Preface

Representation theory plays a central role in Lie theory and has developed in numerous specialized directions over recent decades. Motivation comes from many areas of mathematics and physics, notably the Langlands program. The methods involved are also diverse, including fruitful interactions with "modern" algebraic geometry. Here we focus primarily on algebraic methods in the case of a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ with universal enveloping algebra $U(\mathfrak{g})$, where the prerequisites are relatively modest.

The category $\operatorname{Mod} U(\mathfrak{g})$ of all $U(\mathfrak{g})$-modules is much too large to be understood algebraically. Fortunately, many interesting Lie group representations can be studied effectively in terms of a more limited subcategory where modules are subjected to appropriate finiteness conditions: the *BGG category* $\mathcal{O}$ introduced in the early 1970s by Joseph Bernstein, Israel Gelfand, and Sergei Gelfand. Their papers, stimulated in part by Verma's 1966 thesis [**251**], have led to far-reaching work involving a growing list of researchers. In this book we discuss systematically the early work leading to the Kazhdan–Lusztig Conjecture and its proof around 1980. This is at the core of more recent developments, some of which we go on to introduce in the later chapters. Taken on its own, the study of category $\mathcal{O}$ offers a rewarding tour of the beautiful terrain that lies just beyond the classical Cartan–Weyl theory of finite dimensional representations of $\mathfrak{g}$.

Part I (comprising Chapters 1–8) is written in textbook style, at the level of a second year graduate course in a U.S. university. The emphasis here is on highest weight modules, starting with Verma modules and culminating in the determination of formal characters of simple highest weight modules in the setting of the Kazhdan–Lusztig Conjecture (1979). The proof of this conjecture requires sophisticated ideas from algebraic geometry which go well beyond the algebraic framework of earlier chapters. Thus Chapter 8 marks a shift toward the survey style used in the remainder of the book.

The chapters in Part II can to a large extent be read independently. They supplement the more unified theme of Part I in a variety of ways, often motivated by problems arising in Lie group representations. The book ends with an introduction to the influential work of Beilinson, Ginzburg, and Soergel on Koszul duality.

I have tried to keep prerequisites to a minimum. The reader needs to be comfortable with the basic structure theory of semisimple Lie algebras over $\mathbb{C}$ (summarized in Chapter 0) as well as with standard algebraic methods including elementary homological algebra.

Exercises are scattered throughout the text (mainly in Part I) where I thought they would do the most good. Some of the more straightforward ones are used later in the development. At any rate, the most important exercise for the reader is to engage actively with the ideas presented. Examples are also interspersed, though unfortunately it is difficult to gain much direct insight from low rank cases of the sort which can be done by hand. The deeper parts of the theory have required some imaginative leaps not based on examples alone.

The substantial reference list includes all source material cited, together with related books and survey articles. I have added a somewhat arbitrary sample of other research papers to point the reader in directions such as those sketched in Chapter 13. There is also a list of frequently used symbols, most of which are introduced early in the book. Anyone who consults the literature will encounter a wide array of notational choices; here I have tried to keep things simple and consistent to the extent possible.

The mathematics presented here is not original, though parts of the treatment may be. Many people have provided helpful feedback on earlier versions of the chapters, including Troels Agerholm, Henning Andersen, Brian Boe, Tom Braden, Jon Brundan, Walter Mazorchuk, Wolfgang Soergel, Catharina Stropple, and Geordie Williamson. I am especially indebted to Jens Carsten Jantzen for his detailed suggestions at many stages of the writing. His ideas have left a lasting imprint on the study of category $\mathcal{O}$. Naturally, the final choices made are my own responsibility. Corrections and suggestions from readers are welcome.

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