Preface

These lectures concentrate on some basic facts and ideas of the modern theory of linear elliptic and parabolic partial differential equations (PDEs) in Sobolev spaces. We hope to show that this theory is based on some general and extremely powerful ideas and some *simple* computations. The main objects of study are the Cauchy problem for parabolic equations and the first boundary-value problem for elliptic equations, with some guidelines concerning other boundary-value problems such as the Neumann or oblique derivative problems or problems involving higher order elliptic operators acting on the boundary. The presentation has been chosen in such a way that after having followed the book the reader should acquire a good understanding of a wide variety of results and techniques.

These lecture notes appeared as the result of a two-quarter or a one-semester graduate course I gave at Moscow State University and the School of Mathematics, University of Minnesota, over a number of years and differ significantly from previous drafts. This book also includes some parts of the initial draft and as a whole is most appropriate for a two-quarter or a one-year course. Naturally, one cannot expect that in such a short course all important issues of the theory of elliptic and parabolic equations can be covered. Actually, even the area of second-order *elliptic* partial differential equations is so wide that one cannot imagine a book, let alone a textbook, of reasonable size covering all bases. Restricting further only to the theory of solvability in the *Sobolev* function spaces and *linear* equations still does not make the task realistic. Because of that we will only be concerned with some basic facts and ideas of the modern theory of linear elliptic and parabolic equations in Sobolev spaces. We refer the interested reader to the books [**7**], [**9**], and [**15**], which are classical texts and reference books in elliptic and parabolic PDEs, and the literature therein for additional information on the subject.

I have been educated as a probabilist who in his early stages of research came across the necessity of using some PDE results and realized, with some deep disappointment mixed with astonishment, that at that time there were no simple *introductory* books about the modern theory available to a wide audience. The situation is slightly better now, forty-five years later, but yet by now, as has been pointed out above, the theory became so wide that it is impossible to have just one simple introductory book available to a wide audience. Indeed, one does have several introductory books in different areas of the theory including the book I wrote on Hölder space theory (see [**12**]) and the current textbook on Sobolev space theory.

As with almost any graduate textbook, this one is written for myself and my graduate students who, as I think, should know at least that much of the theory in order to be able to work on problems related to my own interests. That is why the choice of these "basic facts and ideas" and the exposition is by no means exhaustive but rather reflects the author's taste and, in part, his view on what he should have known to be able to work in some areas of mathematics such as the theory of random diffusion processes. I also hope that the contents of the book will be useful to other graduate students and scientists in mathematics, physics, and engineering interested in the theory of partial differential equations.

Comments on the structure of the book

We start with the $\mathcal{L}_2$ theory of elliptic second-order equations in the whole space, first developing it for the Laplacian on the basis of the Fourier transform. Then we go to the $\mathcal{L}_2$ theory for equations with variable coefficients by using partitions of unity, the method of "freezing the coefficients", the method of a priori estimates, and the method of continuity. This is done in Chapter 1.

In Chapter 2 we deal with the $\mathcal{L}_2$ theory of second-order parabolic equations along similar lines. As far as parabolic equations are concerned, in these notes we only concentrate on equations in the whole space and the Cauchy problem.

After that, in Chapter 3, we present some tools from real analysis, helping to pass from $\mathcal{L}_2$ theory to $\mathcal{L}_p$ theory with $p \neq 2$.

In Chapter 4 we derive basic $\mathcal{L}_p$ estimates first for parabolic and then for elliptic equations. The estimates for the elliptic case turn out to follow immediately from the estimates for parabolic equations. On the other hand, for elliptic equations such estimates can be derived directly and we outline how to do this in Section 4.1 and in a few exercises (see Exercises 1.1.5, 1.3.23, and 4.3.2 and the proof of Theorem 4.3.7).

Chapter 5 is devoted to the $\mathcal{L}_p$ theory of elliptic and parabolic equations with continuous coefficients in the whole space. Chapter 6 deals with the same issues but for equations with VMO coefficients, which is quite a new development (only about 16 years old; compare it with the fundamental papers [**1**] of 1959–64).

Chapter 7 is the last one where we systematically consider parabolic equations. There the solvability of parabolic equations with VMO coefficients is proved in Sobolev spaces with mixed norms. Again as in Chapter 2, everything is done only for the equations in the whole space or for the Cauchy problem for equations whose coefficients are only measurable in the time variable. We return to this problem only briefly in Section 13.5 for equations with coefficients independent of time.

Starting from Chapter 8, we only concentrate on elliptic equations in

$$\mathbb{R}^d = \{x = (x^1, ..., x^d) : x^i \in (-\infty, \infty)\}$$

or in domains $\Omega \subset \mathbb{R}^d$. It is worth noticing, however, that almost everything proved for the elliptic equations in Chapter 8 is easily shown to have a natural version valid for parabolic equations in $\mathbb{R} \times \Omega$. Also the Cauchy problem in $(0, \infty) \times \Omega$ can be treated very similarly to what is done in Section 5.2. We do not show how to do that. Here we again run into choosing between what basic facts your graduate students should know and what are other very interesting topics in PDEs. Anyway, the interested reader can find further information about parabolic equations in [**14**], [**15**], and [**18**].

The reader can consult the table of contents as to what issues are investigated in the remaining chapters. We will only give a few more comments.

Chapters 12 and 13 can be studied almost independently of all previous with the exception of Chapter 3. There are many reasons to include their contents in a textbook, although it could be that this is the first time this is done. I wanted my graduate students to be exposed to equations in the function spaces of Bessel potentials since the modern theory of stochastic partial differential equations is using them quite extensively.

Some important topics are scattered throughout the book, the most notable are:

- Equations in divergence form; see Sections 4.4, 8.2, and 13.6.
- Boundary-value problems involving boundary differential operators; see Exercises 1.1.11 and 13.3.15 and Sections 9.3 and 12.3.

- Elliptic equations with measurable coefficients in two dimensions including the Neumann problem; see Exercises 1.4.7, 1.4.8, 1.4.9, 1.6.7, 8.2.6, 8.2.11, 11.5.5, and 11.6.5.

These notes are designed as a textbook and contain about 271 exercises, a few of which (almost all of the 63 exercises marked with an *) are used in the main text. These are the simplest ones. However, many other exercises are quite difficult, despite the fact that their solutions are almost always short. Therefore, the reader should not feel upset if he/she cannot do them even after a good deal of thinking. Perhaps, hints for them should have been provided right after each exercise. We do give the hints to the exercises but only at the end of each chapter just to give the readers an opportunity to test themselves.

Some exercises which are not used in the main text are put in the main text where the reader has enough knowledge to solve them and thus learn more. Some other exercises less directly connected with the text are collected in optional subsections.

The theorems, lemmas, remarks, and such which are part of the main units of the text are numbered serially in a single system that proceeds by section. Exercise 1.7.6 is the sixth numbered unit in the seventh section in the first chapter. In the course of Chapter 1, this exercise is referred to as Exercise 7.6, and in the course of the seventh section of chapter 1 it is referred to as Exercise 6. Similarly and independently of these units formulas are numbered and cross-referenced.

Basic notation

A complete reference list of notation can be found in the index at the end of the book. We always use the summation convention and allow constants denoted by N, usually without indices, to vary from one appearance to another even in the same proof. If we write $N = N(...)$, this means that N depends only on what is inside the parentheses. Usually, in the parentheses we list the objects that are fixed. In this situation one says that the constant N is *under control*. By domains we mean general open sets. On some occasions, we allow ourselves to use different symbols for the same objects, for example,

$$u_{x^i} = \frac{\partial u}{\partial x^i} = D_i u, \quad u_x = \operatorname{grad} u = \nabla u, \quad u_{xx} = (u_{x^i x^j}).$$

Any d-tuple $\alpha = (\alpha_1, ..., \alpha_d)$ of integers $\alpha_k \in \{0, 1, 2...\}$ is called a multi-index. For a multi-index α, $k, j \in \{1, ..., d\}$, and $\xi = (\xi^1, ..., \xi^d) \in \mathbb{R}^d$ we denote

$$D_{kj} u = D_j D_k u = u_{x^k x^j}, \quad |\alpha| = \alpha_1 + ... + \alpha_d,$$

$$D^\alpha = D_1^{\alpha_1} \cdot \ldots \cdot D_d^{\alpha_d}, \quad \xi^\alpha = (\xi^1)^{\alpha_1} \cdot \ldots \cdot (\xi^d)^{\alpha_d}.$$

We also use the notation $Du = u_x$ for the gradient of u, $D^2 u = u_{xx}$ for the matrix of second-order derivatives of u, and $D^n u$ for the set of all nth order derivatives of u. These $D^n u(x)$ for each x can be considered as elements of a Euclidean space of appropriate dimension. By $|D^n u(x)|$ we mean any fixed norm of $D^n u(x)$ in this space.

In the case of parabolic equations we work with

$$\mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}.$$

For functions $u(t, x)$ given on subdomains of $\mathbb{R}^{d+1}$ we use the above notation only for the derivatives in x and denote

$$\partial_t u = \frac{\partial u}{\partial t} = u_t, \quad \partial_t D_k u = u_{tx^k} = u_{x^k t},$$

and so on.

If Ω is a domain in $\mathbb{R}^d$ and $p \in [1, \infty]$, by $\mathcal{L}_p(\Omega)$ we mean the set of all Lebesgue measurable complex-valued functions f for which

$$\|f\|_{\mathcal{L}_p(\Omega)} := \Big(\int_\Omega |f(x)|^p \, dx \Big)^{1/p} < \infty$$

with the standard extension of this formula if $p = \infty$. We also define

$$\mathcal{L}_p = \mathcal{L}_p(\mathbb{R}^d).$$

One knows that $\mathcal{L}_p(\Omega)$ is a Banach space. In the cases when f and g are measurable functions defined in the same domain $D \subset \mathbb{R}^d$, we write "$f = g$ in D" to mean that f equals g *almost everywhere* in D with respect to Lebesgue measure.

By $C_0^\infty(\Omega)$ we mean the set of all infinitely differentiable functions on Ω with compact support (contained) in Ω. By the *support* of a function we mean the closure of the set where the function is different from zero. We call a subset of $\mathbb{R}^d$ *compact* if it is closed and bounded. Of course, saying "compact support" is the same as saying "bounded support" and we keep "compact" just to remind us that we are talking about *closed* sets. We set

$$C_0^\infty = C_0^\infty(\mathbb{R}^d).$$

For $k \in \{0, 1, 2, ...\}$, by $C^k(\Omega)$ we denote the set of all k times continuously differentiable functions on Ω with finite norm

$$\|u\|_{C^k(\Omega)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{C(\Omega)},$$

where

$$\|u\|_{C(\Omega)} = \sup_{x \in \Omega} |u(x)|.$$

As usual, $C(\Omega) = C^0(\Omega)$ and we drop Ω in $C^k(\Omega)$ and $\mathcal{L}_p(\Omega)$ if $\Omega = \mathbb{R}^d$. The subset of C^k consisting of functions on $\mathbb{R}^d$ with compact support is denoted C_0^k. In particular, C_0 is the set of continuous functions with compact support.

By $C^k(\bar{\Omega})$ we denote the subset of $C^k(\Omega)$ consisting of all functions u such that u and $D^\alpha u$ extend to functions continuous in $\bar{\Omega}$ (the closure of Ω) whenever $|\alpha| \le k$. For these extensions we keep the same notation u and $D^\alpha u$, respectively.

If Ω is an unbounded domain, by $C_0^k(\bar{\Omega})$ we mean the subset of $C^k(\bar{\Omega})$ consisting of functions vanishing for $|x|$ sufficiently large. Mainly, the notation $C_0^k(\bar{\Omega})$ will be used for $\Omega = \mathbb{R}^d$ and $\Omega = \mathbb{R}^d_+$, where

$$\mathbb{R}^d_+ = \{(x^1, x') : x^1 > 0, x' = (x^2, ..., x^d) \in \mathbb{R}^{d-1}\}.$$

Generally, speaking about functions on $\mathbb{R}^d$, we mean Lebesgue measurable functions.

To the instructor

We begin with an $\mathcal{L}_2$ theory of second-order elliptic equations of the type

$$a^{ij}(x)u_{x^ix^j}(x) + b^i(x)u_{x^i}(x) + c(x)u(x) = f(x), \quad x \in \mathbb{R}^d,$$

where the coefficients are assumed to be bounded and the matrix $a = (a^{ij})$ symmetric and uniformly positive definite. If the matrix a is uniformly continuous and c is sufficiently large negative, we show that the equation is solvable for $f \in \mathcal{L}_2$. Much later in Section 11.6 on the basis of L_p theory, the restriction on c is replaced with $c \le -\delta$, where $\delta > 0$ is any constant.

Then the general scheme set out in Chapter 1 is repeated several times in the succeeding text in various settings without going into all the minor details each time.

Chapter 3 contains all the tools from real analysis that we use. Here is a major difference between these notes and one of the previous drafts. At some stage, I was presenting the $\mathcal{L}_p$ theory on the basis of the Calderón-Zygmund and Marzinkiewicz theorems interpolating between $p = 1$ and $p = 2$. The present course is based on using the Fefferman-Stein theorem on sharp functions. One could use this theorem along with Stampacchia's interpolation theorem (and Marzinkiewicz's interpolation theorem) to interpolate between $p = 2$ and $p = \infty$, which is done in some texts and in one of the drafts of these lectures. However, there is a shorter way to achieve the goal. The point is that it is possible to obtain *pointwise* estimates of the sharp function $u_{xx}^{\sharp}$ of the second-order derivatives u_{xx} of the unknown solution u through the maximal function of the right-hand side of the equation. This fact has actually been very well known for quite a long time and allows one to get the $\mathcal{L}_p$ estimates of u_{xx} for $p > 2$ just by referring to the Fefferman-Stein and Hardy-Littlewood maximal function theorems.

A disadvantage of this approach is that the students are not exposed to such very powerful and beautiful tools as the Calderón-Zygmund, Stampacchia, and Marzinkiewicz theorems. On the other hand, there are two advantages. First, the Fefferman-Stein theorem is much more elementary than the Calderón-Zygmund theorem (see Chapter 3). Second, the approach based on the pointwise estimates allows us to prove existence theorems for equations with VMO coefficients with almost the same effort as in the case of equations with continuous coefficients.

Despite the fact that, as has been mentioned before, the $\mathcal{L}_p$ theory for elliptic equations can be developed independently of the theory of parabolic equations, in the main text we prefer to derive elliptic estimates from parabolic ones for the following reasons. In the first place, we want the reader to have some insight into the theory of parabolic equations. Secondly, to estimate the $\mathcal{L}_2$ oscillation of u_{xx} over the unit ball B_1 centered at the origin for a C_0^∞ function u, through the maximal function of Δu we split $f := \Delta u$ into two parts: $f = h + g$ where $h \in C_0^\infty$ and $h = f$ in the ball of radius 2 centered at the origin. Then we define v and w as solutions of the equations $\Delta v = h$ and $\Delta w = g$. The trouble is that to find appropriate v and w is not so easy. Say, if $d = 1$ and we want the equations $\Delta v = h$ and $\Delta w = g$ to be satisfied in the whole space, quite often v and v will be unbounded. In the parabolic case this difficulty does not appear because we can find v and w as solutions of the Cauchy problem with zero initial condition, when the initial condition is given for t lying outside the domain where we are estimating the $\mathcal{L}_2$ oscillation of the solution.

One more point worth noting is that one can have a short course on elliptic equations, and after going through Chapter 1, go directly to Chapters 8–10 if one is only interested in the case $p = 2$. If $d = 2$, one can also include Chapter 11. Adding to this list Chapter 3 and Section 4.1 would allow the reader to follow all the material in full generality apart from what concerns parabolic equations and equations with VMO coefficients. In this case one also has to follow the proof of Lemma 6.3.8 in order to get control on the $\mathcal{L}_p$ norm of solutions. Finally, doing Exercise 4.3.2 allows one to include the results of Chapter 6 related to the elliptic equations with VMO coefficients.

If one wants to give a course containing both Hölder space and Sobolev space theories, then one can start with part of the present notes, use the possibility of obtaining basic $C^{2+\alpha}$ estimates by doing Exercises 4.3.2, 4.3.3, and 10.1.8, and then continue lecturing on Hölder space theory following one's favorite texts. By the way, this switch to Hölder space theory can be done right after Chapter 1 if only elliptic equations are to be treated. For parabolic equations this switch is possible after going through Chapters 1 and 2 and doing Exercises 4.3.5, 4.3.6, 10.1.9, and 10.1.10.

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Nicolai Krylov, Minneapolis, August 2007