

Let λ be an ordinal. As an inductive hypothesis, assume that $\Omega_\kappa(\psi^t) = h(\Omega_\kappa(\phi^t))$ for all $\kappa < \lambda$. If λ is a limit ordinal, then

$$\begin{aligned}\Omega_\lambda(\psi^t) &= \bigcap_{\kappa < \lambda} \Omega_\kappa(\psi^t) \\ &= \bigcap_{\kappa < \lambda} h(\Omega_\kappa(\phi^t)) \\ &= h\left(\bigcap_{\kappa < \lambda} \Omega_\kappa(\phi^t)\right) \\ &= h(\Omega_\lambda(\phi^t)).\end{aligned}$$

If λ is a successor ordinal, then $\lambda = \kappa + 1$ for some ordinal κ . Since (h, ρ) is a topological equivalence from ϕ^t to ψ^t , and $\Omega_\kappa(\psi^t) = h(\Omega_\kappa(\phi^t))$ by the inductive hypothesis, the pair $(h|_{\Omega_\kappa(\phi^t)}, \rho|_{\mathbb{R} \times \Omega_\kappa(\phi^t)})$ is a topological equivalence from $\phi^t|_{\Omega_\kappa(\phi^t)}$ to $\psi^t|_{\Omega_\kappa(\psi^t)}$. By Proposition 2.5.10,

$$\begin{aligned}\Omega_\lambda(\psi^t) &= \Omega(\psi^t|_{\Omega_\kappa(\psi^t)}) \\ &= h(\Omega(\phi^t|_{\Omega_\kappa(\phi^t)})) \\ &= h(\Omega_\lambda(\phi^t)).\end{aligned}$$

By the Principle of Transfinite Induction, $\Omega_\lambda(\psi^t) = h(\Omega_\lambda(\phi^t))$ for every ordinal λ . Therefore,

$$C(\psi^t) = h(C(\phi^t)). \quad \square$$

Unfortunately, the center does not satisfy the Accumulation Property. In this respect the center shares a shortcoming of the Poincaré recurrent set.

Example 2.6.10. Let ψ^t be the arrested logistic rotation-dilation flow on the closed unit disk \mathbb{D} in the complex plane. By Example 2.6.4,

$$C(\psi^t) = \{-1, 0, 1\}.$$

If $z \in \text{int}(\mathbb{D}) \setminus \{0\}$, then $\omega(z) = \partial\mathbb{D}$ by Example 2.3.9. Therefore, the center does not contain the ω -limit set of every point in \mathbb{D} .

2.7. Chain Recurrent Points

Conley [11] introduced a more general type of recurrence called chain recurrence. On compact spaces chain recurrence satisfies all of the desirable properties enumerated in the introduction to this chapter.

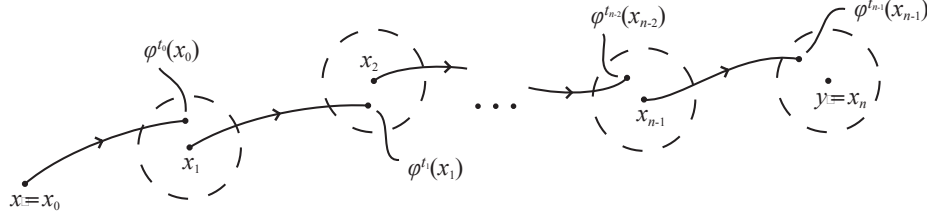


Figure 2.17. An (ϵ, T) -chain for a flow

2.7.1. The Definition of the Chain Recurrent Set.

Definition 2.7.1. Let ϕ^t be a flow on a metric space (X, d) . Given $\epsilon > 0$, $T > 0$, and $x, y \in X$, an (ϵ, T) -**chain** from x to y with respect to ϕ^t and d is a pair of finite sequences $x = x_0, x_1, \dots, x_{n-1}, x_n = y$ in X and t_0, \dots, t_{n-1} in $[T, \infty)$, denoted together by $(x_0, \dots, x_n; t_0, \dots, t_{n-1})$, such that

$$d(\phi^{t_i}(x_i), x_{i+1}) < \epsilon$$

for $i = 0, 1, 2, \dots, n - 1$. See Figure 2.17.

Definition 2.7.2. Let ϕ^t be a flow on a metric space (X, d) . The **forward chain limit set** of $x \in X$ with respect to ϕ^t and d is the set

$$\Omega^+(x) = \bigcap_{\epsilon, T > 0} \{y \in X \mid \exists \text{ an } (\epsilon, T)\text{-chain from } x \text{ to } y \text{ with respect to } \phi^t\}.$$

The **backward chain limit set** of $x \in X$ with respect to ϕ^t and d is the set

$$\Omega^-(x) = \bigcap_{\epsilon, T > 0} \{y \in X \mid \exists \text{ an } (\epsilon, T)\text{-chain from } x \text{ to } y \text{ with respect to } \phi^{-t}\}.$$

Backward and forward chain limit sets are analogous to α - and ω -limit sets, respectively.

Definition 2.7.3. Let ϕ^t be a flow on a metric space (X, d) . Two points $x, y \in X$ are **chain equivalent** with respect to ϕ^t and d if $y \in \Omega^+(x)$ and $x \in \Omega^+(y)$.

Two points x and y are chain equivalent if and only if for every $\epsilon > 0$ and every $T > 0$ there exists an (ϵ, T) -chain from x to y and there exists an (ϵ, T) -chain from y to x . Theorem 2.7.18 provides four equivalent formulations of chain equivalence for flows on compact spaces.

Definition 2.7.4. Let ϕ^t be a flow on a metric space (X, d) . A point $x \in X$ is called **chain recurrent** with respect to ϕ^t and d if x is chain equivalent

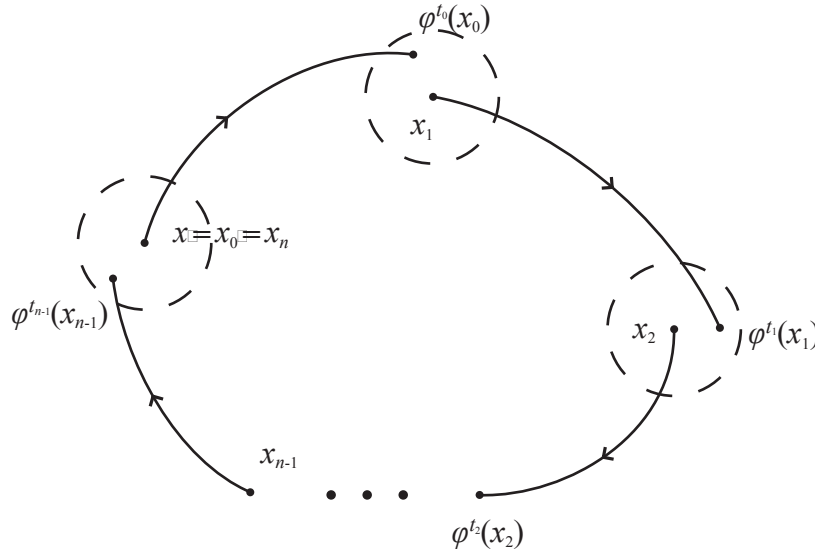


Figure 2.18. Chain recurrence

to itself. The set of all chain recurrent points of ϕ^t , denoted $\mathcal{R}(\phi^t)$, is the **chain recurrent set** of ϕ^t . See Figure 2.18.

While the language of chain limit sets is not necessary to define chain recurrence, it allows us to draw a parallel with Poincaré recurrence. A point x is chain recurrent if and only if $x \in \Omega^+(x)$. Exercise 15 asks you to show that if the phase space is compact, then $x \in \Omega^+(x)$ if and only if $x \in \Omega^-(x)$. Notice the similarity to Poincaré recurrence: A point x is Poincaré recurrent if and only if $x \in \omega(x)$ and $x \in \alpha(x)$.

Conley [11] defines chain recurrence for flows on compact Hausdorff topological spaces using nets. For the sake of clarity and simplicity we restrict our attention to metric spaces with an emphasis on the compact case. Hurley [21], [22] address chain recurrence on spaces which are not compact.

To imagine an (ϵ, T) -chain from x to y imagine following the orbit of x for a time at least T , then jumping a distance less than ϵ to another point, following its orbit for a time at least T , then jumping a distance less than ϵ to another point, and continuing finitely many steps in this manner until jumping onto y from a distance less than ϵ away. A point x is chain recurrent if no matter how long a time you must flow and no matter how short a distance you must jump, it is always possible to return to x . Another

way to think about chain recurrence is to imagine a chain recurrent point as a point which is within ϵ of being periodic for every $\epsilon > 0$.

2.7.2. Examples of Chain Recurrent Sets.

Example 2.7.5. Every periodic point is a chain recurrent point. Let x be a periodic point with period S . Let $\epsilon > 0$ and $T > 0$. If $S > 0$, then there exists an integer n so that $nS > T$. Therefore, $(x, x; nS)$ is an (ϵ, T) -chain from x to itself. If $S = 0$, then x is a fixed point, and $(x, x; T)$ is an (ϵ, T) -chain from x to itself. Hence, $x \in \Omega^+(x)$. Therefore, x is chain recurrent. In particular, every fixed point is a chain recurrent point.

Example 2.7.6. Consider the differential equation

$$\dot{\theta} = \sin^2 \theta$$

in the angular coordinate on S^1 . The flow ϕ^t associated to this differential equation is the arrested rotation. In Example 2.1.6 we saw that ϕ^t has two fixed points, 0 and π . Example 2.5.11 shows that the nonwandering set of ϕ^t is $\{0, \pi\}$. In contrast, we shall demonstrate that the chain recurrent set of ϕ^t is S^1 .

Let $\theta \in S^1$. If θ is a fixed point, then θ is chain recurrent by Example 2.7.5. Denote the usual metric on S^1 by d . Let $\epsilon > 0$ and $T > 0$. If $0 < \theta \bmod 2\pi < \pi$, then $\omega(\theta) = \{\pi\}$ by Example 2.3.5. Thus, there exists $t_0 > T$ so that $d(\phi^{t_0}(\theta), \pi) < \epsilon/2$. Let θ_1 be any point in the $\epsilon/2$ -ball around π such that $\pi < \theta_1 \bmod 2\pi < 2\pi$. By the triangle inequality,

$$d(\phi^{t_0}(\theta), \theta_1) < \epsilon.$$

Since $\pi < \theta_1 \bmod 2\pi < 2\pi$, Example 2.3.5 shows that $\omega(\theta_1) = \{0\}$. Thus, there exists $t_1 > T$ so that $d(\phi^{t_1}(\theta_1), 0) < \epsilon/2$. Example 2.3.5 shows that $\alpha(\theta) = \{0\}$. Thus, there exists θ_2 in the $\epsilon/2$ -ball around 0 and there exists $t_2 > T$ such that $\phi^{t_2}(\theta_2) = \theta$. By the triangle inequality,

$$d(\phi^{t_1}(\theta_1), \theta_2) < \epsilon.$$

Therefore, $(\theta, \theta_1, \theta_2, \theta; t_0, t_1, t_2)$ is an (ϵ, T) -chain from θ to θ . A similar argument shows that if $\pi < \theta \bmod 2\pi < 2\pi$, then there is an (ϵ, T) -chain from θ to θ . Since ϵ and T are arbitrary positive real numbers, $\theta \in \Omega^+(\theta)$. Consequently,

$$\mathcal{R}(\phi^t) = S^1.$$

The following example illustrates a style of argument which plays an important role in the remainder of the text. We shall encounter versions of this argument again in Example 2.7.9, in the proof of Lemma 4.5.4 and in the proof of Proposition 4.5.5.

Example 2.7.7. Let a and b be real numbers with $a < b$. Let $f : [a, b] \rightarrow \mathbb{R}$ be a smooth function such that $f(a) = f(b) = 0$ and $f(x) < 0$ for all $x \in (a, b)$. We shall require this level of generality in Example 2.7.8. If you prefer a concrete example, then the function $f(x) = (x - a)(x - b)$ suffices.

Consider the flow ϕ^t associated to the differential equation

$$\dot{x} = f(x)$$

on $[a, b]$. We shall refer to this flow as the **decreasing segment flow**. By Example 2.1.3 the fixed points of ϕ^t are the zeros of f . Thus,

$$\text{Fix}(\phi^t) = \{a, b\}.$$

By Example 2.7.5 fixed points are chain recurrent. So,

$$\{a, b\} \subseteq \mathcal{R}(\phi^t).$$

We shall prove that $\mathcal{R}(\phi^t) = \{a, b\}$. Notice that $\phi^t(x)$ is a strictly decreasing function of t for all $x \in (a, b)$ because $f(x) < 0$ for each $x \in (a, b)$. This insight is crucial to our argument.

Let $x \in (a, b)$ so that x is not a fixed point. Let $T > 0$. We shall produce an $\epsilon > 0$ so that there is no (ϵ, T) -chain from x to itself.

First, we claim that there exists $\delta > 0$ such that if $y < \phi^T(x) + \delta$, then

$$\phi^t(y) < \phi^T(x)$$

for all $t \geq T$. This means that if you start at a point y less than $\phi^T(x) + \delta$ and flow for at least time T , you must end up at a point less than $\phi^T(x)$. Let $\delta = x - \phi^T(x)$. Since ϕ^t is a strictly decreasing function of t and $T > 0$, we have $\delta > 0$. If $y < \phi^T(x) + \delta = x$, then

$$\phi^t(y) \leq \phi^T(y) < \phi^T(x)$$

for all $t \geq T$.

Now we shall produce an $\epsilon > 0$ so that there is no (ϵ, T) -chain from x to itself. Let $\epsilon = \min\{\delta, x - \phi^T(x)\}$. By means of contradiction, assume that there exists an (ϵ, T) -chain

$$(x = x_0, x_1, \dots, x_n = x; t_0, t_1, \dots, t_{n-1})$$

from x to itself. If d denotes the standard metric on $[a, b]$, then

$$|\phi^{t_0}(x_0) - x_1| = d(\phi^{t_0}(x_0), x_1) < \epsilon \leq \delta.$$

Since $t_0 \geq T$ and $\phi^t(x_0)$ is a decreasing function of t ,

$$x_1 < \phi^{t_0}(x_0) + \delta \leq \phi^T(x_0) + \delta.$$

Since $t_1 \geq T$,

$$\phi^{t_1}(x_1) < \phi^T(x_0).$$

Now, $|\phi^{t_1}(x_1) - x_2| = d(\phi^{t_1}(x_1), x_2) < \epsilon \leq \delta$, so that

$$x_2 < \phi^{t_1}(x_1) + \delta < \phi^T(x_0) + \delta.$$

Thus,

$$\phi^{t_2}(x_2) < \phi^T(x_0).$$

Continuing in this manner we obtain

$$\phi^{t_{n-1}}(x_{n-1}) < \phi^T(x_0) < x_0.$$

Since $x_n = x$ and $t_{n-1} \geq T$,

$$\begin{aligned} |\phi^{t_{n-1}}(x_{n-1}) - x_n| &= |\phi^{t_{n-1}}(x_{n-1}) - x| \\ &> x - \phi^T(x). \end{aligned}$$

On the other hand, by the definition of ϵ ,

$$|\phi^{t_{n-1}}(x_{n-1}) - x_n| < \epsilon \leq x - \phi^T(x).$$

This is a contradiction. Hence, $x \notin \mathcal{R}(\phi^t)$. Therefore,

$$\mathcal{R}(\phi^t) = \{a, b\}.$$

Example 2.7.8. Let $C \subset [0, 1]$ be the Cantor ternary set. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a smooth function such that $f(x) = 0$ for all $x \in C$ and $f(x) < 0$ for all $x \in [0, 1] \setminus C$.

Let ϕ^t be the flow associated to the differential equation

$$\dot{x} = f(x)$$

on $[0, 1]$. By Example 2.1.3 the fixed points of ϕ^t are exactly the zeros of f . Hence, $\text{Fix}(\phi^t) = C$. Since fixed points are chain recurrent, $C \subseteq \mathcal{R}(\phi^t)$. We claim that $\mathcal{R}(\phi^t) = C$.

Let $x \in [0, 1] \setminus C$. By properties of the Cantor ternary set, there exist $a, b \in C$ so that $x \in [a, b]$ and $(a, b) \cap C = \emptyset$. The flow $\phi^t|_{[a, b]}$ is exactly a flow of the type studied in Example 2.7.7. That example shows that $x \notin \mathcal{R}(\phi^t)$. Therefore,

$$\mathcal{R}(\phi^t) = C.$$

Our next example illustrates a more sophisticated version of the argument given in Example 2.7.7 and builds on the analysis of the circle flow in Example 2.7.6.

Example 2.7.9. Consider the system of differential equations

$$\begin{aligned} \dot{r} &= r(1-r)(\sin^2 \theta + 1 - r^2), \\ \dot{\theta} &= \sin^2 \theta + 1 - r^2 \end{aligned}$$

in polar coordinates on the closed unit disk \mathbb{D} in the complex plane. The flow ψ^t corresponding to this system is the arrested logistic rotation-dilation. The flow ψ^t has three fixed points: $-1, 0$ and 1 . We shall demonstrate that

$$\mathcal{R}(\psi^t) = \partial\mathbb{D} \cup \{0\}.$$

Since $r = 1$ on the boundary of \mathbb{D} , the flow restricted to the boundary is the flow of Example 2.7.6 in which every point is chain recurrent. By Example 2.7.5, every fixed point is chain recurrent. So, 0 is chain recurrent. We must show that 0 is the only chain recurrent point in the interior of \mathbb{D} .

Let $z \in \text{int}(\mathbb{D}) \setminus \{0\}$. For each $T > 0$ we seek an $\epsilon > 0$ so that there is no (ϵ, T) -chain from z to itself. Define $L : \mathbb{D} \rightarrow \mathbb{R}$ by

$$L(w) = 1 - |w|.$$

In fact, $L(w)$ is the distance from w to the boundary of \mathbb{D} . Since $0 < r < 1$ on $\text{int}(\mathbb{D}) \setminus \{0\}$,

$$\dot{r} = r(1-r)(\sin^2 \theta + 1 - r^2) > 0$$

on $\text{int}(\mathbb{D}) \setminus \{0\}$. Consequently, $L(\psi^t(w))$ is strictly decreasing as a function of t for each $w \in \text{int}(\mathbb{D}) \setminus \{0\}$.

Let $a = L(\psi^T(z))$ and $b = L(z)$. Then $0 < a < b < 1$. We claim that there exists $\delta > 0$ such that if $L(w) < a + \delta$, then

$$L(\psi^t(w)) < a$$

for all $t \geq T$. Let $\delta = 1 - a - |z|$. If $L(w) < a + \delta$, then $1 - |w| < 1 - |z|$, so that $|w| > |z|$. Because \dot{r} is positive and independent of θ , we have $L(\psi^T(w)) < L(\psi^T(z))$. Since L is decreasing along the orbit of w we obtain

$$L(\psi^t(w)) \leq L(\psi^T(w)) < L(\psi^T(z)) = a$$

for all $t \geq T$.

Let $\eta = \min\{\delta, b - a\}$. Since \mathbb{D} is compact and L is continuous, L is uniformly continuous. Thus, there exists $\epsilon > 0$ such that if $w_1, w_2 \in \mathbb{D}$ and $|w_1 - w_2| < \epsilon$, then

$$|L(w_1) - L(w_2)| < \eta.$$

By means of contradiction, assume that there is an (ϵ, T) -chain

$$(z = z_0, z_1, \dots, z_{n-1}, z_n = z; t_0, \dots, t_{n-1})$$

from z to itself. Since

$$|\psi^{t_0}(z_0) - z_1| < \epsilon,$$

the uniform continuity of L guarantees that

$$|L(\psi^{t_0}(z_0)) - L(z_1)| < \eta \leq \delta.$$

Because $t_0 \geq T$,

$$L(z_1) < L(\psi^{t_0}(z_0)) + \delta \leq L(\psi^T(z_0)) + \delta = a + \delta.$$

Thus,

$$L(\psi^{t_1}(z_1)) < a.$$

Since

$$|\psi^{t_1}(z_1) - z_2| < \epsilon,$$

the uniform continuity of L guarantees that

$$|L(\psi^{t_1}(z_1)) - L(z_2)| < \eta \leq \delta.$$

Because $t_1 \geq T$,

$$L(z_2) < L(\psi^{t_1}(z_1)) + \delta < a + \delta.$$

Thus,

$$L(\psi^{t_2}(z_2)) < a.$$

Continuing in this manner we obtain

$$L(\psi^{t_{n-1}}(z_{n-1})) < a.$$

Since $L(z_n) = L(z) = b$,

$$|L(\psi^{t_{n-1}}(z_{n-1})) - L(z_n)| = L(z) - L(\psi^{t_{n-1}}(z_{n-1})) > b - a.$$

On the other hand, since $|\psi^{t_{n-1}}(z_{n-1}) - z_n| < \epsilon$, the uniform continuity of L implies that

$$|L(\psi^{t_{n-1}}(z_{n-1})) - L(z_n)| < \eta \leq b - a.$$

This is a contradiction. Consequently, there is no (ϵ, T) -chain from z to itself. Therefore,

$$\mathcal{R}(\psi^t) = \partial\mathbb{D} \cup \{0\}.$$

The function L which plays a central role in the analysis of Example 2.7.9 is an example of a complete Lyapunov function. See Section 4.7. Among the key properties of a complete Lyapunov function are that it strictly decreases along orbits in the complement of the chain recurrent set, and that it is constant on collections of chain equivalent points. The identity function on the interval $[a, b]$ plays a similar role in Example 2.7.7. Chapter 4 investigates complete Lyapunov functions and their intimate connection to chain recurrence.

2.7.3. Elementary Properties of the Chain Recurrent Set. The chain recurrent set satisfies the Flow Invariance Property and the Closure Property. We now prove that the chain recurrent set is closed, but defer the proof of invariance to Corollary 3.3.8 where it follows easily from other considerations.

Proposition 2.7.10. *The chain recurrent set of a flow on a metric space is closed.*

Proof. Let ϕ^t be a flow on a metric space with metric d . Let y be a limit point of $\mathcal{R}(\phi^t)$. Let $\epsilon > 0$ and $T > 0$. We shall prove that $y \in \mathcal{R}(\phi^t)$ by constructing an (ϵ, T) -chain from y to itself.

By the continuity of ϕ^T , there exists $\delta > 0$ such that if $d(x, y) < \delta$, then

$$d(\phi^T(x), \phi^T(y)) < \frac{\epsilon}{2}.$$

Since y is a limit point of $\mathcal{R}(\phi^t)$, there exists a chain recurrent point x such that $d(x, y) < \min\{\delta, \epsilon/2\}$. So, $(y, \phi^T(x); T)$ is an (ϵ, T) -chain from y to $\phi^T(x)$.

Since x is chain recurrent, there exists an $(\epsilon/2, 2T)$ -chain

$$(x = x_0, \dots, x_n = x; t_0, \dots, t_{n-1})$$

from x to itself. Then

$$(\phi^T(x), x_1, \dots, x_{n-1}; t_0 - T, t_1, \dots, t_{n-2})$$

is an (ϵ, T) -chain from $\phi^T(x)$ to x_{n-1} .

By the triangle inequality,

$$d(\phi^{t_{n-1}}(x_{n-1}), y) < d(\phi^{t_{n-1}}(x_{n-1}), x) + d(x, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Consequently, $(x_{n-1}, y; t_{n-1})$ is an (ϵ, T) -chain from x_{n-1} to y .

By concatenating these (ϵ, T) -chains,

$$(y, \phi^T(x), x_1, \dots, x_{n-1}, y; T, t_0 - T, t_1, \dots, t_{n-1})$$

is an (ϵ, T) -chain from y to itself. Therefore, $\mathcal{R}(\phi^t)$ is closed. \square

Topological equivalences carry chain recurrent sets to chain recurrent sets under additional hypotheses on the topological equivalence.

Proposition 2.7.11. *Let (X, d_X) and (Y, d_Y) be metric spaces. If (h, ρ) is a topological equivalence from the flow $\phi^t : X \rightarrow X$ to the flow $\psi^t : Y \rightarrow Y$ such that $h : X \rightarrow Y$ is uniformly continuous, $\sup\{\rho(T, z) \mid z \in X\}$ exists for each $T > 0$, and $\sup\{\tilde{\rho}(T, z) \mid z \in X\}$ exists for each $T > 0$, then*

$$\mathcal{R}(\psi^t) = h(\mathcal{R}(\phi^t)).$$

Proof. We begin by showing that $h(\mathcal{R}(\phi^t)) \subseteq \mathcal{R}(\psi^t)$. Let $x \in \mathcal{R}(\phi^t)$. Let $\epsilon > 0$ and $T > 0$. We shall construct an (ϵ, T) -chain from $h(x)$ to itself.

Because h is uniformly continuous on X , there exists $\delta > 0$ such that if $d_X(z_1, z_2) < \delta$, then $d_Y(h(z_1), h(z_2)) < \epsilon$. Let $S = \sup\{\rho(T, z) \mid z \in X\}$. Since $x \in \mathcal{R}(\phi^t)$, there exists a (δ, S) -chain

$$(x = x_0, x_1, \dots, x_n = x; s_0, \dots, s_{n-1})$$

from x to itself. For each $i = 0, \dots, n$ let $y_i = h(x_i)$. Since ρ is a reparametrization, there exists t_i so that $s_i = \rho(t_i, x_i)$ for each $i = 0, \dots, n-1$. We claim that

$$(h(x) = y_0, y_1, \dots, y_n = h(x); t_0, \dots, t_{n-1})$$

is an (ϵ, T) -chain from $h(x)$ to itself.

Since $s_i \geq S$ for $i = 0, \dots, n-1$,

$$\rho(t_i, x_i) = s_i \geq S \geq \rho(T, x_i).$$

Because $\rho(\cdot, x_i)$ is an increasing function, $t_i \geq T$ for $i = 0, \dots, n-1$. Since $d_X(\phi^{s_i}(x_i), x_{i+1}) < \delta$ for $i = 0, \dots, n-1$,

$$\begin{aligned} d_Y(\psi^{t_i}(y_i), y_{i+1}) &= d_Y(\psi^{t_i}(h(x_i)), h(x_{i+1})) \\ &= d_Y(h(\phi^{\rho(t_i, x_i)}(x_i)), h(x_{i+1})) \\ &= d_Y(h(\phi^{s_i}(x_i)), h(x_{i+1})) \\ &< \epsilon \end{aligned}$$

for $i = 0, \dots, n-1$. Thus,

$$(h(x) = y_0, y_1, \dots, y_n = h(x); t_0, \dots, t_{n-1})$$

is an (ϵ, T) -chain from $h(x)$ to itself so that $h(x) \in \mathcal{R}(\psi^t)$. Therefore, $h(\mathcal{R}(\phi^t)) \subseteq \mathcal{R}(\psi^t)$.

Applying the previous argument to the topological equivalence $(h^{-1}, \tilde{\rho})$ we obtain $h^{-1}(\mathcal{R}(\psi^t)) \subseteq \mathcal{R}(\phi^t)$. Thus, $\mathcal{R}(\psi^t) \subseteq h(\mathcal{R}(\phi^t))$. Therefore,

$$\mathcal{R}(\psi^t) = h(\mathcal{R}(\phi^t)). \quad \square$$

As a corollary, the chain recurrent set satisfies the Topological Invariance Property for flows on compact spaces.

Corollary 2.7.12. *Let X and Y be metric spaces. Let $\phi^t : X \rightarrow X$ and $\psi^t : Y \rightarrow Y$ be flows. If X is compact and $h : X \rightarrow Y$ is a topological conjugacy from ϕ^t to ψ^t , then*

$$h(\mathcal{R}(\phi^t)) = \mathcal{R}(\psi^t).$$

Proof. Since X is compact, the homeomorphism h is uniformly continuous. Applying Proposition 2.7.11 with $\rho(t, x) = t$, we obtain $h(\mathcal{R}(\phi^t)) = \mathcal{R}(\psi^t)$. \square

The following example shows that compactness is necessary in the hypotheses of Corollary 2.7.12. If the phase space is not compact, then the chain recurrent set may depend on the metric even if two complete metrics induce the same topology.

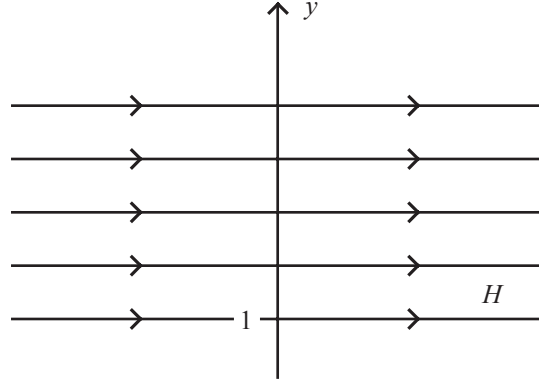


Figure 2.19. A phase portrait of the flow of Example 2.7.13

Example 2.7.13. Let $H = \{(x, y) \in \mathbb{R}^2 \mid y \geq 1\}$. Denote the metric on H induced by the standard metric on \mathbb{R}^2 by d_E , and denote the metric on H induced by the hyperbolic (Riemannian) metric on the upper half-plane $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ by d_H . The metric d_H inherits the following properties from the hyperbolic metric:

$$d_H((x, y_0), (x, y_1)) = |y_1 - y_0|$$

and

$$d_H((x_0, y), (x_1, y)) = \frac{|x_1 - x_0|}{y}$$

for all $(x, y_0), (x, y_1), (x_0, y), (x_1, y) \in H$. Define $\phi : \mathbb{R} \times H \rightarrow H$ by

$$\phi(t, (x, y)) = (x + t, y).$$

Exercise 16 asks you to verify that ϕ is a flow, that the identity map $\iota : (H, d_E) \rightarrow (H, d_H)$ is a topological conjugacy from ϕ^t to itself, and that the chain recurrent set of ϕ^t with respect to d_E is the empty set. See Figure 2.19.

In contrast, we shall demonstrate that the chain recurrent set of ϕ^t with respect to d_H is H . Let $(x, y) \in H$. Let $\epsilon > 0$ and $T > 0$. There exists a positive integer m such that

$$m > \frac{2}{\epsilon} \left(\frac{2T}{\epsilon} - y \right).$$

Let $(x_0, y_0) = (x, y)$. Define

$$(x_{i+1}, y_{i+1}) = \begin{cases} (x_i + T, y_i + \epsilon/2) & \text{if } i = 0, \dots, m-1, \\ (x_i - T, y_i) & \text{if } i = m, \dots, 3m-1, \\ (x_i + T, y_i - \epsilon/2) & \text{if } i = 3m, \dots, 4m-1. \end{cases}$$

Define $t_i = T$ for $i = 0, \dots, 4m - 1$. We claim that

$$\{(x_0, y_0), \dots, (x_{4m}, y_{4m}); t_0, \dots, t_{4m-1}\}$$

is an (ϵ, T) -chain from (x, y) to itself. If $i = 0, \dots, m - 1$, then

$$\begin{aligned} d_H(\phi^{t_i}(x_i, y_i), (x_{i+1}, y_{i+1})) &= d_H((x_i + T, y_i), (x_i + T, y_i + \epsilon/2)) \\ &= \epsilon/2 \\ &< \epsilon. \end{aligned}$$

If $i = m, \dots, 3m - 2$, then $y_i = y + m\epsilon/2 > 2T/\epsilon$. So,

$$\begin{aligned} d_H(\phi^{t_i}(x_i, y_i), (x_{i+1}, y_{i+1})) &= d_H((x_i + T, y_i), (x_i - T, y_i)) \\ &= \frac{2T}{y_i} \\ &< \epsilon. \end{aligned}$$

If $i = 3m - 1, \dots, 4m - 1$, then

$$\begin{aligned} d_H(\phi^{t_i}(x_i, y_i), (x_{i+1}, y_{i+1})) &= d_H((x_i + T, y_i), (x_i + T, y_i - \epsilon/2)) \\ &= \epsilon/2 \\ &< \epsilon. \end{aligned}$$

Therefore, the chain recurrent set of ϕ^t with respect to the metric d_H is H .

The chain recurrent set contains the nonwandering set. Thus, Poincaré recurrent points and central points are chain recurrent.

Proposition 2.7.14. *The chain recurrent set of a flow on a metric space contains the nonwandering set of the flow.*

Proof. Let ϕ^t be a flow on a metric space (X, d) . Let $x \in \Omega(\phi^t)$, and let $\epsilon > 0$ and $T > 0$. By the continuity of ϕ^T , there exists $\delta \in (0, \epsilon)$ such that if $d(x, y) < \delta$, then

$$d(\phi^T(x), \phi^T(y)) < \epsilon.$$

Denote the open δ -ball around x by $B(x, \delta)$. Since X is a metric space, X is Hausdorff. By Lemma 2.5.9 there is a real number $S \geq 2T$ with the property that

$$B(x, \delta) \cap \phi^S(B(x, \delta)) \neq \emptyset.$$

Consequently, there exists $y \in B(x, \delta)$ with $\phi^S(y) \in B(x, \delta)$. Since $d(x, y) < \delta$, the continuity of ϕ^T implies that

$$d(\phi^T(x), \phi^T(y)) < \epsilon.$$

Because $S \geq 2T$, we know that $S - T \geq T$. By the group property of flows,

$$d(\phi^{S-T}(\phi^T(y)), x) = d(\phi^S(y), x) < \delta < \epsilon.$$

Thus,

$$(x, \phi^T(y), x; T, S - T)$$

is an (ϵ, T) -chain from x to itself. Therefore, x is a chain recurrent point for ϕ^t , and

$$\Omega(\phi^t) \subseteq \mathcal{R}(\phi^t). \quad \square$$

Let ϕ^t be a flow on a second-countable metric space. We have now established a chain of inclusions expressing more and more general modes of recurrence:

$$\text{Fix}(\phi^t) \subseteq \text{Per}(\phi^t) \subseteq \text{Rec}(\phi^t) \subseteq C(\phi^t) \subseteq \Omega(\phi^t) \subseteq \mathcal{R}(\phi^t).$$

Since the chain recurrent set contains the nonwandering set, the chain recurrent set contains the α - and ω -limit sets of each point in the phase space. Thus, the chain recurrent set satisfies the Accumulation Property.

Corollary 2.7.15. *If ϕ^t is a flow on a metric space X , then $\omega(x) \subseteq \mathcal{R}(\phi^t)$ and $\alpha(x) \subseteq \mathcal{R}(\phi^t)$ for all $x \in X$.*

Proof. By Propositions 2.5.14 and 2.7.14,

$$\alpha(x) \cup \omega(x) \subseteq \Omega(\phi^t) \subseteq \mathcal{R}(\phi^t)$$

for all $x \in X$. □

2.7.4. Equivalent Formulations of Chain Equivalence. We now describe three alternative formulations of chain equivalence and prove that for a flow on a compact metric space these formulations coincide with Definition 2.7.3. For this purpose it is useful to provide definitions of ϵ -chain and chain equivalence with respect to a map.

Definition 2.7.16. Let (X, d) be a metric space, and let $f : X \rightarrow X$. Given $\epsilon > 0$ and $x, y \in X$, an ϵ -**chain** from x to y is a finite sequence

$$x = x_0, x_1, \dots, x_{n-1}, x_n = y$$

in X such that

$$d(f(x_i), x_{i+1}) < \epsilon$$

for $i = 0, 1, 2, \dots, n - 1$. See Figure 2.20.

Definition 2.7.17. Let X be a metric space, and let $f : X \rightarrow X$. Two points $x, y \in X$ are called **chain equivalent** if for every $\epsilon > 0$ there exists an ϵ -chain from x to y and there exists an ϵ -chain from y to x .

We now describe three alternative formulations of chain equivalence which appear in the literature and prove that for a flow on a compact metric space these formulations coincide with Definition 2.7.3. The essence of this result appears in Hurley [23].

Theorem 2.7.18. *If ϕ^t is a flow on a compact metric space (X, d) and $x, y \in X$, then the following statements are equivalent.*

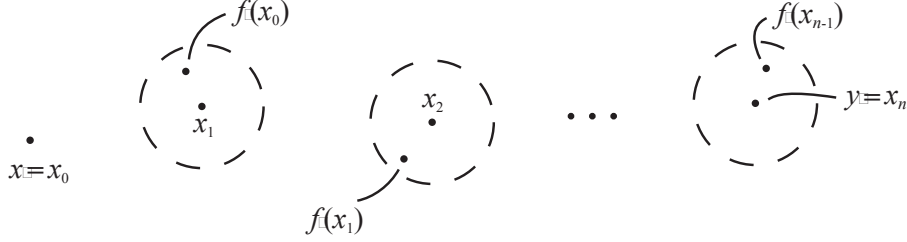


Figure 2.20. An ϵ -chain for a map

- (i) The points x and y are chain equivalent with respect to ϕ^t .
- (ii) For every $\epsilon > 0$ and $T > 0$ there exists an $(\epsilon, 1)$ -chain

$$(x_0, \dots, x_n; t_0, \dots, t_{n-1})$$

from x to y such that

$$t_0 + \dots + t_{n-1} \geq T,$$

and there exists an $(\epsilon, 1)$ -chain

$$(y_0, \dots, y_m; s_0, \dots, s_{m-1})$$

from y to x such that

$$s_0 + \dots + s_{m-1} \geq T.$$

- (iii) For every $\epsilon > 0$ there exists an $(\epsilon, 1)$ -chain from x to y and an $(\epsilon, 1)$ -chain from y to x .
- (iv) The points x and y are chain equivalent with respect to ϕ^1 .

In the spirit of Robinson [37] p. 151, Theorem 2.7.18 (ii) asserts that two points x and y are chain equivalent exactly when, given any specified time T , there are $(\epsilon, 1)$ -chains from x to y and from y to x so that the total time along each chain is at least T . Following Franks [14] p. 1, Theorem 2.7.18 (iii) removes the condition from part (ii) concerning the total time along each chain. Theorem 2.7.18 (iv) states that two points are chain equivalent with respect to a flow exactly when they are chain equivalent with respect to the time-one map of the flow.

In preparation for the proof of Theorem 2.7.18 we establish two lemmas. The first shows that if there exists an $(\epsilon, 1)$ -chain from x to y and an $(\epsilon, 1)$ -chain from y to x , then there is an $(\epsilon, 1)$ -chain from x to y in which all of the times are at most two and the total time along the chain is nearly an integer. The proof exploits Proposition B.0.9 regarding irrational rotations of the circle.

Lemma 2.7.19. *Let ϕ^t be a flow on a metric space (X, d) . If $x, y \in X$ and for every $\epsilon > 0$ there exists an $(\epsilon, 1)$ -chain from x to y and there exists an $(\epsilon, 1)$ -chain from y to x , then for every $\epsilon > 0$ and $\delta > 0$ there exists a positive integer K and an $(\epsilon, 1)$ -chain*

$$(x = x_0, x_1, \dots, x_n = y; t_0, t_1, \dots, t_{n-1})$$

from x to y such that

$$1 \leq t_i < 2$$

for $i = 0, 1, 2, \dots, n-1$, and if $T = t_0 + t_1 + \dots + t_{n-1}$, then

$$|T - K| < \delta.$$

Proof. Let $\epsilon > 0$ and $\delta > 0$. By hypothesis there is an $(\epsilon, 1)$ -chain

$$(x = w_0, w_1, \dots, w_n = y; r_0, r_1, \dots, r_{n-1})$$

from x to y . Without loss of generality, we may assume that $r_i \in [1, 2)$ for $i = 0, 1, 2, \dots, n-1$. If not, then insert points and times into this chain, such as $\phi^1(x)$ and time 1, and reindex the points and times in the chain. Let $R = r_0 + r_1 + \dots + r_{n-1}$. Similarly, there is an $(\epsilon, 1)$ -chain

$$(y = z_0, z_1, \dots, z_m = x; s_0, s_1, \dots, s_{m-1})$$

such that $s_i \in [1, 2)$ for $i = 0, 1, 2, \dots, m-1$. Let $S = s_0 + s_1 + \dots + s_{m-1}$. By the continuity of ϕ^t there exists $\eta > 0$ such that if $|t - r_0| < \eta$, then

$$d(\phi^t(w_0), \phi^{r_0}(w_0)) < \epsilon - d(\phi^{r_0}(w_0), w_1).$$

By the density of the irrational numbers among the real numbers there exists an $\bar{r}_0 \in [r_0, r_0 + \eta)$ such that $R + S - r_0 + \bar{r}_0$ is irrational. By the triangle inequality,

$$\begin{aligned} d(\phi^{\bar{r}_0}(w_0), w_1) &\leq d(\phi^{\bar{r}_0}(w_0), \phi^{r_0}(w_0)) + d(\phi^{r_0}(w_0), w_1) \\ &< \epsilon - d(\phi^{r_0}(w_0), w_1) + d(\phi^{r_0}(w_0), w_1) \\ &= \epsilon. \end{aligned}$$

Thus, $(x = w_0, w_1, \dots, w_n = y; \bar{r}_0, r_1, \dots, r_{n-1})$ is an $(\epsilon, 1)$ -chain from x to y .

Let $\bar{R} = \bar{r}_0 + r_1 + \dots + r_{n-1}$. Since $\bar{R} + S$ is irrational, Proposition B.0.9 implies that there exist positive integers J and K so that

$$|\bar{R} + J(\bar{R} + S) - K| < \delta,$$

We shall construct an $(\epsilon, 1)$ -chain from x to y so that the sum of times in the chain is $\bar{R} + J(\bar{R} + S)$. Begin with the chain

$$(x = w_0, w_1, \dots, w_n = y; \bar{r}_0, r_1, \dots, r_{n-1}),$$

and alternately concatenate the chains

$$(y = z_0, z_1, \dots, z_m = x; s_0, s_1, \dots, s_{m-1})$$

and

$$(x = w_0, w_1, \dots, w_n = y; \bar{r}_0, r_1, \dots, r_{n-1})$$

a total of J times to obtain an $(\epsilon, 1)$ -chain

$$(w_0, \dots, w_n = \underbrace{z_0, \dots, z_m, w_1, \dots, y}_{J \text{ times}}; \bar{r}_0, \dots, r_{n-1}, \underbrace{s_0, \dots, s_{m-1}, \bar{r}_0, \dots, r_{n-1}}_{J \text{ times}})$$

from x to y . The sum $T = \bar{R} + J(\bar{R} + S)$ of the times in this chain has the property that

$$|T - K| < \delta. \quad \square$$

Often we use the fact that the flow ϕ^t is continuous on $\mathbb{R} \times X$ together with the compactness of X to obtain local uniform continuity. The next technical lemma is such a result.

Lemma 2.7.20. *Let ϕ^t be a flow on a compact metric space (X, d) . Then for every $\epsilon > 0$ there is a $\delta > 0$ so that if $|t| < \delta$, then*

$$d(\phi^t(x), x) < \epsilon$$

for every $x \in X$.

Proof. Let $\epsilon > 0$. The compactness of X implies that $[-1, 1] \times X$ is compact. So, since ϕ^t is continuous, ϕ^t is uniformly continuous on $[-1, 1] \times X$. Thus, there exists $\delta \in (0, 1)$ such that if $|t - s| < \delta$ and $d(x, z) < \delta$, then

$$d(\phi^t(x), \phi^s(z)) < \epsilon.$$

In particular, $d(x, x) = 0 < \delta$ for each $x \in X$. Therefore, if $t \in (-\delta, \delta)$, then $x \in X$. \square

We will now prove Theorem 2.7.18

Proof. We shall prove that

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i).$$

- (1) The proof that (i) implies (ii) is immediate.
- (2) The proof that (ii) implies (iii) is also immediate.
- (3) To prove that (iii) implies (iv), assume that for each $\epsilon > 0$ there exists an $(\epsilon, 1)$ -chain from x to y and there exists an $(\epsilon, 1)$ -chain from y to x . We must prove that for each $\epsilon > 0$ there exists an ϵ -chain from x to y with respect to ϕ^1 . The argument that there exists an ϵ -chain from y to x with respect to ϕ^1 is similar.

Let $\epsilon > 0$. By Lemma 2.7.20 there exists $\delta_1 \in (0, 1)$ such that if $t \in (-\delta_1, \delta_1)$, then

$$(2.1) \quad d(\phi^t(z), z) < \frac{\epsilon}{4}$$

for all $z \in X$. By the continuity of ϕ^t and the compactness of $[0, 3] \times X$, there exists $\delta_2 > 0$ such that if $z_1, z_2 \in X$ and $d(z_1, z_2) < \delta_2$, then

$$(2.2) \quad d(\phi^t(z_1), \phi^t(z_2)) < \frac{\epsilon}{2}$$

for all $t \in [0, 3]$.

Let $\eta = \min\{\epsilon/4, \delta_2\}$. By Lemma 2.7.19 there exists an $(\eta, 1)$ -chain

$$(x = x_0, x_1, \dots, x_n = y; t_0, t_1, \dots, t_{n-1})$$

from x to y and there exists a positive integer K such that

$$1 \leq t_i < 2$$

for $i = 0, 1, 2, \dots, n-1$, and

$$|T - K| < \delta_1$$

where $T = t_0 + t_1 + \dots + t_{n-1}$. Denote the chain

$$(x = x_0, x_1, \dots, x_n = y; t_0, t_1, \dots, t_{n-1})$$

by \mathcal{C} . From \mathcal{C} we shall obtain an ϵ -chain

$$x = y_0, y_1, \dots, y_K = y$$

with respect to ϕ^1 .

Let $y_0 = x_0 = x$. The new chain coincides with \mathcal{C} for as many steps as $t_i = 1$. The modifications to \mathcal{C} begin at the first step at which $t_i > 1$. Without loss of generality, we may assume that $t_0 > 1$.

Let $y_1 = \phi^1(y_0)$. Accordingly, $d(\phi^1(y_0), y_1) = 0 < \epsilon$.

The strategy for defining y_2, \dots, y_K is to account for the time elapsed along the new chain versus the time remaining along the chain \mathcal{C} . On the arc from y_0 to $\phi^1(y_0)$, we have used one unit of time. The total time remaining along \mathcal{C} is $T - 1$, while the time remaining along the arc from x_0 to $\phi^{t_0}(x_0)$ is $t_0 - 1$. Since $1 < t_0 < 2$, we have $0 < t_0 - 1 < 1$. To define y_2 , jump to the orbit of x_1 and flow for $1 - (t_0 - 1) = 2 - t_0$ units of time. Observe that $0 < 2 - t_0 < 1 \leq t_1$. It follows from the choice of η that

$$d(\phi^{t_0}(x_0), x_1) < \delta_2.$$

Consequently, inequality (2.2) implies that

$$d(\phi^{2-t_0}(\phi^{t_0}(x_0)), \phi^{2-t_0}(x_1)) < \frac{\epsilon}{2}.$$

Let $y_2 = \phi^{2-t_0}(x_1)$. From the definitions of y_0 , y_1 and y_2 we obtain

$$\begin{aligned} d(\phi^1(y_1), y_2) &= d(\phi^2(y_0), \phi^{2-t_0}(x_1)) \\ &= d(\phi^{2-t_0}(\phi^{t_0}(x_0)), \phi^{2-t_0}(x_1)) \\ &< \frac{\epsilon}{2}. \end{aligned}$$

A total of 2 units of time have elapsed along the chain \mathcal{C} , and $y_2 = \phi^{2-t_0}(x_1)$ lies on the arc joining x_1 to $\phi^{t_1}(x_1)$. The time remaining on this arc is $t_1 - (2 - t_0) = t_0 + t_1 - 2$.

To define y_3 there are three cases to consider. If there is more than one unit of time remaining on the arc joining x_1 to $\phi^{t_1}(x_1)$, then continue to follow the orbit of x_1 for time one. If there is exactly one unit of time remaining, then jump to the arc of the orbit joining x_2 and $\phi^{t_2}(x_2)$. If there is less than one unit of time remaining, then to recover an additional $1 - (t_0 + t_1 - 2) = 3 - (t_0 + t_1)$ units of time, jump to an appropriate point on the orbit of x_2 .

More precisely, the three cases are:

(a) If $t_0 + t_1 - 2 > 1$, then let $y_3 = \phi^1(y_2)$, in which case

$$d(\phi^1(y_2), y_3) = 0 < \epsilon.$$

(b) If $t_0 + t_1 - 2 = 1$, that is, $t_0 + t_1 = 3$, then let $y_3 = x_2$. Since $y_2 = \phi^{2-t_0}(x_1)$, the properties of the chain \mathcal{C} guarantee that

$$d(\phi^1(y_2), y_3) = d(\phi^{t_1}(x_1), x_2) < \epsilon.$$

(c) If $t_0 + t_1 - 2 < 1$, then define $y_3 = \phi^{3-(t_0+t_1)}(x_2)$. Since

$$d(\phi^{t_1}(x_1), x_2) < \delta_2,$$

and $3 - (t_0 + t_1) < 1$, we conclude from inequality (2.2) that

$$\begin{aligned} d(\phi^1(y_2), y_3) &= d(\phi^{3-t_0}(x_1), \phi^{3-(t_0+t_1)}(x_2)) \\ &= d(\phi^{3-(t_0+t_1)}(\phi^{t_1}(x_1)), \phi^{3-(t_0+t_1)}(x_2)) \\ &< \frac{\epsilon}{2}. \end{aligned}$$

The preceding argument suggests how to define the rest of the points in the chain y_0, \dots, y_K . Suppose that we have defined y_0, \dots, y_j . Then y_j lies on the arc of the orbit of x_i joining x_i with $\phi^{t_i}(x_i)$ for some $i = 0, 1, 2, \dots, n-1$. If there is more than one unit of time remaining on the arc starting at x_i , then define y_{j+1} to be on the same arc at time one away from y_j . If there is exactly one unit of time left on the arc starting at x_i , then define y_{j+1} to be x_{i+1} . Finally, if there is less than one unit of time remaining on the arc starting at x_i , then define y_{j+1} to be an appropriate point on the orbit of x_{i+1} .

More precisely, suppose we have defined y_j . There exists a natural number i such that $0 \leq i \leq n-1$ and there exists $\tau \in [0, t_i)$ such that $y_j = \phi^\tau(x_i)$.

- (a) If $t_i - \tau > 1$, then define $y_{j+1} = \phi^1(y_j)$.
- (b) If $t_i - \tau = 1$, then define $y_{j+1} = x_{i+1}$.
- (c) If $t_i - \tau < 1$, then define $y_{j+1} = \phi^{1-(t_i-\tau)}(x_{i+1})$.

The details verifying that

$$d(\phi^1(y_j), y_{j+1}) < \epsilon$$

in each of these cases are similar to those of our earlier argument.

Continue in this manner until we have defined y_{K-1} . The time elapsed along the new ϵ -chain is $K-1$. Recall that the total time along the chain \mathcal{C} is T . Thus, the time remaining is

$$T - (K - 1) = 1 + (T - K)$$

where $|T - K| < \delta_1$.

Let $y_K = y$. We shall prove that $d(\phi^1(y_{K-1}), y) < \epsilon$. The amount of time remaining along the chain \mathcal{C} is within δ_1 of 1 and $0 < \delta_1 < 1$. So, y_{K-1} lies either on the arc of the orbit of x_{n-1} joining x_{n-1} with $\phi^{t_{n-1}}(x_{n-1})$ or lies on the arc of the orbit of x_{n-2} joining x_{n-2} with $\phi^{t_{n-2}}(x_{n-2})$.

If y_{K-1} lies on the arc of the orbit of x_{n-1} joining x_{n-1} to $\phi^{t_{n-1}}(x_{n-1})$, then there exists $\tau \in [0, t_{n-1})$ such that $y_{K-1} = \phi^\tau(x_{n-1})$, and the remaining time along this arc is $t_{n-1} - \tau = 1 + (T - K)$. Therefore,

$$\begin{aligned} d(\phi^1(y_{K-1}), y) &= d(\phi^{\tau+1}(x_{n-1}), y) \\ &= d(\phi^{t_{n-1}-(T-K)}(x_{n-1}), y) \\ &\leq d(\phi^{t_{n-1}-(T-K)}(x_{n-1}), \phi^{t_{n-1}}(x_{n-1})) \\ &\quad + d(\phi^{t_{n-1}}(x_{n-1}), y). \end{aligned}$$

By inequality (2.1) and the conditions on the chain \mathcal{C} , we obtain

$$d(\phi^1(y_{K-1}), y) < \frac{\epsilon}{4} + \frac{\epsilon}{2} < \epsilon.$$

If y_{K-1} lies on the arc of the orbit of x_{n-2} joining x_{n-2} to $\phi^{t_{n-2}}(x_{n-2})$, then there exists $\tau \in [0, t_{n-2})$ such that $y_{K-1} = \phi^\tau(x_{n-2})$. The total remaining time, on the one hand, is the time remaining on the arc joining x_{n-2} with $\phi^{t_{n-2}}(x_{n-2})$ plus the time along the arc joining x_{n-1} with $\phi^{t_{n-1}}(x_{n-1})$, and, on the other hand, is the total time along the chain minus the time elapsed so far. So,

$$t_{n-2} - \tau + t_{n-1} = T - (K - 1).$$

Consequently,

$$\tau + 1 - t_{n-2} = t_{n-1} - (T - K).$$

Since $1 \leq t_{n-1} < 2$ and $|T - K| < \delta_1 < 1$, we obtain

$$\tau + 1 - t_{n-2} = t_{n-1} - (T - K) \leq 3.$$

By inequality (2.2), $d(\phi^{t_{n-2}}(x_{n-2}), x_{n-1}) < \eta \leq \delta_2$ implies that

$$d(\phi^{\tau+1}(x_{n-2}), \phi^{t_{n-1}-(T-K)}(x_{n-1})) < \frac{\epsilon}{2}.$$

By inequality (2.1), $|T - K| < \delta_1$ implies that

$$d(\phi^{t_{n-1}-(T-K)}(x_{n-1}), \phi^{t_{n-1}}(x_{n-1})) < \frac{\epsilon}{4}.$$

Because \mathcal{C} is an $(\eta, 1)$ -chain,

$$d(\phi^{t_{n-1}}(x_{n-1}), y) < \eta \leq \frac{\epsilon}{4}.$$

Thus,

$$\begin{aligned} d(\phi^1(y_{K-1}), y) &= d(\phi^{\tau+1}(x_{n-2}), y) \\ &\leq d(\phi^{\tau+1}(x_{n-2}), \phi^{t_{n-1}-(T-K)}(x_{n-1})) \\ &\quad + d(\phi^{t_{n-1}-(T-K)}(x_{n-1}), \phi^{t_{n-1}}(x_{n-1})) \\ &\quad + d(\phi^{t_{n-1}}(x_{n-1}), y) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon. \end{aligned}$$

Therefore, the sequence $x = y_0, y_1, \dots, y_{K-1}, y_K = y$ satisfies

$$d(\phi^1(y_i), y_{i+1}) < \epsilon$$

for $i = 0, 1, 2, \dots, K - 1$. Thus, $x = y_0, y_1, \dots, y_{K-1}, y_K = y$ is an ϵ -chain from x to y with respect to ϕ^1 . Similarly, there exists an ϵ -chain from y to x . Therefore, x and y are chain equivalent with respect to ϕ^1 .

- (4) To prove that (iv) implies (i), assume that x and y are chain equivalent with respect to ϕ^1 . Let $\epsilon > 0$ and $T > 0$. Without loss of generality, we may assume that T is an integer.

Let $\delta_0 = \epsilon$. Since X is compact and ϕ^1 is continuous, the function ϕ^1 is uniformly continuous on X . Thus, there exists $\delta_1 \in (0, \epsilon)$ such that if $w_1, w_2 \in X$ and $d(w_1, w_2) < \delta_1$, then

$$d(\phi^1(w_1), \phi^1(w_2)) < \frac{\delta_0}{2}.$$

Similarly, for each $j = 2, 3, 4, \dots, 2T - 1$ there exists $\delta_j \in (0, \epsilon)$ such that if $w_1, w_2 \in X$ and $d(w_1, w_2) < \delta_j$, then

$$d(\phi^1(w_1), \phi^1(w_2)) < \frac{\delta_{j-1}}{2}.$$

Define

$$\eta = \min \left\{ \frac{\delta_0}{2}, \dots, \frac{\delta_{2T-1}}{2} \right\}.$$

By hypothesis there exist η -chains

$$x = x_0, x_1, \dots, x_{K-1}, x_K = y$$

and

$$y = y_0, y_1, \dots, y_{L-1}, y_L = x.$$

Begin with the sequence x_0, x_1, \dots, x_{K-1} and concatenate T copies of $y_0, y_1, \dots, y_{L-1}, y_L = x_0, x_1, \dots, x_{K-1}$. End the sequence with $x_K = y$ to obtain a sequence

$$x = x_0, x_1, \dots, x_K = \underbrace{y = y_0, \dots, y_L = x = x_0, x_1, \dots, x_{K-1}, y}_{T \text{ times}}$$

in X with $K + T(K + L) + 1$ entries.

By labeling this new sequence we have an η -chain

$$x = w_0, w_1, \dots, w_{K+T(K+L)} = y$$

with respect to ϕ^1 .

Since K , L and T are positive integers, there exist positive integers n and r such that

$$K + T(K + L) = nT + r$$

and $T \leq r < 2T$.

Define

$$z_i = \begin{cases} w_{iT} & \text{if } 0 \leq i < n, \\ y & \text{if } i = n \end{cases}$$

and

$$t_i = \begin{cases} T & \text{if } 0 \leq i < n - 1, \\ r & \text{if } i = n - 1 \end{cases}$$

for $i = 0, 1, 2, \dots, n$.

We claim that $(z_0, \dots, z_n; t_0, \dots, t_{n-1})$ is an (ϵ, T) -chain from x to y . Let $i = 0, 1, 2, \dots, n$. Observe that by the choice of η ,

$$d(\phi^1(w_{iT}), w_{iT+1}) < \eta \leq \delta_{r-1}.$$

Assume that for some integer k such that $2 \leq k \leq r$ we have

$$d(\phi^{k-1}(w_{iT}), w_{iT+k-1}) < \eta \leq \delta_{r-k+1}.$$

(We have just shown that this is true for $k = 2$.) From the choice of δ_{r-k+1} we find that

$$d(\phi^k(w_{iT}), \phi^1(w_{iT+k-1})) < \frac{\delta_{r-k}}{2}.$$

By the triangle inequality,

$$\begin{aligned}
 d(\phi^k(w_{iT}), w_{iT+k}) &\leq d(\phi^k(w_{iT}), \phi^1(w_{iT+k-1})) \\
 &\quad + d(\phi^1(w_{iT+k-1}), w_{iT+k}) \\
 &< \frac{\delta_{r-k}}{2} + \eta \\
 &\leq \frac{\delta_{r-k}}{2} + \frac{\delta_{r-k}}{2} \\
 &= \delta_{r-k}.
 \end{aligned}$$

Thus, if $k = T$, then

$$d(\phi^T(z_i), z_{i+1}) = d(\phi^T(w_{iT}), w_{(i+1)T}) < \delta_{r-T} \leq \epsilon$$

for $i = 0, 1, 2, \dots, n-2$. If $k = r$, then

$$d(\phi^r(z_{n-1}), z_n) = d(\phi^r(w_{nT}), w_{nT+r}) < \delta_0 = \epsilon.$$

Hence, $(z_0, \dots, z_n, t_0, \dots, t_{n-1})$ is an (ϵ, T) -chain from x to y .

The construction of an (ϵ, T) -chain from y to x is similar. Therefore, (iv) implies (i).

We have now shown that statements (i)–(iv) are equivalent. \square

2.7.5. The Restriction Property of the Chain Recurrent Set. Now we show that if ϕ^t is a flow on a compact metric space, then

$$\mathcal{R}(\phi^t|_{\mathcal{R}(\phi^t)}) = \mathcal{R}(\phi^t),$$

so that the chain recurrent set satisfies the Restriction Property. In other words, for each $x \in \mathcal{R}(\phi^t)$, $\epsilon > 0$ and $T > 0$, there exists an (ϵ, T) -chain

$$(x = x_0, x_1, \dots, x_{n-1}, x_n = x; t_0, \dots, t_{n-1})$$

from x to itself such that $x_i \in \mathcal{R}(\phi^t)$ for $i = 0, 1, 2, \dots, n$. The idea of the proof is due to Robinson [36], although we employ the Hausdorff metric to streamline its presentation.

Proposition 2.7.21. *If ϕ^t is a flow on a compact metric space (X, d) , then*

$$\mathcal{R}(\phi^t|_{\mathcal{R}(\phi^t)}) = \mathcal{R}(\phi^t).$$

Proof. The inclusion $\mathcal{R}(\phi^t|_{\mathcal{R}(\phi^t)}) \subseteq \mathcal{R}(\phi^t)$ is immediate. Thus, it suffices to prove that

$$\mathcal{R}(\phi^t) \subseteq \mathcal{R}(\phi^t|_{\mathcal{R}(\phi^t)}).$$

Let $x \in \mathcal{R}(\phi^t)$. By Theorem 2.7.18 (iv), for each positive integer n there exists a sequence

$$x = x_{n,0}, x_{n,1}, x_{n,2}, \dots, x_{n,k_n} = x$$

such that

$$d(\phi^1(x_{n,i}), x_{n,i+1}) < \frac{1}{n}$$

for $i = 0, 1, 2, \dots, k_n - 1$. Define

$$C_n = \{x_{n,0}, \dots, x_{n,k_n}\}.$$

For each positive integer n , the set C_n is a compact subset of X .

Let \mathcal{H} be the collection of nonempty compact subsets of X . Since X is compact, \mathcal{H} coincides with the collection of nonempty closed bounded subsets of X . Define $D : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ by

$$D(A, B) = \sup\{d(a, B), d(b, A) \mid a \in A, b \in B\}.$$

By Proposition C.0.10 the function D is a metric on \mathcal{H} . We call this metric the Hausdorff metric. By Proposition C.0.15, the space \mathcal{H} is compact with respect to D . Thus, the sequence $\{C_n\}$ in \mathcal{H} contains a subsequence $\{C_{n_j}\}$ which converges to some $C \in \mathcal{H}$. By reindexing this subsequence, we may assume, without loss of generality, that $\{C_n\}$ converges to C . For each $p \in C$ and $\epsilon > 0$ we shall obtain a sequence

$$p = p_0, p_1, \dots, p_{m-1}, p_m = p$$

in C such that

$$d(\phi^1(p_i), p_{i+1}) < \epsilon$$

for $i = 0, 1, 2, \dots, m - 1$. so that every point in C is chain recurrent.

Let $p \in C$ and $\epsilon > 0$. The compactness of X and continuity of ϕ^1 imply that ϕ^1 is uniformly continuous. Thus, there exists $\delta \in (0, \epsilon/3)$ such that if $y_1, y_2 \in X$ and $d(y_1, y_2) < \delta$, then

$$d(\phi^1(y_1), \phi^1(y_2)) < \frac{\epsilon}{3}.$$

Since $\{C_n\}$ converges to C there exists a positive integer n such that

$$\frac{1}{n} < \frac{\epsilon}{3},$$

and

$$D(C_n, C) < \delta.$$

So, there exists $x_{n,j} \in C_n$ such that

$$d(x_{n,j}, p) < \delta.$$

Cyclically permute the elements of C_n by defining

$$y_i = \begin{cases} x_{n,i+j} & \text{if } i = 0, 1, 2, \dots, k_n - j - 1, \\ x_{n,i+j-k_n} & \text{if } i = k_n - j, \dots, k_n. \end{cases}$$

Since $D(C_n, C) < \delta$, there exists $p_j \in C$ such that

$$d(y_j, p_j) < \delta$$

for each $j = 1, \dots, k_n - 1$. Let

$$p_0 = p_{k_n} = p.$$

We will show that $p = p_0, p_1, \dots, p_{k_n} = p$ is an ϵ -chain in C with respect to ϕ^1 .

Let $j \in \{0, 1, 2, \dots, k_n - 1\}$. By the uniform continuity of ϕ^1 we obtain

$$d(\phi^1(p_j), \phi^1(y_j)) < \frac{\epsilon}{3}.$$

From the definition of y_1, \dots, y_{k_n} ,

$$d(\phi^1(y_j), y_{j+1}) < \delta < \frac{\epsilon}{3}.$$

The choice of p_{j+1} guarantees that

$$d(y_{j+1}, p_{j+1}) < \delta < \frac{\epsilon}{3}.$$

The triangle inequality implies that

$$\begin{aligned} d(\phi^1(p_j), p_{j+1}) &\leq d(\phi^1(p_j), \phi^1(y_j)) + d(\phi^1(y_j), y_{j+1}) + d(y_{j+1}, p_{j+1}) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Consequently, $p_0, p_1, p_2, \dots, p_{k_n}$ is an ϵ -chain in C with respect to ϕ^1 from p to itself. Since p is an arbitrary element of C , Theorem 2.7.18 (iv) implies that

$$C \subseteq \mathcal{R}(\phi^t).$$

Since $x \in C_n$ for each positive integer n , we have $x \in C$. Thus, for each $\epsilon > 0$ and $T > 0$ there exists an (ϵ, T) -chain in $\mathcal{R}(\phi^t)$ from x to itself. Hence,

$$x \in \mathcal{R}(\phi^t|_{\mathcal{R}(\phi^t)}).$$

Therefore,

$$\mathcal{R}(\phi^t|_{\mathcal{R}(\phi^t)}) = \mathcal{R}(\phi^t). \quad \square$$

Example 2.7.22. Let ψ^t be the arrested logistic rotation-dilation on the closed unit disk \mathbb{D} in the complex plane. Example 2.7.9 shows that

$$\mathcal{R}(\psi^t) = \partial\mathbb{D} \cup \{0\}.$$

By restricting ψ^t to its chain recurrent set, we obtain the arrested rotation on S^1 described in Example 2.7.6 together with a fixed point at $\{0\}$. Therefore,

$$\mathcal{R}(\psi^t|_{\mathcal{R}(\psi^t)}) = \partial\mathbb{D} \cup \{0\} = \mathcal{R}(\psi^t).$$

In contrast,

$$\Omega(\psi^t|_{\Omega(\psi^t)}) = \{-1, 0, 1\} \neq \Omega(\psi^t)$$

by Example 2.5.16.

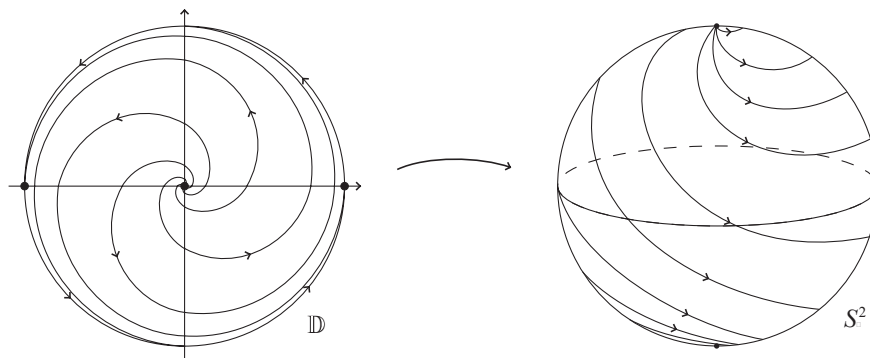


Figure 4.15. The gradient-like quotient of the arrested logistic rotation-dilation

Proposition 4.8.13. *Every strongly gradient-like flow on a compact metric space is gradient-like.*

Proof. Let ϕ^t be a strongly gradient-like flow on a compact metric space. According to the Fundamental Theorem of Dynamical Systems there exists a complete Lyapunov function \mathcal{L} for ϕ^t . By Proposition 4.6.14,

$$\mathcal{R}(\phi^t) = \text{Fix}(\phi^t).$$

Consequently, \mathcal{L} is a strong Lyapunov function with respect to ϕ^t . Therefore, ϕ^t is gradient-like. \square

4.9. Exercises

- (1) For each of the following potential functions V sketch representative orbits of the flow corresponding to the differential equation

$$\dot{x} = -\nabla V(x)$$

on \mathbb{R}^2 .

- (a) $V(x, y) = -x^2 + y^2$,
 (b) $V(x, y) = \frac{-2x^2 + x^4 + 2y^2}{4}$.
 (c) $V(x, y) = \frac{2x^2 - x^4 + 2y^2}{4}$.
 (d) $V(x, y) = \frac{3x^2 - 2x^3 + 3y^2}{6}$.

- (2) Prove that every flow on \mathbb{R} is a gradient flow.