." The book also contains material that has not been published in a monograph form before. The study of this material was initially motivated by various problems in economics and econometrics.

The material is spread out in eight chapters. Chapter 8 is an Appendix and contains some basic notions of functional analysis. Special attention is paid to the properties of linear topologies and the separation of convex sets. The results in this chapter (some of which are presented with proofs) are used throughout the monograph without specific mention.

Chapter 1 presents the fundamental properties of wedges and cones. Here we discuss Archimedean cones, lattice cones, extremal vectors of cones, bases of cones, positive linear functionals and the important decomposability property of cones known as the *Riesz decomposition property.* Chapter 2 introduces cones in topological vector spaces. This chapter illustrates the variety of remarkable results that can be obtained when some link between the order and the topology is imposed. The most important interrelationship between a cone and a linear topology is known as *normality.* We discuss normal cones in detail and obtain several characterizations. In normed spaces, the normality of the cone amounts to the norm boundedness of the order intervals generated by the cone. In Chapter 2 we also introduce ideals and present some of their useful order and topological properties.

Chapter 3 studies in detail cones in finite dimensional vector spaces. The results here are much sharper. For instance, as we shall see, every closed cone of a finite dimensional vector space is normal. The reader will find in this chapter a study (together with a geometrical description) of the polyhedral cones as well as a discussion of the properties of linear inequalities—including a proof of "the Principle of Linear Programming." The chapter culminates with a study of pull-back cones and establishes the following "universality" property of $C[0,1]$: every closed cone of a finite dimensional vector space is the pull-back cone of the cone of $C[0,1]$ via a one-to-one operator from the space to $C[0,1]$.

Chapter 4 investigates the fixed points and eigenvalues of an important class of positive operators known as *Krein operators.* A *Krein space* is an ordered Banach space having order units and a closed cone. A positive operator T on a Krein space is a *Krein operator* if for any $x > 0$ the vector $T^n x$ is an order unit for some n. Many integral operators are Krein operators. These operators possess some useful fixed points that are investigated in this chapter.

Chapters 5, 6, and 7 contain new material that, as far as we know, has not appeared before in any monograph. Chapter 5 develops in detail the theory of $\mathcal{K}$-lattices. An ordered vector space L is called a $\mathcal{K}$-*lattice*, where $\mathcal{K}$ is a super cone of L, i.e., $\mathcal{K} \supseteq L_+$, if for every nonempty subset A of L the collection of all L_+-upper bounds of A is nonempty and has a $\mathcal{K}$-infimum. As can be seen immediately from this definition, the notion of a $\mathcal{K}$-lattice has applications to optimization theory. Chapter 5 also introduces the notion of the Riesz–Kantorovich transform for an m-tuple of order bounded operators that is used to investigate the fundamental duality properties of ordered bounded operators from an ordered vector space to a Dedekind complete Riesz space. Subsequently, using the theory of $\mathcal{K}$-lattices, we define an important order extension of $\mathcal{L}_b(L, N)$, the ordered vector space of all order bounded operators from L to a Dedekind complete Riesz space. This extension allows us to enrich the lattice structure of $\mathcal{L}_b(L, N)$ in a useful manner.

Chapter 6 specializes the theory of $\mathcal{K}$-lattices to the space of all ordered bounded linear functionals on an ordered vector space. Among other things, this chapter introduces an important order extension of L' called the "super topological dual of L'" and studies its fundamental properties. Moreover, in this chapter the reader will find several interesting optimization results. In essence, Chapter 6 brings, via the concept of a $\mathcal{K}$-lattice, the theory of ordered vector spaces to the theory of linear minimization. In other words, this chapter can be viewed as contributing new functional analytic tools to the study of linear minimization problems.

Finally, in Chapter 7 we present a comprehensive investigation of piecewise affine functions. It turns out that their structure is intimately related to order and lattice properties that are discussed in detail in this chapter. The main result here is that the collection of piecewise affine functions coincides with the Riesz subspace generated by the affine functions. Piecewise affine (or piecewise linear) functions are very important in approximation theory and have been studied extensively in one-dimensional settings. However, even until now, in dimensions more than one there seems to be no satisfactory theory of piecewise affine functions. They are defined on finite dimensional spaces, and no attempt has been made to generalize their theory to infinite dimensional settings. This provides the opportunity for several future research directions.

At the end of each section there is a list of exercises of varying degrees of difficulty designed to help the reader comprehend the material in the section. There are almost three hundred and fifty exercises in the book. Hints to selected exercises are also given. The inclusion of the exercises makes the book, on one hand, suitable for graduate courses and, on the other hand, a reference source on ordered vector spaces and cones. It is our hope (and belief) that this monograph will not only be beneficial to mathematicians but also to other scientists in many disciplines, theoretical and applied, as well.

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