# Contents


Preface xi

Chapter 1. Second-order elliptic equations in $W_2^2(\mathbb{R}^d)$ 1

§1. The simplest equation $\lambda u - \Delta u = f$ 2

§2. Integrating the determinants of Hessians (optional) 7

§3. Sobolev spaces $W_p^k(\Omega)$ 8

§4. Second-order elliptic differential operators 14

§5. Multiplicative inequalities 17

§6. Solvability of elliptic equations with continuous coefficients 20

§7. Higher regularity of solutions 25

§8. Sobolev mollifiers 32

§9. Singular-integral representation of $u_{xx}$ 38

§10. Hints to exercises 42

Chapter 2. Second-order parabolic equations in $W_2^{1,k}(\mathbb{R}^{d+1})$ 45

§1. The simplest equation $u_t + a^{ij}(t)D_{ij}u - \lambda u = f$ 45

§2. Sobolev spaces $W_p^{r,k+2r}(\Omega)$ 51

§3. Parabolic equations with continuous coefficients 56

§4. Local or interior estimates 60

§5. The Cauchy problem 67

§6. Hints to exercises 70

Chapter 3. Some tools from real analysis 73

§1. Partitions and stopping times 73

§2. Maximal and sharp functions 78

2:1. The Fefferman-Stein theorem 78

2:2. Exercises (optional) 83

§3. Comparing maximal and sharp functions in the Euclidean space 87

§4. Hints to exercises 91

Chapter 4. Basic $\mathcal{L}_p$ estimates for parabolic and elliptic equations 93

§1. An approach to elliptic equations 94

§2. Preliminary estimates of $L$-caloric functions 98

§3. Solvability of model equations 104

§4. Divergence form of the right-hand side for the Laplacian 113

§5. Hints to exercises 115

Chapter 5. Parabolic and elliptic equations in $W_p^{1,k}$ and $W_p^k$ 117

§1. Better regularity for equations with coefficients independent of $x$ 117

§2. Equations with continuous coefficients. The Cauchy problem 119

Chapter 6. Equations with VMO coefficients 125

§1. Estimating $\mathcal{L}_q$ oscillations of $u_{xx}$ 125

§2. Estimating sharp functions of $u_{xx}$ 130

§3. A priori estimates for parabolic and elliptic equations with VMO coefficients 134

§4. Solvability of parabolic and elliptic equations with VMO coefficients. The Cauchy problem 139

§5. Hints to exercises 143

Chapter 7. Parabolic equations with VMO coefficients in spaces with mixed norms 145

§1. Estimating sharp functions of $\|u_{xx}(t,\cdot)\|_{\mathcal{L}_q}$ 146

§2. Existence and uniqueness results 149

§3. Hints to exercises 155

Chapter 8. Second-order elliptic equations in $W_p^2(\Omega)$ 157

§1. Spaces of functions vanishing on the boundary 158

§2. Equations in half spaces 160

§3. Domains of class $C^k$. Equations near the boundary 165

§4. Partitions of unity and the regularizer 171

§5. Solvability of equations in domains for large $\lambda$ 174

§6. Hints to exercises 179

Chapter 9. Second-order elliptic equations in $W_p^k(\Omega)$ 181

§1. Finite differences. Better regularity of solutions in $\mathbb{R}_+^d$ for model equations 181

§2. Equations in domains 187

§3. The oblique derivative problem in $\mathbb{R}_+^d$ 190

§4. Local regularity of solutions 196

§5. Hints to exercises 199

Chapter 10. Sobolev embedding theorems for $W_p^k(\Omega)$ 201

§1. Embedding for Campanato and Slobodetskii spaces 203

1:1. Embeddings in $C^\alpha$ 203

1:2. Exercises (optional) 207

§2. Embedding $W_p^1(\Omega) \subset C^{1-d/p}(\Omega)$. Morrey's theorem 209

§3. The Gagliardo-Nirenberg theorem 215

§4. General embedding theorems 216

§5. Compactness of embeddings. Kondrashov's theorem 223

§6. An application of Riesz's theory of compact operators 227

§7. Hints to exercises 229

Chapter 11. Second-order elliptic equations $Lu - \lambda u = f$ with $\lambda$ small 231

§1. Maximum principle for smooth functions 232

1:1. Maximum principle 232

1:2. Exercises (optional) 234
§2. Resolvent operator for $\lambda$ large 236
§3. Solvability of equations in smooth domains 241
§4. The way we proceed further 245
§5. Decay at infinity of solutions of $Lu = f$ in $\mathbb{R}^d$ 246
§6. Equations in $\mathbb{R}^d$ with $\lambda$ small 249
§7. Traces of $W_p^k(\Omega)$ functions on $\partial\Omega$ 255
§8. The maximum principle in $W_p^2$. Green's functions 262
§9. Hints to exercises 263

Chapter 12. Fourier transform and elliptic operators 267
§1. The space $\mathcal{S}$ 268
§2. The notion of elliptic differential operator 269
§3. Comments on the oblique derivative and other boundary-value problems. Instances of pseudo-differential operators 272
§4. Pseudo-differential operators 274
§5. Green's functions 280
§6. Existence of Green's functions 285
§7. Estimating $G$ and its derivatives 288
7:1. Differentiability of $G$ and estimates of its derivatives 288
7:2. Exercises (optional) 292
§8. Boundedness of the zeroth-order pseudo-differential operators in $\mathcal{L}_p$ 293
§9. Operators related to the Laplacian 297
9:1. The operators $(1 - \Delta)^{\gamma/2}$, Cauchy's operator, the Riesz and Hilbert transforms, the Cauchy-Riemann operator 297
9:2. Exercises (optional) 302
§10. An embedding lemma 304
§11. Hints to exercises 308

Chapter 13. Elliptic operators and the spaces $H_p^\gamma$ 311
§1. The space $\mathcal{H}$ 311

§2. Some properties of the space $\mathcal{H}$ 315

§3. The spaces $H_p^\gamma$ 317

3:1. Definition, solvability of elliptic equations in $H_p^\gamma$, the equality $H_p^\gamma = W_p^\gamma$ 317

3:2. Exercises (optional) 323

§4. Higher-order elliptic differential equations with continuous coefficients in $H_p^\gamma$ 328

§5. Second-order parabolic equations. Semigroups (optional) 333

§6. Second-order divergence type elliptic equations with continuous coefficients 335

§7. Nonzero Dirichlet condition and traces 339

§8. Sobolev embedding theorems for $H_p^\gamma$ spaces 342

§9. Sobolev mollifiers 346

§10. Hints to exercises 350

Bibliography 353

Index 355