Preface

This is a book about C*-algebras, various types of approximation, and a few of the surprising applications that have been recently discovered. In short, we will study *approximation theory in the context of operator algebras*.

Approximation is ubiquitous in mathematics; when the object of interest cannot be studied directly, we approximate by tractable relatives and pass to a limit. In our context this is particularly important because C*-algebras are (almost always) infinite dimensional and we can say precious little without the help of approximation theory. Moreover, most concrete examples enjoy some sort of finite-dimensional localization; hence it is very important to exploit these features to the fullest. Indeed, over the years approximation theory has been at the heart of many of the deepest, most important results: Murray and von Neumann's uniqueness theorem for the hyperfinite II_1-factor and Connes's remarkable extension to the injective realm; Haagerup's discovery that reduced free group C*-algebras have the metric approximation property; Higson and Kasparov's resolution of the Baum-Connes conjecture for Haagerup's groups; Popa's work on subfactors and Cartan subalgebras; Voiculescu's whole free entropy industry, which is defined via approximation; Elliott's classification program, which collapses without approximate intertwining arguments; and one can't forget the influential work of Choi, Effros, and Kirchberg on nuclear and exact C*-algebras.

Approximation is everywhere; it is powerful, important, the backbone of countless breakthroughs. We intend to celebrate it. This subject is a functional analyst's delight, a beautiful mixture of hard and soft analysis, pure joy for the technically inclined. Our wheat may be other texts' chaff, but we see no reason to hide our infatuation with the grace and power which is approximation theory. We don't mean to suggest that mastering technicalities is the point of operator algebras – it isn't. We simply hope to elevate them from a necessary ally to a revered friend. Also, one shouldn't think these pages are a one-stop shopping place for all aspects of approximation theory – they aren't. The main focus is nuclearity and exactness, with several related concepts and a few applications thrown in for good measure.

From the outset of this project, we were torn between writing user-friendly notes which students would appreciate – many papers in this subject are notoriously difficult to read – or sticking to an expert-oriented, research-monograph level of exposition. In the end, we decided to split the difference. Part 1 of these notes is written with the beginner in mind, someone who just finished a first course in operator algebras (C^*- and W^*-algebras). We wanted the basic theory to be accessible to students working on their own; hence Chapters 2 - 10 have a lot of detail and proceed at a rather slow pace.[1] Chapters 11 - 17 and all of the appendices are written at a higher level, something closer to that found in the literature.

Here is a synopsis of the contents.

Part 1: Basic Theory

The primary objective here is an almost-comprehensive treatment of nuclearity and exactness.[2] Playing the revisionist-historian role, we define these classes in terms of finite-dimensional approximation properties and later demonstrate the tensor product characterizations. We also study several related ideas which contribute to, and benefit from, nuclearity and exactness.

The first chapter is just a collection of results that we need for later purposes. We often utilize the interplay between C^*-algebras and von Neumann algebras; hence this chapter reviews a number of "basic" facts on both sides. (Some are not so basic and others are so classical that many students never learn them.)

Chapter 2 contains definitions, simple exercises designed to get the reader warmed up, and a few basic examples (AF algebras, C^*-algebras of amenable groups, type I algebras).

[1]Except for a few sections in Chapters 4 and 5, where much more is demanded of the reader. This was necessary to keep the book to a reasonable length.

[2]The most egregious omission is probably Kirchberg's $\mathcal{O}_2$-embedding theorem for separable exact C^*-algebras. We felt there were not enough general (i.e., outside of the classification program) applications to warrant including the difficult proof. The paper [**107**] is readily available and has a self-contained, well-written proof. Rørdam's book [**168**] has a nearly complete proof and a forthcoming book of Kirchberg and Wassermann will certainly contain all the details. Another significant omission is a discussion of general locally compact groups; we stick to the discrete case. The ideas are adequately exposed in this setting and we don't think beginners benefit from more generality.

In Chapter 3 we give a long introduction to the theory of C*-tensor products. Most of the chapter is devoted to definitions and a thorough discussion of the subtleties which make C*-tensor products both interesting and hazardous. However, the last two sections contain important theorems, taking us back to the original definitions of nuclearity and exactness.

In the next two chapters we show that many natural examples of C*-algebras admit some sort of finite-dimensional approximants. In Chapter 4 we discuss a number of general constructions which one finds in the literature (crossed products by amenable actions, free products, etc.). Chapter 5 is an introduction to exact discrete groups and some related topics which are relevant to noncommutative geometry. Both of these chapters contain redundancies in the sense that we start with special cases and gradually tack on generality. The Bourbakians may protest, but we feel this approach is pedagogically superior.

Someone who works through Chapters 2 - 5 will have a pretty good feel for most aspects of nuclearity and exactness. There is, however, one important permanence property which requires much more work: Both nuclearity and exactness pass to quotients. In some sense, the next four chapters are required to prove these fundamental facts. This doesn't mean we've taken the most direct route, however. On the contrary, we take our sweet time and present a number of related approximation properties which are of independent interest and play crucial roles in the quotient results.

Chapter 6 contains the basics of amenable tracial states. These "invariant means" on C*-algebras can be characterized in terms of approximation or tensor products. They also yield a simple proof of the deep fact that every finite injective von Neumann algebra is semidiscrete.

In Chapter 7 we study quasidiagonal C*-algebras. They are also defined via approximation, but the flavor is quite different from nuclearity or exactness. Most of the basic theory is presented, including Voiculescu's homotopy invariance theorem, though much of it isn't necessary for applications to exactness. (For this we only need Dadarlat's approximation theorem for exact quasidiagonal C*-algebras; see Section 7.5.)

This leads naturally to Chapter 8: AF Embeddability. For applications, the most important fact is that every exact C*-algebra is a subquotient of an AF C*-algebra. We give the proof in the beginning of the chapter so those only interested in exactness can quickly proceed forward. For others, we have included the homotopy invariance theorem for AF embeddability and a short survey of related results.

In Chapter 9 we put all the pieces together, completing the basic-theory portion of the book. The main result gives two more tensor product characterizations of exactness, from which corollaries flow: Exact C*-algebras are locally reflexive (another important finite-dimensional approximation property), nuclearity and exactness pass to quotients, and a few others.

Finally, we conclude Part 1 with a chapter summarizing permanence properties. This is just for ease of reference, in case one forgets whether or not extensions of exact C*-algebras are exact.

Part 2: Special Topics

The next four chapters are a disjoint collection of related concepts. They are logically independent and meant to spark the reader's interest – much more could be written about any one of them.

Chapter 11 is primarily about simple quasidiagonal C*-algebras. Motivated by Elliott's classification program, we spend time discussing the generalized inductive limit approach (of Blackadar and Kirchberg) to nuclear quasidiagonal C*-algebras. We also prove a theorem of Popa, showing that quasidiagonality is often detectable internally. Finally, we present Connes's amazing uniqueness theorem for the injective II_1-factor, exploiting Popa's techniques.

Chapter 12 introduces some properties of discrete groups that have been extremely important over the years. First, we discuss Kazhdan's property (T), prove that $\mathrm{SL}(3,\mathbb{Z})$ has this property, and demonstrate Connes's result that II_1-factors with property (T) have few outer automorphisms. Next, we define Haagerup's approximation property – the antithesis of property (T) – and prove that a group which acts properly on a tree (e.g., a free group) enjoys this property. The latter sections of this chapter discuss related approximation properties and their interrelations.

Chapter 13 – on Lance's weak expectation property and the local lifting property for C*-algebras – gives a streamlined approach to some of Kirchberg's influential work around these ideas. We also reproduce Junge and Pisier's theorem on the tensor product of $\mathbb{B}(\ell^2)$ with itself.

Part 2 concludes with Chapter 14: Weakly Exact von Neumann Algebras. This concept was first suggested by Kirchberg; the theorems and proofs are similar to C*-results found in Part 1 of the book. It is not yet clear if this theory will bear fruit like its C*-predecessor, but it seemed like a natural topic to include.

Part 3: Applications

The last three chapters, comprising Part 3, are devoted to applications. We hope to convince you that approximation properties are useful; seemingly unrelated problems will crack wide open when pried with the right technical tool.

Chapter 15 contains solidity and prime factorization results for certain group von Neumann algebras. The solidity results generalize one of the celebrated achievements of free probability theory, while the prime factorization results are natural analogues of some spectacular recent work in dynamical systems. Both depend in a crucial way on some of the C^*-ideas and tensor product techniques contained in Part 1.

Chapter 16 resolves a problem in single operator theory which, at present, appears to require exact quasidiagonal C^*-algebras. We need the fact that exactness implies local reflexivity – one of the deepest, most difficult theorems in C^*-algebras – and it is hard to imagine an operator-theoretic proof which could circumvent this fact.

The final chapter is based on some work of Simon Wassermann. He observed that property (T) groups together with quasidiagonal ideas lead to natural examples for which the Brown-Douglas-Fillmore semigroup is not a group. Approximation properties, or the lack thereof, are at the heart of the argument.

So, that's what you'll find in this book. To the student: We hope these notes are reasonably accessible and a helpful introduction to an area of active research. To the veteran: We hope this will be a useful reference for C^*-approximation theory. As mentioned earlier, Kirchberg and Wassermann are working on an exact C^*-algebra text and there will certainly be overlap between these notes and theirs. However, the emphasis and selection of topics will likely differ; with any luck, the union of our books will satisfy the needs of most. We should also mention that a web page correcting this book's inevitable errors can be found at www.ams.org/bookpages/gsm-88.

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