

Fundamental Facts

We stated in the preface that these notes should be accessible to anyone with a “first course” in operator algebras under their belt. An excellent first course would consist of the material contained in [127], for example, and we assume familiarity with that book. However, we’ll need numerous other facts that may or may not have made it into *your* first course; the purpose of this chapter is to summarize the requisite results.

It goes without saying that advanced students and seasoned researchers should skip this chapter, referring back if necessary. Indeed, the only things required before starting Chapter 2 are basic properties of completely positive maps and Arveson’s Extension Theorem (Sections 1.5 and 1.6). We advise the novice to nail down this material and then to jump ahead to Chapter 2 – mathematics books need not be read linearly.¹

1.1. Notation

We use \mathcal{H} , \mathcal{K} and \mathcal{L} to denote generic complex Hilbert spaces. The n -dimensional Hilbert space is usually denoted ℓ_n^2 , while ℓ^2 is the separable, infinite-dimensional Hilbert space. Here $\mathbb{B}(\mathcal{H})$ is the algebra of bounded linear operators on \mathcal{H} , $\mathbb{K}(\mathcal{H})$ denotes the compacts and $Q(\mathcal{H}) = \mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})$ is the Calkin algebra. (An abstract copy of the compacts will be denoted by \mathbb{K} .) Also, Tr is the canonical (typically unbounded, densely defined) trace on $\mathbb{B}(\mathcal{H})$. We let \mathcal{S}_1 (resp. \mathcal{S}_2) denote the trace class (resp. Hilbert-Schmidt) operators, with canonical norm $\|T\|_1 = \text{Tr}(|T|)$ (resp. $\|T\|_2 = \sqrt{\text{Tr}(|T|^2)}$).

¹If you feel obliged to go carefully through this chapter, be advised that tensor products are required in a few places. Thus, first reading the beginning of Chapter 3 might be necessary. How’s that for nonlinear?

Here $\mathbb{M}_n(\mathbb{C})$ is the $n \times n$ complex matrices and tr is its unique tracial state. A collection of $n \times n$ matrices $\{e_{i,j}\}_{1 \leq i,j \leq n}$ is called a *system of matrix units* if $e_{i,j}e_{s,t} = \delta_{j,s}e_{i,t}$. It is often convenient to index our matrices over a finite set F , in which case we write $\mathbb{M}_F(\mathbb{C}) (= \mathbb{B}(\ell^2(F)))$ and let $\{e_{p,q}\}_{p,q \in F}$ denote the canonical matrix units.

We reserve A, B, C and D for C^* -algebras while M and N will typically denote von Neumann algebras. We let A_{sa} be the self-adjoint elements, A_1 the closed unit ball and A_+ the positive elements in A . The symbols E and F will denote operator systems (or operator spaces). We usually use I and J for ideals in C^* -algebras (e.g. $I \triangleleft A$), though they are occasionally index sets too. All ideals are assumed closed and two-sided.

The set of states on A – positive linear functionals of norm one – will be denoted $S(A)$. If $\varphi \in S(A)$, we let $L^2(A, \varphi)$ be the GNS (Gelfand-Naimark-Segal) Hilbert space and $\pi_\varphi: A \rightarrow \mathbb{B}(L^2(A, \varphi))$ be the GNS representation. For an element $a \in A$, we let $\hat{a} \in L^2(A, \varphi)$ denote its natural image.

1.2. C^* -algebras

Quasicontral approximate units. Quasicontral approximate units are an indispensable tool. See [53, Theorem I.9.16] for a proof of the following fact.

Theorem 1.2.1. *Let $I \triangleleft A$ be an ideal. Then I has an approximate unit $\{e_i\} \subset I$ such that $\|e_i a - a e_i\| \rightarrow 0$, as $i \rightarrow \infty$, for all $a \in A$. In fact, if $\{f_k\} \subset I$ is any approximate unit for I , then a quasicontral approximate unit can always be extracted from its convex hull.*

We don't need it too many times, but it is worth mentioning that quasicontral approximate units allow a particular type of approximate decomposition.

Proposition 1.2.2. *Let A be unital and $\{e_i\} \subset I \triangleleft A$ be a quasicontral approximate unit. For every pair $a, b \in A$ such that $a - b \in I$ we have*

$$\|a - \left((1 - e_i)^{\frac{1}{2}} b (1 - e_i)^{\frac{1}{2}} + e_i^{\frac{1}{2}} a e_i^{\frac{1}{2}} \right)\| \rightarrow 0.$$

Proof. First notice that for every $x \in A$ and polynomial p we have $\|p(e_i)x - xp(e_i)\| \rightarrow 0$ (by some standard estimates). Since continuous functions can be approximated by polynomials, it follows that $\|e_i^{\frac{1}{2}}x - xe_i^{\frac{1}{2}}\| \rightarrow 0$ and $\|(1 - e_i)^{\frac{1}{2}}x - x(1 - e_i)^{\frac{1}{2}}\| \rightarrow 0$.

Next, observe that

$$\|(1 - e_i)^{\frac{1}{2}}a(1 - e_i)^{\frac{1}{2}} - (1 - e_i)^{\frac{1}{2}}b(1 - e_i)^{\frac{1}{2}}\| = \|(1 - e_i)^{\frac{1}{2}}(a - b)(1 - e_i)^{\frac{1}{2}}\| \rightarrow 0$$

since $\lim_i \|(1 - e_i)x\|$ is equal to the norm of $x + I \in A/I$. Putting these observations together, we obtain asymptotic approximations

$$(1 - e_i)^{\frac{1}{2}} b (1 - e_i)^{\frac{1}{2}} + e_i^{\frac{1}{2}} a e_i^{\frac{1}{2}} \approx (1 - e_i)^{\frac{1}{2}} a (1 - e_i)^{\frac{1}{2}} + e_i^{\frac{1}{2}} a e_i^{\frac{1}{2}} \approx a(1 - e_i) + a e_i = a$$

and the proof is complete. \square

Uniqueness of GNS representations. Hopefully you already know the uniqueness statement for GNS representations, but here is a technical variation (with exactly the same proof).

Proposition 1.2.3. *Let $\varphi \in S(A)$ be a state on A , $A_0 \subset A$ be a norm dense $*$ -subalgebra and $\rho: A_0 \rightarrow \mathbb{B}(\mathcal{H})$ be a $*$ -homomorphism with the property that there exists a unit vector $v \in \mathcal{H}$ such that $A_0 v$ is dense in \mathcal{H} and $\varphi(x) = \langle \rho(x)v, v \rangle$ for all $x \in A_0$. Then ρ extends to a representation of A (which is unitarily equivalent to π_φ).*

Proof. One defines a linear map $U: \hat{A}_0 \rightarrow A_0 v$ by declaring $U\hat{a} = \rho(a)v$. Check that this is well-defined and isometric from a dense subspace of $L^2(A, \varphi)$ to a dense subspace of \mathcal{H} ; thus it extends uniquely to a unitary. The extension of ρ is obtained by conjugating π_φ by this unitary. \square

1.3. Von Neumann algebras

Though these notes are primarily concerned with C^* -algebras, we will need von Neumann algebras from time to time. The C^* -purists should be forewarned that we intend to delve into W^* -theory whenever possible (even when it isn't absolutely necessary).

Structure of von Neumann algebras. The basic decomposition theory of von Neumann algebras will be important. We won't give any proper definitions, but thanks to well-known theorems our approach is legal (i.e., our definitions are equivalent to the "real" definitions; see [183, Definition V.1.17]).

We let $\prod_j B_j$ denote the ℓ^∞ -direct sum of C^* -algebras $\{B_j\}_{j \in J}$, i.e., the set of tuples $(b_j)_{j \in J}$ such that $b_j \in B_j$ and $\sup_j \|b_j\| < \infty$.

Definition 1.3.1. The von Neumann algebra M is *type I* if it is isomorphic to

$$\prod_{j \in J} \mathcal{A}_j \bar{\otimes} \mathbb{B}(\mathcal{H}_j)$$

for some set J of cardinal numbers, where each \mathcal{A}_j is an abelian von Neumann algebra and \mathcal{H}_j is a Hilbert space of dimension j .²

²If you haven't seen it, the definition of von Neumann tensor products is given in Remark 3.3.5.

Definition 1.3.2. The von Neumann algebra M is *type* II_1 if it has no summand of type I and there exists a separating family of normal tracial states (i.e., for every $0 < x \in M$ there exists a normal tracial state τ on M such that $\tau(x) > 0$).

Roughly speaking, the next type is just an increasing union of II_1 corners.

Definition 1.3.3. The von Neumann algebra M is *type* II_∞ if M has no summand of type I or II_1 but there exists an increasing net of projections $\{p_i\}_{i \in I} \subset M$, converging strongly to 1_M , such that $p_i M p_i$ is of type II_1 for every $i \in I$.

Finally, a von Neumann algebra is said to be of *type* III if it has no summand of any of the types defined above. The following decomposition theorem is fundamental ([183, Theorem V.1.19]).

Theorem 1.3.4. *Every von Neumann algebra M has a unique decomposition*

$$M \cong M_I \oplus M_{\text{II}_1} \oplus M_{\text{II}_\infty} \oplus M_{\text{III}}$$

as a direct sum of algebras of type I, II_1 , II_∞ and III (some of these summands may be 0).

Preduals and Sakai's Theorem. Recall that $\mathbb{B}(\mathcal{H})$ is canonically isomorphic to the dual Banach space of the trace class operators $\mathcal{S}_1 \subset \mathbb{B}(\mathcal{H})$. Hence every von Neumann algebra $M \subset \mathbb{B}(\mathcal{H})$ is also a dual Banach space (namely, the dual of the quotient of \mathcal{S}_1 by the pre-annihilator of M). A fundamental result of Sakai (see [183, Corollary III.3.9]) implies that the induced weak-* topology is canonical, i.e., independent of the normal representation $M \subset \mathbb{B}(\mathcal{H})$. (Recall that a map $\varphi: M \rightarrow N$ of von Neumann algebras is *normal* if $\varphi(\sup x_i) = \sup \varphi(x_i)$ for all norm bounded, monotone increasing nets of self-adjoint elements $\{x_i\} \subset M_{sa}$.)

Theorem 1.3.5. *For a von Neumann algebra M , let M_* be the Banach space of normal linear functionals on M . Then M is (isometrically) isomorphic to the dual of M_* . Moreover, M_* is the unique predual in the sense that if X is a Banach space with the property that M is isometrically isomorphic to X^* , then X is isometrically isomorphic to M_* .*

Definition 1.3.6. The canonical weak-* topology on M (coming from M_*) is called the *ultraweak topology*.

Point-ultraweak limits. Let M be a von Neumann algebra and M_* be its predual. For a Banach space X , let $\mathbb{B}(X, M)$ be the bounded linear maps from X to M . It turns out that $\mathbb{B}(X, M)$ also has a predual. Let $\mathbb{B}(X, M)_* \subset \mathbb{B}(X, M)^*$ be the closed linear span of the linear functionals

$x \otimes \xi \in \mathbb{B}(X, M)^*$, where $x \in X$, $\xi \in M_*$ and $x \otimes \xi$ is defined by $x \otimes \xi(T) = \xi(T(x))$. Then $\mathbb{B}(X, M)$ is isometrically isomorphic to the dual of $\mathbb{B}(X, M)_*$, whence it receives a weak-* topology. On bounded sets, this topology agrees with the *point-ultraweak* (aka *point- σ -weak*) topology. That is, for a bounded net convergence works as follows:

$$T_\lambda \rightarrow T \iff \xi(T_\lambda(x)) \rightarrow \xi(T(x)), \forall x \in X, \forall \xi \in M_*.$$

Thus the unit ball of $\mathbb{B}(X, M)$ is compact, by Alaoglu's Theorem, in the point-ultraweak topology. Hence we obtain the following theorem (cf. Theorem A.8).

Theorem 1.3.7. *Let X be a Banach space, M be a von Neumann algebra and $T_\lambda: X \rightarrow M$ be a bounded net of linear maps. Then $\{T_\lambda\}_{\lambda \in \Lambda}$ has a cluster point in the point-ultraweak topology.*

Representation theory. In contrast to the C^* -case, representation theory of von Neumann algebras is almost trivial: one can cut by a projection in the commutant and that's about it. Of course, the precise statement is slightly more complicated (see [183, Theorem IV.5.5]).

Theorem 1.3.8. *Let $M \subset \mathbb{B}(\mathcal{H})$ be a von Neumann algebra and $\pi: M \rightarrow \mathbb{B}(\mathcal{K})$ be a normal representation. There exist a Hilbert space $\tilde{\mathcal{K}}$ and a projection $P_\pi \in \mathbb{B}(\mathcal{H} \otimes \tilde{\mathcal{K}})$ such that P_π commutes with $M \otimes 1 \subset \mathbb{B}(\mathcal{H} \otimes \tilde{\mathcal{K}})$ and π is unitarily equivalent to the representation $M \rightarrow P_\pi \mathbb{B}(\mathcal{H} \otimes \tilde{\mathcal{K}}) P_\pi$, $m \mapsto P_\pi(m \otimes 1)$.*

1.4. Double duals

The Banach space double dual of a C^* -algebra A is a wild beast; it should be approached with humility, even trepidity. Whatever it takes, though, one must become acquainted with A^{**} as it's an extremely useful universe in which to work. See [183, Section III.2] for more.

The enveloping von Neumann algebra. Recall that the *universal representation* of a C^* -algebra A is

$$\pi_u = \bigoplus_{\varphi \in S(A)} \pi_\varphi: A \rightarrow \mathbb{B} \left(\bigoplus_{\varphi \in S(A)} L^2(A, \varphi) \right) = \mathbb{B}(\mathcal{H}_u).$$

By definition, the *enveloping von Neumann algebra of A* is the double commutant $\pi_u(A)''$. Thanks to the next result, we need not distinguish between the double dual and the enveloping von Neumann algebra; we'll use A^{**} to denote either throughout this book.

Theorem 1.4.1. *The enveloping von Neumann algebra of A is isometrically isomorphic to the double dual A^{**} . Hence the ultraweak topology on $\pi_u(A)'' (= A^{**})$ restricts to the weak topology on A (by Sakai's Theorem).*

Here is an often used consequence: If $a_i, a \in A$ and $a_i \rightarrow a$ in the ultraweak topology, then a belongs to the *norm* closure of the convex hull of the a_i 's (thanks to the Hahn-Banach Theorem).

Central covers. Since every representation can be decomposed as a direct sum of cyclic representations (i.e., GNS representations), it is easily seen that A^{**} enjoys the following *universal property*: for each nondegenerate representation $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$ there exists a unique normal extension $\tilde{\pi}: A^{**} \rightarrow \mathbb{B}(\mathcal{H})$ such that $\tilde{\pi}|_A = \pi$ and $\tilde{\pi}(A^{**}) = \pi(A)''$. The kernel of $\tilde{\pi}$ is weakly closed (by normality); hence it's a von Neumann algebra. As such, it has a unit e_π which is a central projection in A^{**} .

Definition 1.4.2. Let $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$ be a nondegenerate representation. The *central cover* of π , denoted $c(\pi)$, is defined to be $e_\pi^\perp = 1_{A^{**}} - e_\pi$.

The following isomorphisms are immediate from the definition:

$$c(\pi)A^{**} = c(\pi)A^{**}c(\pi) \cong \tilde{\pi}(A^{**}) = \pi(A)''.$$

We only need them on a few occasions, but here are some necessary facts.

Proposition 1.4.3. *If π_1 and π_2 are irreducible representations, then the following are equivalent:*

- (1) $c(\pi_1)c(\pi_2) \neq 0$;
- (2) $c(\pi_1) = c(\pi_2)$;
- (3) π_1 and π_2 are unitarily equivalent.

Proposition 1.4.4. *For two representations $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$ and $\rho: A \rightarrow \mathbb{B}(\mathcal{K})$, the following are equivalent:*

- (1) $c(\pi)c(\rho) = 0$;
- (2) $(\pi \oplus \rho)(A)'' = \pi(A)'' \oplus \rho(A)''$.

Two representations $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$ and $\rho: A \rightarrow \mathbb{B}(\mathcal{K})$ are said to be *quasi-equivalent* if there exists an isomorphism $\theta: \pi(A)'' \rightarrow \rho(A)''$ such that $\theta(\pi(a)) = \rho(a)$ for all $a \in A$. Of course, unitarily equivalent representations are quasi-equivalent in this sense, but the converse is false. (Any representation of $\pi(A)''$ which is not unitarily equivalent to the original – for example, one could modify the commutant – will yield a quasi-equivalent representation of A which is not unitarily equivalent to π .) Here are two simple facts.

Proposition 1.4.5. *The representations π and ρ are quasi-equivalent if and only if $c(\pi) = c(\rho)$. Since every central projection in A^{**} defines a representation of A , there is a one-to-one correspondence between central projections $p \in \mathcal{Z}(A^{**})$ and quasi-equivalence classes of representations.*

Proposition 1.4.6. *The representation π is quasi-equivalent to a subrepresentation of ρ if and only if $c(\pi) \leq c(\rho)$.*

Lusin's Theorem, excision and Glimm's Lemma. The (difficult) proof of the following noncommutative extension of Lusin's Theorem can be found in [183] (II.4.15) or [142] (2.7.3).

Theorem 1.4.7 (Lusin's Theorem). *Let $A \subset \mathbb{B}(\mathcal{H})$ be a nondegenerate C^* -algebra with $A'' = M$. For every finite set of vectors $\mathfrak{F} \subset \mathcal{H}$, $\varepsilon > 0$, projection $p_0 \in M$ and self-adjoint $y \in M$, there exist a self-adjoint $x \in A$ and a projection $p \in M$ such that $p \leq p_0$, $\|p(h) - p_0(h)\| < \varepsilon$ for all $h \in \mathfrak{F}$, $\|x\| \leq \min\{2\|yp_0\|, \|y\|\} + \varepsilon$ and $xp = yp$.*

We will need a slight sharpening of Kadison's Transitivity Theorem.

Corollary 1.4.8 (Strong transitivity). *Let $A^{**} \subset \mathbb{B}(\mathcal{H}_u)$ be the universal representation and $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$ be an irreducible representation with normal extension $\tilde{\pi}$ to A^{**} . For each self-adjoint $a \in A^{**}$ and finite-rank projection $Q \in \mathbb{B}(\mathcal{H})$, we can find a self-adjoint net $(c_i)_{i \in I} \subset A$ such that $c_i \rightarrow a$ in the strong operator topology, $\|c_i\| \leq \|a\| + 1$ and $\tilde{\pi}(a)Q = \pi(c_i)Q$, for all $i \in I$. If $0 \leq a \leq Q$ (in the decomposition $A^{**} = \mathbb{B}(\mathcal{H}) \oplus (1 - c(\pi))A^{**}$), then the c_i 's can be taken positive.*

Proof. Let $\mathfrak{F} \subset \mathcal{H}_u$ be any finite set of vectors and $\tilde{Q} \in \mathbb{B}(\mathcal{H})$ be any finite-rank projection dominating Q . Applying Lusin's Theorem to $a, \tilde{Q} \oplus (1 - c(\pi))$ and any $\varepsilon > 0$, we can find $c \in A$ and a projection $P \in A^{**}$ such that

- (1) $P \leq \tilde{Q} \oplus (1 - c(\pi))$;
- (2) $\|P(v) - \tilde{Q} \oplus (1 - c(\pi))(v)\| < \varepsilon$, for all $v \in \mathfrak{F}$;
- (3) $aP = cP$;
- (4) $\|c\| \leq \|a\| + 1$.

Writing $P = c(\pi)P \oplus (1 - c(\pi))P$, we claim that it is no loss of generality to assume $Q \leq c(\pi)P = \tilde{Q}$; it is easily seen that this implies the lemma. The fact that $P \leq \tilde{Q} \oplus (1 - c(\pi))$ implies $c(\pi)P \leq \tilde{Q}$. On the other hand, \mathfrak{F} can be any finite set of vectors – hence we could throw in a basis for the range of \tilde{Q} . But then for small ε we would have $\|c(\pi)P - \tilde{Q}\| < 1$, which implies $c(\pi)P = \tilde{Q}$ as desired.

Now, suppose that $0 \leq a \leq Q$. Applying the first part of the proof to $a^{1/2}$, we find a self-adjoint net (b_i) such that $b_i \rightarrow a^{1/2}$, $\|b_i\| \leq (\|a\| + 1)^{1/2}$

and $\tilde{\pi}(a^{1/2})Q = \pi(b_i)Q$, for all $i \in I$. Since a and Q commute, one has

$$\pi(b_i^2)Q = \pi(b_i)(\tilde{\pi}(a^{1/2})Q) = (\pi(b_i)Q)\tilde{\pi}(a^{1/2}) = \tilde{\pi}(a)Q.$$

Hence $c_i = b_i^2$ satisfy the required conditions. \square

Lemma 1.4.9. *Let A be a C^* -algebra, φ be a pure state and $L = \{a \in A : \varphi(a^*a) = 0\}$ be the associated left ideal. Then we have $\ker \varphi = L + L^*$.*

Proof. Let $x \in \ker \varphi$ be nonzero. Let (π, \mathcal{H}, ξ) be the GNS triplet and \mathcal{K} be the 2-dimensional subspace spanned by ξ and $\pi(x)\xi$. Since $\xi \perp \pi(x)\xi$, Kadison's Transitivity Theorem provides us with a positive element $b \in A$ such that $\pi(b)\xi = \xi$ and $\pi(b)(\pi(x)\xi) = 0$. It follows that $bx \in L$ and $(1-b)x \in L^*$. \square

Here is half of the Akemann-Anderson-Pedersen Excision Theorem (see [1] for the other half).

Theorem 1.4.10 (Excision). *Let A be a C^* -algebra and φ be a pure state. There exists a net $(e_i) \subset A$ such that $0 \leq e_i \leq 1$, $\varphi(e_i) = 1$ and $\lim_i \|e_i a e_i - \varphi(a)e_i^2\| = 0$ for every $a \in A$.*

Proof. We first assume that A is unital. Let L be the left ideal associated to φ and (c_i) be a right approximate unit for L (i.e., (c_i) is an approximate unit for the hereditary subalgebra $L \cap L^*$ and for every $a \in L$ we have $\|a - ac_i\| \rightarrow 0$). Let $e_i = 1 - c_i$. Since $a - \varphi(a) \in \ker \varphi = L + L^*$, we have $\lim \|e_i(a - \varphi(a))e_i\| = 0$ and $\varphi(e_i) = 1$.

Now suppose that A is nonunital and take e_i as above for \tilde{A} . Let (b_j) be a quasicentral approximate unit for A such that $\varphi(b_j) = 1$. (Existence follows from Kadison's Transitivity Theorem.) Then, $b_j e_i b_j$ does the job. \square

Here is a nonstandard proof of a fundamental fact.

Lemma 1.4.11 (Glimm's Lemma). *Let $A \subset \mathbb{B}(\mathcal{H})$ be a separable C^* -algebra containing no nonzero compact operators on \mathcal{H} . If φ is a state on A , then there exist orthonormal vectors (ξ_n) such that $\omega_{\xi_n}(a) \rightarrow \varphi(a)$ for all $a \in A$, where $\omega_{\xi_n}(T) = \langle T\xi_n, \xi_n \rangle$.*

Proof. ³ Let $\mathfrak{F} \subset A$ be a finite subset of norm-one elements, $\varepsilon > 0$, $\mathcal{K}_0 \subset \mathcal{H}$ be finite dimensional and $P_{\mathcal{K}_0}$ be the orthogonal projection onto \mathcal{K}_0 . It suffices to show the existence of a unit vector $\xi \in \mathcal{K}_0^\perp$ such that $|\omega_\xi(a) - \varphi(a)| < 6\varepsilon$ for all $a \in \mathfrak{F}$. By the Krein-Milman Theorem, there exists a convex combination $\psi = \sum_{k=1}^n \lambda_k \psi_k$ of pure states ψ_k such that $\varphi \approx_{\mathfrak{F}, \varepsilon} \psi$. By excision, for each k , there exists a norm-one positive element $e_k \in A$

³We thank Akitaka Kishimoto for showing us this short proof.

such that $\|e_k(a - \psi_k(a))e_k\| < \varepsilon$ for every $a \in \mathfrak{F}$. Let $P_{\mathcal{K}_0}^\perp = 1 - P_{\mathcal{K}_0}$ and notice that $P_{\mathcal{K}_0}^\perp e_1 P_{\mathcal{K}_0}^\perp - e_1$ is a compact operator; hence $\|P_{\mathcal{K}_0}^\perp e_1 P_{\mathcal{K}_0}^\perp\| = 1$.

Let $\zeta_1 \in \mathcal{K}_0^\perp$ be a unit vector such that $\|e_1 \zeta_1 - \zeta_1\| < \varepsilon$. Now, let \mathcal{K}_1 be the finite-dimensional subspace spanned by \mathcal{K}_0 and $\{a\zeta_1\}_{a \in \mathfrak{F}} \cup \{a^* \zeta_1\}_{a \in \mathfrak{F}}$. Reasoning as above, there exists a unit vector $\zeta_2 \in \mathcal{K}_1^\perp$ such that $\|e_2 \zeta_2 - \zeta_2\| < \varepsilon$. Repeating this n times, we get vectors ζ_1, \dots, ζ_n and set $\xi = \sum \lambda_k^{1/2} \zeta_k$. Then, for every $a \in \mathfrak{F}$, we have

$$\begin{aligned} \omega_\xi(a) &= \sum \lambda_k \omega_{\zeta_k}(a) \\ &\approx_{2\varepsilon} \sum \lambda_k \omega_{e_k \zeta_k}(a) \\ &\approx_\varepsilon \sum \lambda_k \psi_k(a) \|e_k \zeta_k\|^2 \\ &\approx_{2\varepsilon} \psi(a) \approx_\varepsilon \varphi(a). \end{aligned}$$

□

1.5. Completely positive maps

Completely positive maps (and their cousins, completely bounded maps) are the heart and soul of C^* -approximation theory. For a positively complete treatment of these morphisms see [141].

Definitions, examples and Stinespring's Theorem.

Definition 1.5.1. An *operator system* E is a closed self-adjoint subspace of a unital C^* -algebra A such that $1_A \in E$. The $n \times n$ matrices over E , $\mathbb{M}_n(E)$, inherit an order structure from $\mathbb{M}_n(A)$: an element in $\mathbb{M}_n(E)$ is positive if and only if it is positive in $\mathbb{M}_n(A)$. Note that the existence of a unit guarantees that E is spanned by positive elements.

A map φ from an operator system E to a (not necessarily unital) C^* -algebra B is said to be *completely positive* if $\varphi_n: \mathbb{M}_n(E) \rightarrow \mathbb{M}_n(B)$, defined by

$$\varphi_n([a_{i,j}]) = [\varphi(a_{i,j})],$$

is positive (i.e., maps positive matrices to positive matrices) for every n . We denote by $\text{CP}(E, B)$ the set of completely positive maps from an operator system E into B .

Following well-established precedent, we use c.p. to abbreviate “completely positive,” u.c.p. for “unital completely positive” and c.c.p. for “contractive completely positive.”

Example 1.5.2. A $*$ -homomorphism π between C^* -algebras is c.p. since the inflations π_n are also $*$ -homomorphisms (hence preserve positivity). More generally, a map φ of the form $\varphi(a) = V^* \pi(a) V$ for some $*$ -homomorphism

π and an operator V is c.p. (One should verify this. Don't forget that $a \geq 0 \Leftrightarrow a = x^*x$.) A positive linear functional f on an operator system E is c.p. Indeed, for $\xi = (\xi_1, \dots, \xi_n) \in \ell_n^2$ and $a = [a_{i,j}] \geq 0$ in $\mathbb{M}_n(E)$ we have

$$\langle f_n(a)\xi, \xi \rangle = f\left(\sum_{i,j=1}^n \bar{\xi}_i \xi_j a_{i,j}\right) = f\left([\bar{\xi}_1 \ \cdots \ \bar{\xi}_n] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}\right) \geq 0.$$

The transpose map on $\mathbb{M}_n(\mathbb{C})$ is positive but not c.p., since its norm increases after inflation (cf. Theorem 1.5.3 and Proposition 3.5.1).

Directly generalizing the GNS construction, we have Stinespring's Dilation Theorem for c.p. maps. The details of the proof can be found in many places; however, we need the explicit construction and hence we reproduce the main ingredients.

Theorem 1.5.3 (Stinespring). *Let A be a unital C^* -algebra and $\varphi: A \rightarrow \mathbb{B}(\mathcal{H})$ be a c.p. map. Then, there exist a Hilbert space $\widehat{\mathcal{H}}$, a $*$ -representation $\pi: A \rightarrow \mathbb{B}(\widehat{\mathcal{H}})$ and an operator $V: \mathcal{H} \rightarrow \widehat{\mathcal{H}}$ such that*

$$\varphi(a) = V^* \pi(a) V$$

for every $a \in A$. In particular, $\|\varphi\| = \|V^*V\| = \|\varphi(1)\|$ (which, applied to φ_n , implies $\|\varphi_n\| = \|\varphi(1)\|$ as well).

Proof. Define a sesquilinear form $\langle \cdot, \cdot \rangle$ on $A \odot \mathcal{H}$ (this is the algebraic tensor product – see Chapter 3) by

$$\left\langle \sum_j b_j \otimes \eta_j, \sum_i a_i \otimes \xi_i \right\rangle = \sum_{i,j} \langle \varphi(a_i^* b_j) \eta_j, \xi_i \rangle_{\mathcal{H}}.$$

This form turns out to be positive semidefinite and so one mods out by the zero subspace and completes to get a Hilbert space $\widehat{\mathcal{H}}$ (just as in the usual GNS construction). We denote by $(\sum_i a_i \otimes \xi_i)^\wedge$ the element in $\widehat{\mathcal{H}}$ corresponding to $\sum_i a_i \otimes \xi_i \in A \odot \mathcal{H}$. Let $V: \mathcal{H} \rightarrow \widehat{\mathcal{H}}$ be the contraction defined by

$$V(\xi) = (1_A \otimes \xi)^\wedge.$$

For $a \in A$, we define a linear operator $\pi(a)$ on $(A \odot \mathcal{H})^\wedge \subset \widehat{\mathcal{H}}$ by

$$\pi(a) \left(\left(\sum_i b_i \otimes \xi_i \right)^\wedge \right) = \left(\sum_i a b_i \otimes \xi_i \right)^\wedge.$$

As expected, π is a $*$ -representation such that $\varphi(a) = V^* \pi(a) V$ for every $a \in A$. \square

Remark 1.5.4 (Nonunital Stinespring). Stinespring's Dilation Theorem holds for non-unital C^* -algebras too. This follows from Proposition 2.2.1 in the next chapter, for example.

Remarks 1.5.5. We call the triplet $(\pi, \widehat{\mathcal{H}}, V)$ in Theorem 1.5.3 a *Stinespring dilation* of φ . When φ is unital, $V^*V = \varphi(1) = 1$ and hence V is an isometry in this case. The projection $VV^* \in \mathbb{B}(\widehat{\mathcal{H}})$ is called the *Stinespring projection*. In general there could be many different Stinespring dilations, but we may always assume that a dilation $(\pi, \widehat{\mathcal{H}}, V)$ is minimal in the sense that $\pi(A)V\mathcal{H}$ is dense in $\widehat{\mathcal{H}}$ (which holds for the construction used in the proof above). Under this minimality condition, a Stinespring dilation is unique up to unitary equivalence.

When we come to c.p. maps and *maximal* tensor products, the following result will be crucial (see Theorem 3.5.3): If $(\pi, \widehat{\mathcal{H}}, V)$ is a minimal Stinespring dilation of $\varphi: A \rightarrow \mathbb{B}(\mathcal{H})$, then the commutant $\varphi(A)' \subset \mathbb{B}(\mathcal{H})$ also lifts to $\mathbb{B}(\widehat{\mathcal{H}})$.

Proposition 1.5.6. *Let $(\pi, \widehat{\mathcal{H}}, V)$ be the minimal Stinespring dilation of a c.c.p. map $\varphi: A \rightarrow \mathbb{B}(\mathcal{H})$. Then, there exists a $*$ -homomorphism*

$$\rho: \varphi(A)' \rightarrow \pi(A)' \subset \mathbb{B}(\widehat{\mathcal{H}})$$

such that

$$\varphi(a)x = V^*\pi(a)\rho(x)V$$

for every $a \in A$ and $x \in \varphi(A)'$.

Proof. For $x \in \varphi(A)'$, we define a linear operator $\rho(x)$ on the span of $\pi(A)V\mathcal{H}$ by

$$\rho(x) \left(\sum_i \pi(a_i)V\xi_i \right) = \sum_i \pi(a_i)Vx\xi_i.$$

Once we prove that $\rho(x)$ is well-defined and bounded for every $x \in \varphi(A)'$, it is not too hard to check that ρ gives rise to a $*$ -representation of $\varphi(A)'$ on $\widehat{\mathcal{H}}$ such that $\rho(\varphi(A)') \subset \pi(A)'$ and $\varphi(a)x = V^*\pi(a)\rho(x)V$ for every $a \in A$ and $x \in \varphi(A)'$.

So, let $x \in \varphi(A)'$ and $\sum_i \pi(a_i)V\xi_i \in \pi(A)V\mathcal{H}$ be given. If we set $\xi = [\xi_1, \dots, \xi_n]^T \in \mathcal{H}^n$ and let $\text{diag}(x)$ denote the $n \times n$ matrix with x 's down the diagonal and zeroes elsewhere, then we have

$$\begin{aligned} \|\rho(x) \sum_i \pi(a_i)V\xi_i\|_{\widehat{\mathcal{H}}}^2 &= \sum_{i,j} \langle x^* \varphi(a_i^* a_j) x \xi_j, \xi_i \rangle_{\mathcal{H}} \\ &= \langle \text{diag}(x)^* \varphi_n([a_i^* a_j]) \text{diag}(x) \xi, \xi \rangle_{\mathcal{H}^n} \\ &\leq \|x\|^2 \langle \varphi_n([a_i^* a_j]) \xi, \xi \rangle_{\mathcal{H}^n} \\ &= \|x\|^2 \left\| \sum_i \pi(a_i)V\xi_i \right\|_{\widehat{\mathcal{H}}}^2, \end{aligned}$$

where we use the fact that $\text{diag}(x)$ and $\varphi_n([a_i^* a_j]) \in \mathbb{M}_n(\varphi(A))$ commute in the third line above. Therefore, we have $\|\rho(x)\| \leq \|x\|$ as desired. \square

Multiplicative domains.

Proposition 1.5.7. *Let A and B be C^* -algebras and $\varphi: A \rightarrow B$ be a c.c.p. map.*

- (1) *(Schwarz Inequality) The inequality $\varphi(a)^*\varphi(a) \leq \varphi(a^*a)$ holds for every $a \in A$.*
- (2) *(Bimodule Property) Given $a \in A$, if $\varphi(a^*a) = \varphi(a)^*\varphi(a)$ and $\varphi(aa^*) = \varphi(a)\varphi(a)^*$, then $\varphi(ba) = \varphi(b)\varphi(a)$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for every $b \in A$.*
- (3) *The subspace*

$$A_\varphi = \{a \in A : \varphi(a^*a) = \varphi(a)^*\varphi(a) \text{ and } \varphi(aa^*) = \varphi(a)\varphi(a)^*\}$$

is a C^ -subalgebra of A .*

Proof. Let $B \subset \mathbb{B}(\mathcal{H})$ be a faithful $*$ -representation and $(\pi, \widehat{\mathcal{H}}, V)$ be a Stinespring dilation of $\varphi: A \rightarrow B \subset \mathbb{B}(\mathcal{H})$. Then, for every $a \in A$, we have

$$\varphi(a^*a) - \varphi(a)^*\varphi(a) = V^*\pi(a)^*(1_{\widehat{\mathcal{H}}} - VV^*)\pi(a)V \geq 0$$

since V is a contraction. This proves (1). Moreover, $\varphi(a^*a) - \varphi(a)^*\varphi(a) = 0$ is equivalent to $(1_{\widehat{\mathcal{H}}} - VV^*)^{1/2}\pi(a)V = 0$, which in turn implies

$$\varphi(ba) - \varphi(b)\varphi(a) = V^*\pi(b)(1_{\widehat{\mathcal{H}}} - VV^*)\pi(a)V = 0$$

for every $b \in A$. By symmetry, this proves (2). Assertion (3) follows from (2). \square

Definition 1.5.8. Let $\varphi: A \rightarrow B$ be a c.c.p. map. The C^* -subalgebra A_φ in Proposition 1.5.7 is called the *multiplicative domain* of φ .

Evidently A_φ is the largest subalgebra of A on which φ restricts to a $*$ -homomorphism. Note also that if $\|a\| \leq 1$ and $\varphi(a)$ is a unitary element, then a is in the multiplicative domain of φ (by the Schwarz inequality).

Conditional expectations. Conditional expectations are important examples of c.c.p. maps. Here's the definition:

Definition 1.5.9. Let $B \subset A$ be C^* -algebras. A *projection from A onto B* is a linear map $E: A \rightarrow B$ such that $E(b) = b$ for every $b \in B$. A *conditional expectation from A onto B* is a c.c.p. projection E from A onto B such that $E(bxb') = bE(x)b'$ for every $x \in A$ and $b, b' \in B$ (i.e., E is a B -bimodule map).

Theorem 1.5.10 (Tomiya). *Let $B \subset A$ be C^* -algebras and E be a projection from A onto B . Then, the following are equivalent:*

- (1) *E is a conditional expectation;*

- (2) E is c.c.p.;
(3) E is contractive.

Proof. We only have to prove that the last condition implies the first, so assume E is contractive. Passing to double duals, we may assume that A and B are von Neumann algebras. We first prove that E is a B -bimodule map. Since von Neumann algebras are the (norm) closed linear span of their projections, it suffices to check the module property on projections. Let $p \in B$ be a projection and let $p^\perp = 1_A - p$. Since $pE(p^\perp x) = E(pE(p^\perp x))$ for every $x \in A$, we have that for any $t \in \mathbb{R}$,

$$\begin{aligned} (1+t)^2 \|pE(p^\perp x)\|^2 &= \|pE(p^\perp x + tpE(p^\perp x))\|^2 \\ &\leq \|p^\perp x + tpE(p^\perp x)\|^2 \\ &\leq \|p^\perp x\|^2 + t^2 \|pE(p^\perp x)\|^2. \end{aligned}$$

It follows that $\|pE(p^\perp x)\|^2 + 2t\|pE(p^\perp x)\|^2 \leq \|p^\perp x\|^2$ for all $t \in \mathbb{R}$; hence, $pE(p^\perp x) = 0$. In particular, $E(1_B^\perp x) = 1_B E(1_B^\perp x) = 0$. The same reasoning shows $(1_B - p)E(px) = 0$. It follows that

$$E(px) = pE(px) = pE(x - p^\perp x) = pE(x)$$

for every projection $p \in B$ and $x \in A$. Switching to the other side, one shows $E(xp) = E(x)p$ as well – hence E is a B -bimodule map.

Since E is unital – indeed, $bE(1_A) = E(b) = b$ for any $b \in B$ – and contractive, E is necessarily positive (this follows from the corresponding fact for functionals). To prove that E is c.p., let a positive element $[x_{i,j}] \in \mathbb{M}_n(A)$ be given. Let $\pi: B \rightarrow \mathbb{B}(\mathcal{H})$ be any $*$ -representation with a cyclic vector ξ . Then, for any $b_1, \dots, b_n \in B$, we have

$$\sum_{i,j} \langle \pi(E(x_{i,j}))\pi(b_j)\xi, \pi(b_i)\xi \rangle = \langle \pi(E(\sum_{i,j} b_i^* x_{i,j} b_j))\xi, \xi \rangle \geq 0$$

since $\sum_{i,j} b_i^* x_{i,j} b_j \geq 0$ in A . It follows that $[\pi(E(x_{i,j}))]_{i,j} \geq 0$ in $\mathbb{M}_n(\pi(B))$. Since π is an arbitrary cyclic representation, we conclude $[E(x_{i,j})]_{i,j} \geq 0$ in $\mathbb{M}_n(B)$. \square

The following basic fact is extremely useful.

Lemma 1.5.11. *Let M be a von Neumann algebra with a faithful normal tracial state τ and let $1_M \in N \subset M$ be a von Neumann subalgebra. Then, there exists a unique trace-preserving, normal conditional expectation E from M onto N .*

Proof. The restriction of τ to N will also be denoted by τ . Let $a, y \in M$ be arbitrary and $a = u|a|$ be the polar decomposition of a . We claim that

$$|\tau(ya)| = |\tau(yu|a|)| \leq \tau(yu|a|u^*y^*)^{1/2} \tau(|a|)^{1/2} \leq \|y\| \tau(|a|).$$

Indeed, the first inequality is due to Cauchy-Schwarz, while the second follows from the general fact that $\tau(x|a|x^*) = \tau(|a|^{1/2}x^*x|a|^{1/2}) \leq \|x^*x\|\tau(|a|)$, for any $x \in M$.

For each $a \in N$, define $\tau_a \in N_*$ by $\tau_a(y) = \tau(ya)$. The inequality above implies that $\|\tau_a\| = \tau(|a|)$. Also note that $\{\tau_a : a \in N\}$ is a norm-dense linear subspace in N_* . (If it were not dense, we could find $0 \neq n \in N$ such that $\tau_a(n) = 0$ for all $a \in N$, which is impossible since τ is faithful.)

Now we construct the map $E: M \rightarrow N$: For each $x \in M$, define $E(x) \in N = (N_*)^*$ to be the unique linear functional such that $E(x)(\tau_a) = \tau(xa)$, for all $a \in N$. (Recall that $|\tau(xa)| \leq \|x\|\tau(|a|) = \|x\|\|\tau_a\|$; hence $\|E(x)\| \leq \|x\|$.) Note that

$$\tau(E(x)a) = \tau_a(E(x)) = E(x)(\tau_a) = \tau(xa),$$

for all $a \in N$. A routine exercise shows that E is a trace-preserving normal projection from M onto N ; since E is also contractive, it must be a conditional expectation.

To prove uniqueness, assume E' is another trace-preserving conditional expectation. Then for every $x \in M$ and $a \in N$, we have

$$\tau(E'(x)a) = \tau(E'(xa)) = \tau(xa) = \tau(E(xa)) = \tau(E(x)a),$$

and hence $E' = E$. □

With the same hypotheses as the last lemma, let $L^2(M, \tau)$ be the GNS Hilbert space for (M, τ) and $L^2(N, \tau)$ be the Hilbert subspace corresponding to N . Then, the trace-preserving conditional expectation E extends to the orthogonal projection e_N from $L^2(M, \tau)$ onto $L^2(N, \tau)$ in such a way that $e_N x e_N = E(x)e_N$ for every $x \in M$. In fact, one can give an alternate proof of the lemma by observing that $e_N x e_N \in N$, for every $x \in M$, as an element in $\mathbb{B}(L^2(N, \tau))$. More precisely, one must know that the commutant of the right N action on $L^2(N, \tau)$ coincides with N (see Section 6.1) and then check that $e_N x e_N$ commutes with the right N action.

The case of matrices. When either the domain or range is a matrix algebra, there are useful one-to-one correspondences which we will often invoke. Proofs are included for completeness, but the important part is the explicit maps defining the correspondences.

Proposition 1.5.12. *Let A be a C^* -algebra and $\{e_{i,j}\}$ be matrix units of $\mathbb{M}_n(\mathbb{C})$. A map $\varphi: \mathbb{M}_n(\mathbb{C}) \rightarrow A$ is c.p. if and only if $[\varphi(e_{i,j})]$ is positive in $\mathbb{M}_n(A)$. In other words,*

$$\text{CP}(\mathbb{M}_n(\mathbb{C}), A) \ni \varphi \longmapsto [\varphi(e_{i,j})] \in \mathbb{M}_n(A)_+$$

is a bijective correspondence.

Proof. Since $[e_{i,j}] \in \mathbb{M}_n(\mathbb{M}_n(\mathbb{C}))$ is positive (it's a multiple of a rank-one projection), the “only if” part is trivial. To prove the “if” part, assume $a = [\varphi(e_{i,j})] \geq 0$ in $\mathbb{M}_n(A)$ and let $a^{1/2} = [b_{i,j}]$. It follows that

$$\varphi(e_{i,j}) = \sum_{k=1}^n b_{k,i}^* b_{k,j}.$$

Let $A \subset \mathbb{B}(\mathcal{H})$ be a faithful representation and define $V: \mathcal{H} \rightarrow \ell_n^2 \otimes \ell_n^2 \otimes \mathcal{H}$ by

$$V\xi = \sum_{j,k=1}^n \zeta_j \otimes \zeta_k \otimes b_{k,j}\xi,$$

where $\{\zeta_j\}_{j=1}^n$ is the standard orthonormal basis for ℓ_n^2 . Then, for $T = [t_{i,j}] \in \mathbb{M}_n(\mathbb{C})$, we have

$$\begin{aligned} \langle V^*(T \otimes 1 \otimes 1)V\eta, \xi \rangle &= \langle (T \otimes 1 \otimes 1)V\eta, V\xi \rangle \\ &= \sum_{i,j,k,l=1}^n \langle T\zeta_j, \zeta_i \rangle \langle \zeta_k, \zeta_l \rangle \langle b_{k,j}\eta, b_{l,i}\xi \rangle \\ &= \sum_{i,j=1}^n t_{i,j} \langle \sum_{k=1}^n b_{k,i}^* b_{k,j}\eta, \xi \rangle \\ &= \langle \varphi(\sum_{i,j} t_{i,j} e_{i,j})\eta, \xi \rangle \\ &= \langle \varphi(T)\eta, \xi \rangle \end{aligned}$$

for every $\xi, \eta \in \mathcal{H}$. Therefore, $\varphi(T) = V^*(T \otimes 1 \otimes 1)V$ for every $T \in \mathbb{M}_n(\mathbb{C})$ and φ is c.p. \square

Example 1.5.13. Let $a_1, \dots, a_n \in A$ be given and define a linear map $\varphi: \mathbb{M}_n(\mathbb{C}) \rightarrow A$ by $\varphi(e_{i,j}) = a_i a_j^*$. The previous result easily implies that φ is completely positive. Indeed,

$$\begin{aligned} [\varphi(e_{i,j})] &= \begin{bmatrix} a_1 a_1^* & a_1 a_2^* & \cdots & a_1 a_n^* \\ a_2 a_1^* & a_2 a_2^* & \cdots & a_2 a_n^* \\ \vdots & \vdots & \ddots & \vdots \\ a_n a_1^* & a_n a_2^* & \cdots & a_n a_n^* \end{bmatrix} \\ &= \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & 0 \end{bmatrix}^* \\ &\geq 0. \end{aligned}$$

There is a similar characterization of c.p. maps from a C^* -algebra A into $\mathbb{M}_n(\mathbb{C})$. For a linear map $\varphi: A \rightarrow \mathbb{M}_n(\mathbb{C})$, we define a linear functional $\widehat{\varphi}$ on $\mathbb{M}_n(A)$ by

$$\widehat{\varphi}([a_{i,j}]) = \sum_{i,j=1}^n \varphi(a_{i,j})_{i,j}.$$

The notation $\varphi(a_{i,j})_{i,j}$ means the (i,j) th entry of the matrix $\varphi(a_{i,j})$. Yes, the formula is slightly complicated, but it is explicit and very important.

Proposition 1.5.14. *Let A be a unital C^* -algebra. A map $\varphi: A \rightarrow \mathbb{M}_n(\mathbb{C})$ is c.p. if and only if $\widehat{\varphi}$ is positive on $\mathbb{M}_n(A)$. In other words,*

$$\text{CP}(A, \mathbb{M}_n(\mathbb{C})) \ni \varphi \longmapsto \widehat{\varphi} \in \mathbb{M}_n(A)_+^*$$

is a bijective correspondence.

Proof. Let $\{\zeta_i\}_{i=1}^n$ be the standard orthonormal basis for ℓ_n^2 and let $\zeta = [\zeta_1, \dots, \zeta_n]^T \in (\ell_n^2)^n$. Since

$$\widehat{\varphi}([a_{i,j}]) = \langle \varphi_n([a_{i,j}])\zeta, \zeta \rangle$$

for $[a_{i,j}] \in \mathbb{M}_n(A)$, positivity of φ_n implies that of $\widehat{\varphi}$. This proves the “only if” part. To prove the “if” part, assume $\widehat{\varphi}$ is positive and let (π, \mathcal{H}, ξ) be the GNS triplet of $\widehat{\varphi}$. Let $\{e_{i,j}\}$ be the standard matrix units for $\mathbb{M}_n(\mathbb{C})$, which we also view as elements in $\mathbb{M}_n(A)$. Then, for the operator $V: \ell_n^2 \rightarrow \mathcal{H}$ defined by $V\zeta_j = \pi(e_{1,j})\xi$, it is not hard to check

$$\varphi(a) = V^* \pi \left(\begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} \right) V.$$

It follows that φ is c.p. □

Lemma 1.5.15. *Let $E \subset A$ be an operator subsystem and $\psi: E \rightarrow \mathbb{C}$ be a positive linear functional. Then $\|\psi\| = \psi(1)$. Hence, any norm-preserving extension of ψ to A is also positive.*

Proof. Fix $x \in E$ such that $\|x\| \leq 1$ and $|\psi(x)|$ is close to $\|\psi\|$. Multiplying by a complex scalar of norm one, we may assume that $0 < \psi(x) \in \mathbb{R}$. Since positive maps are automatically self-adjoint, we have

$$\psi(x) = \frac{1}{2}\psi(x + x^*)$$

and thus we may assume x is self-adjoint. However, in this case we have the operator inequality $x \leq \|x\|1$ and hence

$$\psi(x) \leq \psi(1)\|x\|.$$

Since a functional satisfying the equation $\psi(1) = \|\psi\|$ is necessarily positive, we are done. □

Corollary 1.5.16. *Let $E \subset A$ be an operator subsystem and $\varphi: E \rightarrow \mathbb{M}_n(\mathbb{C})$ be a c.p. map. Then φ extends to a c.p. map $A \rightarrow \mathbb{M}_n(\mathbb{C})$.*

Proof. Given φ , we can define a linear functional $\widehat{\varphi}$ on $\mathbb{M}_n(E)$ (as above) and it is positive (same proof as above). By the previous lemma, we can extend to a positive functional on all of $\mathbb{M}_n(A)$ and then apply the one-to-one correspondence in reverse to get our desired extension. \square

1.6. Arveson's Extension Theorem

The following theorem, due to Arveson, is absolutely fundamental and probably gets invoked more than any other result in these notes.

Theorem 1.6.1. *Let A be a unital C^* -algebra and $E \subset A$ be an operator subsystem. Then, every c.c.p. map $\varphi: E \rightarrow \mathbb{B}(\mathcal{H})$ extends to a c.c.p. map $\bar{\varphi}: A \rightarrow \mathbb{B}(\mathcal{H})$.*

Proof. Let $P_i \in \mathbb{B}(\mathcal{H})$ be an increasing net of finite-rank projections which converge to the identity in the strong operator topology. For each i , we regard the c.c.p. map $\varphi_i: E \rightarrow P_i\mathbb{B}(\mathcal{H})P_i$, $\varphi_i(e) = P_i\varphi(e)P_i$ as taking values in a matrix algebra. Thus, by Corollary 1.5.16, we may assume that each φ_i is actually defined on all of A . Now we regard φ_i as taking values in $\mathbb{B}(\mathcal{H})$ and apply compactness of the unit ball of $\mathbb{B}(A, \mathbb{B}(\mathcal{H}))$ in the point-ultraweak topology (Theorem 1.3.7) to find a cluster point $\Phi: A \rightarrow \mathbb{B}(\mathcal{H})$. It is readily verified that Φ is c.p. and extends φ . \square

Remark 1.6.2 (Injectivity and Arveson's Theorem). Arveson's Extension Theorem is equivalent to the statement that $\mathbb{B}(\mathcal{H})$ is injective in the category of operator systems with c.c.p. maps as morphisms. This is even true in the category of operator spaces with completely bounded maps as morphisms.

It follows from Arveson's Theorem that a von Neumann algebra $M \subset \mathbb{B}(\mathcal{H})$ is *injective* if and only if there is a conditional expectation from $\mathbb{B}(\mathcal{H})$ onto M . It also follows that injectivity is independent of the choice of faithful representation $M \subset \mathbb{B}(\mathcal{H})$.

Corollary 1.6.3. *Let $E \subset \mathbb{B}(\mathcal{H})$ be an ultraweakly closed operator system and let $\varphi: E \rightarrow \mathbb{M}_n(\mathbb{C})$ be a c.c.p. map. There exists a net of ultraweakly continuous c.c.p. maps $\varphi_\lambda: E \rightarrow \mathbb{M}_n(\mathbb{C})$ which converges to φ in the point-norm topology (i.e., $\|\varphi_\lambda(x) - \varphi(x)\| \rightarrow 0$ for all $x \in E$).*

Proof. By Arveson's Extension Theorem, we may assume that $E = \mathbb{B}(\mathcal{H})$. Since φ is c.p., the corresponding functional $\widehat{\varphi} \in \mathbb{M}_n(\mathbb{B}(\mathcal{H}))^*$ is positive. Hence, there exists a net $\widehat{\varphi}_\lambda$ of positive normal linear functionals which converges pointwise to $\widehat{\varphi}$. Then, the corresponding c.p. maps $\varphi_\lambda: \mathbb{B}(\mathcal{H}) \rightarrow \mathbb{M}_n(\mathbb{C})$ are normal and converge to φ in the point-norm topology (which is

easily seen from explicit form of the correspondence). Unfortunately the φ_λ 's need not be contractive, but they are “almost contractive” (since $\varphi_\lambda(1) \rightarrow \varphi(1)$ in norm); hence we can fiddle with their norms a bit to correct this deficiency. \square

1.7. Voiculescu’s Theorem

Voiculescu’s Theorem is analogous to the Hahn-Banach Theorem in two ways: It gets used all of the time; and it really refers to a collection of related results and corollaries.⁴ Here, we collect all the forms we need, though we only prove those which haven’t yet appeared in a book.

Finite-dimensional case. Exploiting the duality between c.p. maps $A \rightarrow \mathbb{M}_n(\mathbb{C})$ and states on $\mathbb{M}_n(A)$, it is not too hard to deduce the next result from Glimm’s lemma (Lemma 1.4.11).

Proposition 1.7.1. *Let \mathcal{H} be separable, $1 \in A \subset \mathbb{B}(\mathcal{H})$ be a separable C^* -algebra and $\varphi: A \rightarrow \mathbb{M}_n(\mathbb{C})$ be a u.c.p. map such that $\varphi|_{A \cap \mathbb{K}(\mathcal{H})} = 0$. Then there exist isometries $V_k: \ell_n^2 \rightarrow \mathcal{H}$ with the following properties:*

- (1) *the ranges of the V_k ’s are pairwise orthogonal;*
- (2) *$\|\varphi(a) - V_k^* a V_k\| \rightarrow 0$ for every $a \in A$.*

General case.

Definition 1.7.2. Two maps $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$ and $\sigma: A \rightarrow \mathbb{B}(\mathcal{K})$ are called *approximately unitarily equivalent* if there is a sequence of unitary operators $U_n: \mathcal{H} \rightarrow \mathcal{K}$ such that

$$\|\sigma(a) - U_n \pi(a) U_n^*\| \rightarrow 0$$

for all $a \in A$. If it also happens that $\sigma(a) - U_n \pi(a) U_n^*$ is a compact operator, for all $a \in A$ and $n \in \mathbb{N}$, then we say that π and σ are *approximately unitarily equivalent relative to the compacts*.

Note that approximate unitary equivalence relative to the compacts is a *much* stronger notion as it implies that after passing to the Calkin algebra, the representations π and σ are actually unitarily equivalent. See [11] or [53, Corollary II.5.5] for a proof of the next result.

Theorem 1.7.3 (Voiculescu’s Theorem). *Let \mathcal{H} and \mathcal{K} be separable Hilbert spaces and $A \subset \mathbb{B}(\mathcal{H})$ be a separable C^* -algebra such that $1_{\mathcal{H}} \in A$. Let $\iota: A \hookrightarrow \mathbb{B}(\mathcal{H})$ denote the canonical inclusion and let $\rho: A \rightarrow \mathbb{B}(\mathcal{K})$ be any unital representation such that $\rho|_{A \cap \mathbb{K}(\mathcal{H})} = 0$. Then ι and $\iota \oplus \rho$ are approximately unitarily equivalent relative to the compacts.*

⁴Thirdly, some authors assume familiarity with all possible formulations and don’t bother explaining which version is being invoked.

Definition 1.7.4. A representation $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$ is called *essential* if $\pi(A)$ contains no nonzero compact operators.

Essential representations are easy to construct: if $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$ is any representation, then its infinite inflation (i.e., the direct sum of infinitely many copies of π) will be essential.

Corollary 1.7.5. *Let $\pi_i: A \rightarrow \mathbb{B}(\mathcal{H}_i)$, $i = 1, 2$, be faithful essential representations. If A is unital and both π_1, π_2 are unital, then they are approximately unitarily equivalent relative to the compacts. If A is nonunital, then π_1 and π_2 are always approximately unitarily equivalent relative to the compacts.*

In particular, the previous corollary implies that if A is simple and unital, then it has precisely one unital representation, up to approximate unitary equivalence relative to the compacts, since all representations will be faithful and essential.

Technical variations. We'll need some technical variations of Voiculescu's Theorem, but they require a bit more terminology.

If $\pi: \mathbb{B}(\mathcal{H}) \rightarrow Q(\mathcal{H})$ is the canonical mapping onto the Calkin algebra, A is a unital C^* -algebra and $\varphi: A \rightarrow \mathbb{B}(\mathcal{H})$ is a unital completely positive map, then we say that φ is a *representation modulo the compacts* if $\pi \circ \varphi: A \rightarrow Q(\mathcal{H})$ is a $*$ -homomorphism. If $\pi \circ \varphi$ is injective, then we say that φ is a *faithful representation modulo the compacts*. In this situation we define constants $\eta_\varphi(a)$ by

$$\eta_\varphi(a) = 2 \max\{\|\varphi(a^*a) - \varphi(a^*)\varphi(a)\|^{1/2}, \|\varphi(aa^*) - \varphi(a)\varphi(a^*)\|^{1/2}\}$$

for every $a \in A$.

Theorem 1.7.6. *Let A be a unital separable C^* -algebra and $\varphi: A \rightarrow \mathbb{B}(\mathcal{H})$ be a faithful representation modulo the compacts on a separable space \mathcal{H} . If $\sigma: A \rightarrow \mathbb{B}(\mathcal{K})$ is any faithful unital essential representation on a separable space \mathcal{K} , then there exist unitaries $U_n: \mathcal{H} \rightarrow \mathcal{K}$ such that*

$$\limsup_{n \rightarrow \infty} \|\sigma(a) - U_n \varphi(a) U_n^*\| \leq \eta_\varphi(a)$$

for every $a \in A$.

Proof. It suffices to show the existence of a representation σ satisfying the conclusion of the theorem, since all such representations are approximately unitarily equivalent.

Let $\rho: A \rightarrow \mathbb{B}(\mathcal{L})$ be the Stinespring dilation of φ , $V: \mathcal{H} \rightarrow \mathcal{L}$ be the associated isometry, $P = VV^* \in \mathbb{B}(\mathcal{L})$ be the Stinespring projection and

$P^\perp = 1 - P$. It follows from the identity

$$(P^\perp \rho(a) P)^* (P^\perp \rho(a) P) = V(\varphi(a^* a) - \varphi(a^*) \varphi(a)) V^*$$

that

$$\|P^\perp \rho(a) P\| = \|\varphi(a^* a) - \varphi(a^*) \varphi(a)\|^{1/2},$$

for all $a \in A$.

Now write $\mathcal{L} = P\mathcal{L} \oplus P^\perp\mathcal{L}$ and decompose the representation ρ accordingly. That is, consider the matrix decomposition

$$\rho(a) = \begin{bmatrix} \rho(a)_{11} & \rho(a)_{12} \\ \rho(a)_{21} & \rho(a)_{22} \end{bmatrix},$$

where $\rho(a)_{21} = P^\perp \rho(a) P$ and $\rho(a)_{12} = \rho(a^*)_{21}^*$. Thanks to orthogonal domains and ranges, the norm of the matrix

$$\begin{bmatrix} 0 & \rho(a)_{12} \\ \rho(a)_{21} & 0 \end{bmatrix}$$

is equal to $\frac{1}{2}\eta_\phi(a)$.

Now comes the trick. We consider the Hilbert space $P^\perp\mathcal{L} \oplus P\mathcal{L}$ and the representation $\rho': A \rightarrow \mathbb{B}(P^\perp\mathcal{L} \oplus P\mathcal{L})$ given in matrix form as

$$\rho'(a) = \begin{bmatrix} \rho(a)_{22} & \rho(a)_{21} \\ \rho(a)_{12} & \rho(a)_{11} \end{bmatrix}.$$

Using the obvious identification of the Hilbert spaces

$$P\mathcal{L} \oplus \left(\bigoplus_{\mathbb{N}} P^\perp\mathcal{L} \oplus P\mathcal{L} \right) \quad \text{and} \quad \bigoplus_{\mathbb{N}} \mathcal{L} = \bigoplus_{\mathbb{N}} (P\mathcal{L} \oplus P^\perp\mathcal{L}),$$

a standard calculation shows that

$$\|\rho(a)_{11} \oplus \rho'^\infty(a) - \rho^\infty(a)\| \leq \eta_\phi(a)$$

for all $a \in A$, where $\rho'^\infty = \bigoplus_{\mathbb{N}} \rho'$ and $\rho^\infty = \bigoplus_{\mathbb{N}} \rho$. Note also that $\rho(a)_{11} = V\varphi(a)V^*$.

Let $C = \varphi(A) + \mathbb{K}(\mathcal{H})$ and observe that C is actually a separable, unital C^* -subalgebra of $\mathbb{B}(\mathcal{H})$ with $\pi(C) \cong A$ (again, $\pi: \mathbb{B}(\mathcal{H}) \rightarrow Q(\mathcal{H})$ is the quotient map). Note that $\iota \oplus \rho'^\infty \circ \pi$ is approximately unitarily equivalent relative to the compacts to ι , where $\iota: C \hookrightarrow \mathbb{B}(\mathcal{H})$ is the inclusion. Let $W_n: \mathcal{H} \rightarrow \mathcal{H} \oplus \left(\bigoplus_{\mathbb{N}} (P^\perp\mathcal{L} \oplus P\mathcal{L}) \right)$ be unitaries such that

$$\|\varphi(a) \oplus \rho'^\infty(a) - W_n \varphi(a) W_n^*\| \rightarrow 0$$

for all $a \in A$.

Let

$$\tilde{V}: \mathcal{H} \oplus \left(\bigoplus_{\mathbb{N}} (P^\perp\mathcal{L} \oplus P\mathcal{L}) \right) \rightarrow \bigoplus_{\mathbb{N}} \mathcal{L}$$

be the unitary $V \oplus 1$. Note that $\tilde{V}(\varphi(a) \oplus \rho'^{\infty}(a))\tilde{V}^* = V\varphi(a)V^* \oplus \rho'^{\infty}(a) = \rho(a)_{11} \oplus \rho'^{\infty}(a)$. We now complete the proof by defining $\mathcal{K} = \bigoplus_{\mathbb{N}} \mathcal{L}$, $\sigma = \rho^{\infty} = \bigoplus_{\mathbb{N}} \rho$, and $U_n = \tilde{V}W_n: \mathcal{H} \rightarrow \bigoplus_{\mathbb{N}} \mathcal{L} = \mathcal{K}$. \square

Corollary 1.7.7. *Let $\varphi: A \rightarrow \mathbb{M}_n(\mathbb{C}) \subset \mathbb{B}(\mathcal{K})$ be a u.c.p. map where $\mathbb{M}_n(\mathbb{C}) \subset \mathbb{B}(\mathcal{K})$ is a unital inclusion and \mathcal{K} is infinite dimensional. Let $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$ be a faithful unital essential representation. Then there exists a sequence of unitaries $U_n: \mathcal{H} \rightarrow \mathcal{H} \oplus \mathcal{K}$ such that*

$$\limsup_{n \rightarrow \infty} \|(\pi(a) \oplus \varphi(a)) - U_n \pi(a) U_n^*\| \leq \eta_{\varphi}(a)$$

for every $a \in A$.

Proof. Note that if \mathcal{K} had finite dimension, then this result would follow from the previous result, since $\pi \oplus \varphi$ would be a faithful homomorphism modulo the compacts. However, we have assumed \mathcal{K} to be infinite dimensional; hence there is something to prove.

Let $\tilde{\varphi}: A \rightarrow \mathbb{M}_n(\mathbb{C}) = \mathbb{B}(\ell_n^2)$ be the map φ but now regarded as taking values in $\mathbb{B}(\ell_n^2)$ (instead of $\mathbb{B}(\mathcal{K})$). As noted above, we can find unitaries $V_n: \mathcal{H} \rightarrow \mathcal{H} \oplus \ell_n^2$ such that

$$\limsup_{n \rightarrow \infty} \|(\pi(a) \oplus \tilde{\varphi}(a)) - V_n \pi(a) V_n^*\| \leq \eta_{\varphi}(a)$$

for every $a \in A$.

Since $\mathbb{M}_n(\mathbb{C}) \subset \mathbb{B}(\mathcal{K})$ is a unital inclusion, we can find an isomorphism $\mathbb{B}(\mathcal{K}) \cong \mathbb{B}(\ell_n^2 \otimes \mathcal{L})$ that maps $\mathbb{M}_n(\mathbb{C}) \subset \mathbb{B}(\mathcal{K})$ to $\mathbb{M}_n(\mathbb{C}) \otimes 1_{\mathcal{L}}$. Under this isomorphism we may identify φ with $\tilde{\varphi} \otimes 1_{\mathcal{L}}$ and hence the unitaries $V_n \otimes 1_{\mathcal{L}}$ will conjugate $\pi \otimes 1_{\mathcal{L}}$ to $(\pi \oplus \tilde{\varphi}) \otimes 1_{\mathcal{L}}$ which we further identify with $\pi \otimes 1_{\mathcal{L}} \oplus \varphi$. The proof is finished once we observe that π is approximately unitarily equivalent to $\pi \otimes 1_{\mathcal{L}}$ and $\pi \otimes 1_{\mathcal{L}} \oplus \varphi$ is approximately unitarily equivalent to $\pi \oplus \varphi$. \square

There is one more version of Voiculescu's Theorem that we'll need, but not until the end of the book. See [53, II.5.3] for a proof.

Theorem 1.7.8. *Let $A \subset \mathbb{B}(\mathcal{H})$ be a separable C^* -algebra and $\varphi: A \rightarrow \mathbb{B}(\mathcal{K})$ be a c.c.p. map such that $\varphi(x) = 0$ for all $x \in A \cap \mathbb{K}(\mathcal{H})$. Then there exist isometries $V_k: \mathcal{K} \rightarrow \mathcal{H}$ such that $\varphi(a) - V_k^* a V_k \in \mathbb{K}(\mathcal{K})$ and $\lim \|\varphi(a) - V_k^* a V_k\| = 0$, for all $a \in A$.*