

Euler's Formula for $\sin(z)$

6.0. Motivation

In this chapter we take a little break and show how the complex analysis we have learned so far can be used to prove Euler's infinite-product representation of the sine function:

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

We should note that this is a special case of general results about factorizations of entire functions to be discussed later; here we give two *ad hoc* proofs, one using the Residue Theorem and one by Liouville's Theorem.

How could someone come up with a formula like that in the first place? A person might start with the observation that $\sin(\pi z)$ vanishes precisely at the integers, so it could be that a formula something like

$$\sin(\pi z) = \prod_{n=-\infty}^{\infty} (z - n)$$

works; the product on the right apparently vanishes at the integers, so if sine were a polynomial this would be at worst off by a constant factor.

Unfortunately a person realizes very quickly that the product $\prod_{n=-\infty}^{\infty} (z - n)$ simply does not converge; the factors do not even tend to 1, which seems a likely prerequisite for convergence of an infinite product. We could try to fix this by multiplying each factor by some constant to

make the product converge *somewhere*. The simplest approach would be to multiply each factor (or each factor with $n \neq 0$) by a constant so as to make the factor equal 1 at the origin. So we want to replace the factor $(z - n)$ with $(z - n)/(0 - n)$, or $1 - z/n$. That gives us a tentative

$$\sin(\pi z) = \left(\prod_{n=-\infty}^{-1} \left(1 - \frac{z}{n}\right) \right) (z) \left(\prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) \right).$$

It is *possible* that that is within a constant factor of being correct. If so then what would the constant be? If you divide both sides by z and then let $z \rightarrow 0$, you get $\pi = 1$. So we fix that:

$$\sin(\pi z) = \left(\prod_{n=-\infty}^{-1} \left(1 - \frac{z}{n}\right) \right) (\pi z) \left(\prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) \right).$$

This is actually correct. Almost. The products still do not converge. Whether an infinite product converges has to do with how fast the factors tend to 1, and (as we will see below) the fact that the series $\sum_{n=1}^{\infty} 1/n$ diverges implies that the product $\prod_{n=1}^{\infty} (1 - z/n)$ diverges. But it turns out that this product converges if we modify it by grouping certain pairs of factors together: $(1 - z/n)$ and $(1 + z/n)$ are both too far from 1 for convergence, but their product is $(1 - z^2/n^2)$, which is much closer to 1. And so we combine pairs of factors to get

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

which turns out to be correct.

Of course we have not yet given much indication why this product should actually equal $\sin(\pi z)$. It is not hard to see that the product converges to an entire function, and if we define

$$P(z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

then it is not hard to show that $P(z) = 0$ if and only if $z \in \mathbb{Z}$, and one can even show without much trouble that $P(z+1) = -P(z)$, but $\sin(\pi z)$ is not the only function with all these properties (for example if g is entire then the function $\sin(\pi z)e^{g(\sin^2(\pi z))}$ has all the properties listed). But it seems that it could be so; now we just have to prove it. There are various ways to do that.

6.1. Proof by the Residue Theorem

One might imagine proving that Euler's infinite product equals $\sin(\pi z)$ by taking the logarithm of both sides — we need only show that

$$\log(\sin(\pi z)) = \log(\pi z) + \sum_{n=1}^{\infty} \log\left(1 - \frac{z^2}{n^2}\right).$$

Of course this cannot quite work because these functions do not quite *have* (“single-valued”) logarithms; at the very least we would need to be very careful about saying exactly what branches of the logarithm we are using (note that the factors in the product have zeroes, and we are not going to be able to define branches of the logarithm holomorphic near those zeroes . . .). This begins to seem somewhat tricky.

Before abandoning this idea we might note that although most holomorphic functions do not have (holomorphic) logarithms, and in particular these ones do not, the nonexistent logarithm of a holomorphic function *does* have a derivative! Seriously: Two branches of $\log(f)$ differ by a constant, and that constant goes away when we differentiate. So, in spite of our problems with interpreting the previous formula, if we differentiate it we get something that might make sense, might be true, and might then lead to Euler's formula for $\sin(\pi z)$. Noting that the derivative of $\log(f)$ is f'/f , we take the derivative of all the terms in the last formula and we get

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{-2z/n^2}{1 - z^2/n^2} = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right).$$

Could this be right? Yes: Note that the sum on the right has poles at the integers and residue 1 at each pole, just like the function $\pi \cot(\pi z)$.

The actual proof starts here: We will show using the Residue Theorem that the function $\pi \cot(\pi z)$ is given by the infinite sum above, and we will deduce Euler's formula for $\sin(\pi z)$ (after a few preliminaries about infinite products and the derivatives of nonexistent logarithms).

The infinite-sum representation for the cotangent follows from a Cauchy's Integral Formula–Residue Theorem hybrid:

Theorem 6.1.0 (Cauchy Integral Formula for functions with simple poles). *Suppose that $V \subset \mathbb{C}$ is open and Γ is a cycle in V such that $\text{Ind}(\Gamma, a) = 0$ for all $a \in \mathbb{C} \setminus V$. Suppose that S is a (relatively) closed subset of V , $S \cap \Gamma^* = \emptyset$, and every point of S is isolated. Suppose that $f \in H(V \setminus S)$ and f has a simple pole or removable singularity at every point of S . Then for $z \in V \setminus (S \cup \Gamma^*)$ we have*

$$\text{Ind}(\Gamma, z)f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w-z} dw + \sum_{p \in S} \frac{\text{Ind}(\Gamma, p)\text{Res}(f, p)}{z-p}.$$

(Note that the sum has only finitely many nonzero terms, as in Theorem 4.17. Note also that the validity of the theorem depends on the fact that f has at worst a *simple* pole at each point of S .)

Proof. Fix $z \in V \setminus (S \cup \Gamma^*)$ and let

$$g(w) = \frac{f(w)}{w - z}.$$

Now g is holomorphic in V except for (possible) simple poles at z and the points of S . Exercise 4.7(ii) shows that $\text{Res}(g, z) = f(z)$ and $\text{Res}(g, p) = \text{Res}(f, p)/(p - z)$ for $p \in S$. So the Residue Theorem (Theorem 4.17) shows that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma} g(w) dw &= \text{Ind}(\Gamma, z) \text{Res}(g, z) + \sum_{p \in S} \text{Ind}(\Gamma, p) \text{Res}(g, p) \\ &= \text{Ind}(\Gamma, z) f(z) + \sum_{p \in S} \frac{\text{Ind}(\Gamma, p) \text{Res}(f, p)}{p - z}. \quad \square \end{aligned}$$

For our application we need to know that the cotangent function remains bounded if we stay away from its poles:

Lemma 6.1.1. *There exists a constant M such that*

$$|\cot(\pi z)| \leq M$$

whenever $|\text{Im}(z)| \geq 1$ or $\text{Re}(z) = n + 1/2$ ($n \in \mathbb{Z}$).

Proof. If $z = x + iy$ then

$$\cot(\pi z) = i \frac{e^{-2\pi y} e^{2\pi i x} + 1}{e^{-2\pi y} e^{2\pi i x} - 1} = i \frac{1 + e^{2\pi y} e^{-2\pi i x}}{1 - e^{2\pi y} e^{-2\pi i x}}.$$

The first expression shows that $\cot(\pi z) \rightarrow -i$ uniformly in x as $y \rightarrow \infty$, while the second shows that $\cot(\pi z) \rightarrow i$ uniformly in x as $y \rightarrow -\infty$. So there exists a number A such that

$$|\cot(\pi z)| \leq 2 \quad (|y| \geq A).$$

Let

$$K = \{x + iy : 0 \leq x \leq 1, 1 \leq |y| \leq A\} \cup \left\{ \frac{1}{2} + iy : |y| \leq 1 \right\}.$$

Since K is compact and the function $\cot(\pi z)$ is continuous on K , there exists $M \geq 2$ such that $|\cot(\pi z)| \leq M$ for $z \in K$. Since $\cot(\pi(z+1)) = \cot(\pi z)$, it follows that $|\cot(\pi z)| \leq M$ for $z \in \bigcup_{n \in \mathbb{Z}} (n + K)$ and hence for $z \in \{x + iy : |y| \geq A\} \cup \bigcup_{n \in \mathbb{Z}} (n + K)$. \square

Definition. If $z \in \mathbb{C}$ and $s > 0$ then the boundary of the square with lower-left corner z and side length s is the cycle

$$[z, z + s] \dot{+} [z + s, z + s + is] \dot{+} [z + s + is, z + is] \dot{+} [z + is, z].$$

Theorem 6.1.2. If $z \in \mathbb{C} \setminus \mathbb{Z}$ then

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right).$$

Proof. For N a positive integer let Q_N be the boundary of the square with lower-left corner $-(N + \frac{1}{2}) - i(N + \frac{1}{2})$ and side length $2N + 1$. An application of Theorem 6.1.0 shows that if $z \in \mathbb{C} \setminus \mathbb{Z}$, $|\operatorname{Re}(z)| < N + \frac{1}{2}$ and $|\operatorname{Im}(z)| < N + \frac{1}{2}$ then

$$\pi \cot(\pi z) = \sum_{n=-N}^N \frac{1}{z-n} + \frac{1}{2\pi i} \int_{Q_N} \frac{\pi \cot(\pi w)}{w-z} dw,$$

so we need only show that

$$\lim_{N \rightarrow \infty} \int_{Q_N} \frac{\pi \cot(\pi w)}{w-z} dw = 0.$$

Now Lemma 6.1.1 shows that $|\cot(\pi z)| \leq M$ on Q_N^* , but this is not quite enough; a direct application of the ML inequality shows only that the integral is bounded, not that it tends to 0. But the ML inequality is a rather crude device for estimating the size of integrals, because it ignores the fact that the absolute value of the integrand may not be constant and also ignores any possible cancellation. It turns out that our integral is much smaller than the ML inequality would appear to indicate, because of cancellation:

We notice that the integral of an *even* function over Q_N must equal 0. Now, $\cot(\pi w)/(w-z)$ is not an even function of w , but if z is fixed and w is large then it is very close to the even function $\cot(\pi w)/w$. So we estimate our integral by comparing it to the integral of $\cot(\pi w)/w$, as follows:

$$\int_{Q_N} \frac{\cot(\pi w)}{w-z} dw = \int_{Q_N} \cot(\pi w) \left(\frac{1}{w-z} - \frac{1}{w} \right) dw = z \int_{Q_N} \frac{\cot(\pi w)}{w(w-z)} dw.$$

The ML inequality suffices to show that this last integral tends to 0:

Note that $|w| > N$ for $w \in Q_N^*$. Hence if $N > 2|z|$ we have $|w-z| > N/2$ and hence

$$\left| \frac{1}{w(w-z)} \right| < \frac{2}{N^2}$$

for $w \in Q_N^*$. So the ML inequality and Lemma 6.1.1 show that

$$\left| \int_{Q_N} \frac{\cot(\pi w)}{w(w-z)} dw \right| < \frac{2M}{N^2} (8N+4)$$

for $N > 2|z|$; hence

$$\lim_{N \rightarrow \infty} \int_{Q_N} \frac{\cot(\pi w)}{w(w-z)} dw = 0. \quad \square$$

Note that the proof shows that

$$\pi \cot(\pi z) = \lim_{N \rightarrow \infty} \sum_{n=-N}^N \frac{1}{z-n}.$$

It would be a bad idea to rewrite this as

$$\pi \cot(\pi z) = \sum_{n=-\infty}^{\infty} \frac{1}{z-n},$$

because that might be taken to mean

$$\pi \cot(\pi z) = \lim_{N, M \rightarrow \infty} \sum_{n=-M}^N \frac{1}{z-n},$$

and this last equation is not correct, because the indicated limit does not exist!

- **Exercise 6.1.** Show that the series

$$\sum_{n=0}^{\infty} \frac{1}{z-n}$$

diverges for all $z \in \mathbb{C} \setminus \mathbb{Z}$.

Hint: If $\sum a_n$ diverges and $\sum (a_n - b_n)$ converges then $\sum b_n$ diverges.

- **Exercise 6.2.** Show that the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right)$$

converges absolutely for all $z \in \mathbb{C} \setminus \mathbb{Z}$.

We need to say a little bit about infinite products. The main technical detail follows:

Lemma 6.1.3. *If $z_1, \dots, z_n \in \mathbb{C}$ and $\sum_{j=1}^{n-1} |1 - z_j| < 1/2$ then*

$$\left| 1 - \prod_{j=1}^n z_j \right| \leq 2 \sum_{j=1}^n |1 - z_j|.$$

Proof. The proof is by induction on n ; the case $n = 1$ is clear. (By convention a sum of the form $\sum_{j=1}^0$ is equal to 0, being the sum of no terms.)

Suppose we know that $\left| 1 - \prod_{j=1}^n z_j \right| \leq 2 \sum_{j=1}^n |1 - z_j|$ whenever $\sum_{j=1}^{n-1} |1 - z_j| < 1/2$ and suppose as well that $\sum_{j=1}^n |1 - z_j| < 1/2$. Then

$$\left| \prod_{j=1}^n z_j \right| \leq 1 + \left| 1 - \prod_{j=1}^n z_j \right| \leq 1 + 2 \sum_{j=1}^n |1 - z_j| \leq 2,$$

and so

$$\begin{aligned} \left| 1 - \prod_{j=1}^{n+1} z_j \right| &= \left| 1 - \prod_{j=1}^n z_j + \prod_{j=1}^n z_j - \prod_{j=1}^{n+1} z_j \right| \\ &\leq \left| 1 - \prod_{j=1}^n z_j \right| + \left| \prod_{j=1}^n z_j \right| |1 - z_{n+1}| \\ &\leq 2 \sum_{j=1}^n |1 - z_j| + 2|1 - z_{n+1}| \\ &= 2 \sum_{j=1}^{n+1} |1 - z_j|. \end{aligned} \quad \square$$

Lemma 6.1.4. (i) *If $z_1, \dots \in \mathbb{C}$ and $\sum_{j=1}^{\infty} |1 - z_j| < \infty$ then*

$$\prod_{j=1}^{\infty} z_j = \lim_{n \rightarrow \infty} \prod_{j=1}^n z_j$$

exists; furthermore, $\prod_{j=1}^{\infty} z_j \neq 0$ unless $z_j = 0$ for some j .

(ii) *If (f_j) is a sequence of complex-valued functions on some set S and the sum $\sum_{j=1}^{\infty} |1 - f_j|$ converges uniformly on S then $P_n = \prod_{j=1}^n f_j$ tends to $P = \prod_{j=1}^{\infty} f_j$ uniformly on S ; if $z \in S$ then $P(z) \neq 0$ unless $f_j(z) = 0$ for some j .*

Note. It is possible for an infinite product to equal 0 even if none of the factors vanish (consider $\prod_{j=1}^{\infty} 1/2$). Some authors say that such a sum “diverges to 0”. (And other authors say it converges to 0; for this reason we will try to avoid saying that an infinite product converges or diverges.)

Proof. (i) Choose N so that $\sum_{j=N+1}^{\infty} |1 - z_j| < 1/2$. To show that $\lim_{n \rightarrow \infty} \prod_{j=1}^n z_j$ exists it is sufficient to show that (p_n) is a Cauchy sequence, where

$$p_n = \prod_{j=N+1}^{N+n} z_j.$$

But Lemma 6.1.3 shows that $|p_n| \leq 2$, and hence another application of the lemma shows that if $n > m$ then

$$|p_n - p_m| = |p_m| \left| 1 - \prod_{j=N+m+1}^{N+n} z_j \right| \leq 4 \sum_{j=N+m+1}^{N+n} |1 - z_j|;$$

hence $|p_n - p_m| \rightarrow 0$ as $n, m \rightarrow \infty$.

If none of the z_j vanish then $\prod_{j=1}^N z_j \neq 0$, and Lemma 6.1.4 shows that $|1 - \prod_{j=N+1}^{\infty} z_j| < 1$, so $\prod_{j=N+1}^{\infty} z_j \neq 0$; hence $\prod_{j=1}^{\infty} z_j \neq 0$.

(ii) The proof of the second part of the lemma is the same as the proof of the first part. \square

Now the sum

$$\sum_{n=1}^{\infty} \left| \frac{z^2}{n^2} \right|$$

converges uniformly on compact subsets of the plane; thus Lemma 6.1.4(ii) shows that the partial products

$$P_N(z) = \pi z \prod_{n=1}^N \left(1 - \frac{z^2}{n^2} \right)$$

converge uniformly on compact subsets of the plane to

$$P(z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2} \right),$$

an entire function which vanishes only at the integers.

Note that we also have $P'_N \rightarrow P'$, by Proposition 3.5.

In general, if f is a differentiable function, we define the *logarithmic derivative* $L(f)$ by

$$L(f) = \frac{f'}{f},$$

at least at points where f is nonzero. The logarithmic derivative of f is the derivative of the logarithm of f when f has a logarithm; this makes the formula

$$L(fg) = L(f) + L(g)$$

plausible. In fact the product rule shows that $L(fg) = L(f) + L(g)$ for any differentiable nonvanishing functions f and g .

It follows that

$$L\left(\prod_{n=1}^N f_n\right) = \sum_{n=1}^N L(f_n)$$

on the set where none of the f_n vanish. (This sometimes gives a convenient and efficient way to calculate derivatives of products of several functions, by the way.)

We will use the following continuity property of L :

Proposition 6.1.5. *Suppose that V is a connected open set, $f_1, f_2, \dots \in H(V)$ and $f_n \rightarrow f$ uniformly on compact subsets of V . Suppose that f is not identically zero. Then*

$$L(f_n) \rightarrow L(f)$$

uniformly on K , if K is any compact subset of V on which f has no zero.

Proof. This is immediate from Proposition 3.5. \square

And we will use the fact that $L(f)$ determines f , at least up to a constant factor:

Lemma 6.1.6. *Suppose that V is a connected open subset of \mathbb{C} , f and g are holomorphic functions in V neither of which vanishes identically, and*

$$L(f) = L(g)$$

on the set where neither f nor g vanishes. Then $f = cg$ for some constant c .

Proof. We may suppose that neither f nor g has a zero in V (because the set where they are both nonzero is in general a dense connected open subset of V). Now the fact that $L(f) = L(g)$ shows that $L(f/g) = 0$, and hence $(f/g)' = 0$; thus f/g is constant. \square

We can finally prove Euler's formula. Assume first that $z \in \mathbb{C} \setminus \mathbb{Z}$. Note that

$$L\left(1 - \frac{z^2}{n^2}\right) = L\left(1 - \frac{z}{n}\right) + L\left(1 + \frac{z}{n}\right) = \frac{1}{z-n} + \frac{1}{z+n}.$$

If P_N and P are as above then it follows that

$$L(P_N)(z) = \sum_{n=-N}^N \frac{1}{z-n},$$

and now Proposition 6.1.5 shows that

$$L(P) = \lim_{N \rightarrow \infty} \sum_{n=-N}^N \frac{1}{z-n} = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right).$$

But now Theorem 6.1.2 shows that

$$L(P)(z) = \pi \cot(\pi z) = L(\sin(\pi z)),$$

so that

$$P(z) = c \sin(\pi z)$$

by Lemma 6.1.6. Dividing by πz we see that

$$\frac{c \sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \quad (z \neq 0),$$

and letting $z \rightarrow 0$ here shows that $c = 1$.

This proves Euler's formula for $z \in \mathbb{C} \setminus \mathbb{Z}$; the case $z \in \mathbb{Z}$ follows by continuity. \square

6.2. Estimating Sums Using Integrals

In the second proof of Euler's formula we will need to use an integral to estimate a certain infinite sum. Let us say a little bit about various ways one might do that. In this section we will assume that $\phi : (0, \infty) \rightarrow (0, \infty)$ is continuous, and we will consider the question of what we can say about the sum

$$\sum_{n=1}^{\infty} \phi(n)$$

in terms of integrals.

The simplest approach is something found in a lot of calculus books under the heading “Integral Test”. Suppose that ϕ is nonincreasing. This shows that

$$\phi(n) = \int_{n-1}^n \phi(n) dt \leq \int_{n-1}^n \phi(t) dt,$$

and hence that

$$\sum_{n=1}^{\infty} \phi(n) \leq \int_0^{\infty} \phi(t) dt.$$

This can sometimes be useful even if it does not appear that way at first. For example, if we want to know that

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges then a direct application of the inequality above does not help, because $\int_0^{\infty} dt/t^2 = \infty$. But of course the same argument shows that

$$\sum_{n=1}^{\infty} \phi(n) = \phi(1) + \sum_{n=1}^{\infty} \phi(n) \leq \phi(1) + \int_1^{\infty} \phi(t) dt,$$

which shows that $\sum_{n=1}^{\infty} 1/n^2$ converges. Noting that a similar argument shows that

$$\int_1^{\infty} \phi(t) dt \leq \sum_{n=1}^{\infty} \phi(n)$$

we obtain the “integral test”: If $\phi : [1, \infty) \rightarrow (0, \infty)$ is continuous and nonincreasing then $\sum_{n=1}^{\infty} \phi(n)$ converges if and only if $\int_1^{\infty} \phi(t) dt < \infty$.

If we just want to check whether a sum converges then that will often suffice. However, sometimes we need more precise information about the *size* of our sum than is given by the integral test. If you think about it for a second, you decide that $\int_{n-1}^n \phi(t) dt$ is probably a fairly poor approximation to $\phi(n)$; taking the integral from $n - \frac{1}{2}$ to $n + \frac{1}{2}$ should often give a better approximation. Of course now the problem of estimating the error in this approximation arises.

- **Exercise 6.3.** Suppose that ϕ'' is continuous on $[-1/2, 1/2]$.

(i) Show that

$$\int_{-1/2}^{1/2} \phi(t) dt - \phi(0) = \frac{1}{2} \int_{-1/2}^{1/2} \left(|t| - \frac{1}{2} \right)^2 \phi''(t) dt.$$

(ii) Deduce that

$$\left| \int_{-1/2}^{1/2} \phi(t) dt - \phi(0) \right| \leq \frac{1}{8} \int_{-1/2}^{1/2} |\phi''(t)| dt.$$

Hint: For the first part, write

$$\frac{1}{2} \int_{-1/2}^{1/2} \left(|t| - \frac{1}{2} \right)^2 \phi''(t) dt = \frac{1}{2} \int_0^{1/2} \left(t - \frac{1}{2} \right)^2 (\phi''(t) + \phi''(-t)) dt$$

and integrate by parts until done.

The exercise shows that

$$\left| \int_{1/2}^{\infty} \phi(t) dt - \sum_{n=1}^{\infty} \phi(n) \right| \leq \frac{1}{8} \int_{1/2}^{\infty} |\phi''(t)| dt.$$

This can be useful, although dealing with the error term $\frac{1}{8} \int_{1/2}^{\infty} |\phi''|$ can be inconvenient. It is too bad that we do not have a nice simple inequality like $\phi(n) \leq \int_{n-1/2}^{n+1/2} \phi$, so we could get a bound for our sum in terms of an integral with no error term, as in our first integral test above . . .

But wait. If $\phi'' \geq 0$ then the first part of the exercise above shows that we *do* have

$$\phi(n) \leq \int_{n-1/2}^{n+1/2} \phi(t) dt;$$

we have been forgetting about the basic principle that convexity is where inequalities come from!

Or in any case I forgot this in the first version of this proof; maybe some of us have never seen it stated explicitly. Here it is:

Convexity is the source of many useful inequalities.

Now you've seen it; try to remember this the next time you have an inequality you wish you could prove.

We should mention a technicality: By definition the continuous function ϕ is *convex* if

$$\phi(x) \leq \frac{\phi(x+t) + \phi(x-t)}{2}$$

for all x, t such that the interval $[x-t, x+t]$ is contained in the domain of ϕ . If it happens that ϕ'' is continuous, then ϕ is convex if and only if $\phi'' \geq 0$.

But convexity is actually a weaker condition: The function $\phi(x) = |x|$ shows that a convex function need not have a second derivative.

And in fact convexity is exactly the condition we need here, not $\phi'' \geq 0$; if ϕ is convex then

$$\phi(n) = 2 \int_0^{\frac{1}{2}} \phi(n+t) dt \leq 2 \int_0^{\frac{1}{2}} \frac{\phi(n+t) + \phi(n-t)}{2} dt = \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \phi(t) dt,$$

a much simpler argument than the previous exercise.

Let us summarize the things we have proved here for future reference; they can all be useful in various places:

Theorem 6.2.0. *Suppose that $\phi : (0, \infty) \rightarrow (0, \infty)$ is continuous.*

(i) *If ϕ is nonincreasing then*

$$\int_1^{\infty} \phi(t) dt \leq \sum_{n=1}^{\infty} \phi(n) \leq \int_0^{\infty} \phi(t) dt.$$

(ii) *If ϕ'' is continuous then*

$$\left| \int_{1/2}^{\infty} \phi(t) dt - \sum_{n=1}^{\infty} \phi(n) \right| \leq \frac{1}{8} \int_{1/2}^{\infty} |\phi''(t)| dt.$$

(iii) *If ϕ is convex then*

$$\sum_{n=1}^{\infty} \phi(n) \leq \int_{1/2}^{\infty} \phi(t) dt.$$

We will see in the next section that the difference between the $\int_0^{\infty} \phi$ in part (i) and the $\int_{1/2}^{\infty} \phi$ in part (iii) can make a big difference.

6.3. Proof Using Liouville's Theorem

One can use Liouville's Theorem to prove Euler's formula; we define

$$P(z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2} \right),$$

and it follows from Exercise 3.8 in Chapter 3 that we need only show that P is an entire function such that $P(z+1) = -P(z)$ and $|P(z)| \leq ce^{\pi|\operatorname{Im}(z)|}$.

(The exercise then shows that $P(z) = \alpha \sin(\pi z)$ for some constant α ; as noted above we can then divide both sides by z and then consider the limit as $z \rightarrow 0$ to show that $\alpha = 1$.)

The fact that P is an entire function is proved as in Section 6.1; this follows from Lemma 6.1.4, since the sum $\sum_{n=1}^{\infty} |z^2/n^2|$ converges uniformly for z in any compact subset of the plane. (The fact that Lemma 6.1.4 is used in both proofs is not a surprise; the lemma is simply the basic tool one uses in checking convergence of infinite products.)

The proof that $P(z+1) = -P(z)$ is not hard. Let

$$P_N(z) = z \prod_{n=1}^N \left(1 - \frac{z^2}{n^2}\right).$$

A little rearrangement shows that

$$P_N(z) = \frac{(-1)^N}{(N!)^2} \prod_{n=-N}^N (z - n).$$

Hence

$$P_N(z+1) = \frac{(-1)^N}{(N!)^2} \prod_{n=-N}^N (z - (n-1)) = \frac{(-1)^N}{(N!)^2} \prod_{n=-N-1}^{N-1} (z - n),$$

so that if $z \notin \mathbb{Z}$ we have

$$\frac{P_N(z+1)}{P_N(z)} = \frac{z - (-N-1)}{z - N}.$$

This shows that $P_N(z+1)/P_N(z) \rightarrow -1$ and hence $P(z+1) = -P(z)$ for $z \notin \mathbb{Z}$; the fact that $P(z+1) = -P(z)$ for all z follows by continuity.

The interesting part is the proof that $|P(z)| \leq ce^{\pi|\operatorname{Im}(z)|}$. It seems natural to take the logarithm, since sums can be easier to deal with than products. So we begin by noting that

$$\log \left(\left| \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \right| \right) = \sum_{n=1}^{\infty} \log \left(\left| 1 - \frac{z^2}{n^2} \right| \right) \leq \sum_{n=1}^{\infty} \log \left(1 + \frac{|z|^2}{n^2} \right).$$

(At first it looks like we threw away too much in that last inequality, so that we could not hope to get a good enough bound starting here. But we are only going to be using this inequality for z close to the imaginary axis; for such z the inequality is close to equality.) Now we estimate that sum by comparing it to an integral, as in the previous section:

Fix $z \neq 0$ and let $\phi(t) = \log(1 + |z|^2/t^2)$. In the first version of the proof I used part (i) of Theorem 6.1.7:

$$\sum_{n=1}^{\infty} \log\left(1 + \frac{|z|^2}{n^2}\right) \leq \int_0^{\infty} \log\left(1 + \frac{|z|^2}{t^2}\right) dt = \pi|z|.$$

(The integral can be evaluated by elementary calculus; see below for details.)

This shows that

$$\prod_{n=1}^{\infty} \left(1 + \frac{|z|^2}{n^2}\right) \leq e^{\pi|z|},$$

which looks like what we want. Unfortunately there is a missing factor on the left-hand side; all we have proved here is that

$$|P(z)| \leq \pi|z|e^{\pi|z|}.$$

Now, this last inequality can in fact be used to prove Euler's formula, but the simple Exercise 3.8 does not quite apply, we need a somewhat clumsier argument.

A miracle happened when I realized I should be applying the convexity of ϕ : The improvement in the estimate of the sum by the integral was exactly enough to take care of that extra factor of $z!$

You can easily verify that $\phi'' \geq 0$. Hence part (iii) of Theorem 6.1.7 shows that

$$\sum_{n=1}^{\infty} \log\left(1 + \frac{|z|^2}{n^2}\right) \leq \int_{1/2}^{\infty} \log\left(1 + \frac{|z|^2}{t^2}\right) dt.$$

Now a simple integration by parts shows that

$$\int \log\left(1 + \frac{|z|^2}{t^2}\right) dt = t \log\left(1 + \frac{|z|^2}{t^2}\right) + 2|z| \arctan\left(\frac{t}{|z|}\right),$$

at least for $t \in (0, \infty)$. We need to figure out how this antiderivative behaves as $t \rightarrow +\infty$. Since $\log(1) = 0$ and $\log'(1) = 1$, it follows that $\log(1 + |z|^2/t^2)$ is approximately $|z|^2/t^2$ for large t , so that $t \log(1 + |z|^2/t^2) \rightarrow 0$. It is clear that $\arctan(t/|z|) \rightarrow \pi/2$, so we obtain

$$\begin{aligned} \int_{1/2}^{\infty} \log\left(1 + \frac{|z|^2}{t^2}\right) dt &= \pi|z| - (\log(1 + 4|z|^2)/2 + 2|z| \arctan(1/(2|z|))) \\ &\leq \pi|z| - \log(4|z|^2)/2 \\ &= \pi|z| - \log(|z|) - \log(2) \\ &< \pi|z| - \log(|z|). \end{aligned}$$

That $-\log(|z|)$ saves the day, showing that

$$|P(z)| \leq \pi e^{\pi|z|}.$$

Since $P(z+2) = P(z)$, it follows (see Exercise 6.4 below) that

$$|P(z)| \leq ce^{\pi|\operatorname{Im}(z)|},$$

and this proves Euler's formula, as noted at the start of this section.

Exercises

6.4. Suppose that $P : \mathbb{C} \rightarrow \mathbb{C}$ is continuous, $P(z+2) = P(z)$, and $|P(z)| \leq e^{\pi|z|}$. Show that there exists c such that $|P(z)| \leq ce^{\pi|\operatorname{Im}(z)|}$ for all z .

6.5. Prove Wallis' formula:

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{4n^2}{(2n-1)(2n+1)} \right) = \left(\frac{2}{1} \right) \left(\frac{2}{3} \right) \left(\frac{4}{3} \right) \left(\frac{4}{5} \right) \cdots.$$

Note. After you prove that $\frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{4n^2}{(2n-1)(2n+1)} \right)$ there is still a *little* bit of work to do in showing that $\prod_{n=1}^{\infty} \left(\frac{4n^2}{(2n-1)(2n+1)} \right) = \left(\frac{2}{1} \right) \left(\frac{2}{3} \right) \left(\frac{4}{3} \right) \left(\frac{4}{5} \right) \cdots!$ Yes, it is clear that

$$\frac{4n^2}{(2n-1)(2n+1)} = \frac{2n}{2n-1} \frac{2n}{2n+1},$$

but that is not *quite* enough by itself, there is a little argument required. For example, it is also true that

$$\frac{4n^2}{(2n-1)(2n+1)} = \frac{4n}{2n-1} \frac{n}{2n+1},$$

but it does not follow from that that

$$\frac{\pi}{2} = \left(\frac{4}{1} \right) \left(\frac{1}{3} \right) \left(\frac{8}{3} \right) \left(\frac{2}{5} \right) \cdots;$$

in fact *that* product does not converge.

6.6. Find $\sum_{n=1}^{\infty} \frac{1}{n^2}$, using the infinite series for $\cot(\pi z)$.

The next exercise shows that, at least for continuous functions, the definition of convexity we gave above is equivalent to an apparently stronger condition. (We take the domain to be all of \mathbb{R} just for convenience.)

6.7. Suppose that $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies

$$\phi(x) \leq \frac{\phi(x+y) + \phi(x-y)}{2}$$

for all $x, y \in \mathbb{R}$. Show that

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

whenever $0 \leq t \leq 1$.

Hint: First note that the hypothesis is equivalent to

$$\phi\left(\frac{x+y}{2}\right) \leq \frac{\phi(x) + \phi(y)}{2},$$

which gives the conclusion for $t = 1/2$. Since ϕ is continuous, you may assume that $t = k/2^n$ for non-negative integers n, k . Now the result follows by induction on n ; for example, the identity

$$\frac{\frac{x+y}{2} + y}{2} = \frac{1}{4}x + \frac{3}{4}y$$

shows how the case $t = 1/4$ follows from the case $t = 1/2$:

$$\begin{aligned} \phi\left(\frac{1}{4}x + \frac{3}{4}y\right) &= \phi\left(\frac{\frac{x+y}{2} + y}{2}\right) \\ &\leq \frac{\phi\left(\frac{x+y}{2}\right) + \phi(y)}{2} \leq \frac{\frac{\phi(x) + \phi(y)}{2} + \phi(y)}{2} = \frac{1}{4}\phi(x) + \frac{3}{4}\phi(y). \end{aligned}$$

Preliminaries for the Picard Theorems (excerpt)

(Notations established previously: If Ω is an open subset of the plane then $\text{Aut}(\Omega)$ is the group of invertible holomorphic maps from Ω to itself; now if $p : \Omega \rightarrow V$ is a holomorphic covering map then $\text{Aut}(\Omega, p)$ is the subgroup $\{\phi \in \text{Aut}(\Omega) : p \circ \phi = p\}$.)

[...]

Theorem 19.4.3. *Suppose Ω is simply connected and $p_j : \Omega \rightarrow V_j$ is a holomorphic covering map for $j = 0, 1$. Then V_1 and V_0 are conformally equivalent if and only if $\text{Aut}(\Omega, p_0)$ and $\text{Aut}(\Omega, p_1)$ are conjugate in $\text{Aut}(\Omega)$.*

Proof. [...] □

For $r > 1$ define $A_r = \{z : 1 < |z| < r\}$. The next theorem gives a demonstration of the power of Theorem 19.4.3.

Theorem 19.4.4. *If $1 < r_1 < r_2$ then A_{r_1} and A_{r_2} are not conformally equivalent.*

Recall that two square matrices A and B are said to be *similar* if there exists a matrix C such that $A = CBC^{-1}$. Recall as well the trivial fact that similar matrices have the same eigenvalues. And recall from Section 10.7 that $\text{Aut}(\Pi^+)$ is isomorphic to $SL_2(\mathbb{R})/G$, where $SL_2(\mathbb{R})$ is the group of all real 2×2 matrices with determinant 1 and G is the subgroup $\{I, -I\}$.

Proof. For $\delta > 0$ define $\phi_\delta \in \text{Aut}(\Pi^+)$ by $\phi_\delta(z) = \delta z$; note that the matrix in $SL_2(\mathbb{R})$ corresponding to ϕ_δ (or rather one of the two matrices in $SL_2(\mathbb{R})$ corresponding to ϕ_δ) is

$$M_\delta = \begin{bmatrix} \delta^{1/2} & 0 \\ 0 & \delta^{-1/2} \end{bmatrix}.$$

Theorem 19.1.2 shows that for $j = 1, 2$ there is a holomorphic covering map $p_j : \Pi^+ \rightarrow A_{r_j}$ such that $\text{Aut}(\Pi^+, p_j)$ is generated by ϕ_{δ_j} , where $\delta_j > 1$ and $\delta_1 \neq \delta_2$. We need only show that $\text{Aut}(\Pi^+, p_1)$ is not conjugate to $\text{Aut}(\Pi^+, p_2)$ in $\text{Aut}(\Pi^+)$.

Suppose to the contrary that $\text{Aut}(\Pi^+, p_1) = \chi \text{Aut}(\Pi^+, p_2) \chi^{-1}$ for some $\chi \in \text{Aut}(\Pi^+)$. Then $\chi \circ \phi_{\delta_2} \circ \chi^{-1}$ must be a generator of $\text{Aut}(\Pi^+, \delta_1)$, so that $\chi \circ \phi_{\delta_2} \circ \chi^{-1} = \phi_{\delta_1}$ or $\chi \circ \phi_{\delta_2} \circ \chi^{-1} = \phi_{1/\delta_1}$. Hence M_{δ_2} must be similar to one of the four matrices $\pm M_{\delta_1}, \pm M_{1/\delta_1}$. But this is impossible because similar matrices have the same eigenvalues. \square

[...]

The Picard Theorems (excerpt)

Notation will be as in the previous two chapters: Γ is the group of all $\phi \in \text{Aut}(\Pi^+)$ which can be written in the form $\phi(z) = (az + b)/(cz + d)$ with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$, $\Gamma(2)$ is the subgroup of Γ where a and d are odd and b and c are even, and $\lambda : \Pi^+ \rightarrow \mathbb{C} \setminus \{0, 1\}$ is a holomorphic covering map with $\text{Aut}(\Pi^+, \lambda) = \Gamma(2)$.

The fact that λ is a holomorphic covering map from Π^+ onto $\mathbb{C} \setminus \{0, 1\}$ makes the Little Picard Theorem very simple:

Theorem 20.0 (Little Picard Theorem). *If f is a nonconstant entire function then the range $f(\mathbb{C})$ is either \mathbb{C} or $\mathbb{C} \setminus \{\alpha\}$ for some $\alpha \in \mathbb{C}$.*

In other words, if the range of an entire function omits two complex values then the function is constant.

Proof. Suppose that f is an entire function and f omits the two values $\alpha, \beta \in \mathbb{C}$, with $\alpha \neq \beta$. We need to show that f is constant.

Considering $(f - \alpha)/(\beta - \alpha)$ in place of f , we may assume that the two values omitted are 0 and 1; thus

$$f(\mathbb{C}) \subset \mathbb{C} \setminus \{0, 1\}.$$

Now Theorem 19.0.5 shows that there exists $\tilde{f} : \mathbb{C} \rightarrow \Pi^+$ with $\lambda \circ \tilde{f} = f$. Liouville's Theorem shows that $1/(i + \tilde{f})$ is constant; hence \tilde{f} is constant and so f is constant. \square

I think it was Littlewood who said that this would be the world's shortest PhD thesis. A one-line proof of a very deep theorem. (Of course the

existence of a covering map from the upper half-plane onto $\mathbb{C} \setminus \{0, 1\}$ was not entirely trivial.)

Now for the Big Picard Theorem. There are various statements that commonly go by this name:

[...]

As previously we set $\mathbb{D}' = \mathbb{D} \setminus \{0\}$. [The Big Picard Theorem] is equivalent to the following:

Theorem 20.3 (Big Picard Theorem). *If $f \in H(\mathbb{D}')$ and $f(\mathbb{D}') \subset \mathbb{C} \setminus \{0, 1\}$ then f has a pole or a removable singularity at 0.*

[...]

Traditional proofs of the Big Picard Theorem are totally unlike the usual proof of the Little Picard Theorem, but we shall see that it is possible to give a proof of the Big Picard Theorem that is very much like a generalization of the proof of the Little Picard Theorem; where the first proof used nothing but the most basic properties of covering maps, the second proof uses the slightly more sophisticated results from Chapter 19.

The idea behind the proof may be clearer if we start with two results dealing with more familiar notions like harmonic conjugates and exponentials: Our proof of the Big Picard Theorem is to the traditional proof of the Little Picard Theorem exactly as the proof of Theorem B below is to the proof of Theorem A:

Theorem A. *If $u : \mathbb{C} \rightarrow \mathbb{R}$ is a bounded harmonic function then u is constant.*

Proof. Since \mathbb{C} is simply connected, there exists a real-valued harmonic function v such that $u + iv$ is holomorphic. Let $h = e^{u+iv}$. Then h is an entire function, and h is bounded, since $|h| = e^u$. Thus h is constant, and hence $u = \log |h|$ is constant. \square

Theorem B. *If $f : \mathbb{D}' \rightarrow \mathbb{R}$ is harmonic and bounded then u extends to a function harmonic in \mathbb{D} .*

The problem with Theorem B is that we cannot say that u has a harmonic conjugate, since \mathbb{D}' is not simply connected. (This is precisely analogous to the problem with Theorem 20.3: We cannot say that there exists $\tilde{f} : \mathbb{D}' \rightarrow \Pi^+$ with $\lambda \circ \tilde{f} = f$ because \mathbb{D}' is not simply connected.)

One can give a slightly informal proof of Theorem B as follows: There does exist a “multi-valued” harmonic conjugate v (this is an informal way of saying that there is a holomorphic function $u + iv$ defined in some disk

contained in \mathbb{D}' which admits unrestricted continuation in \mathbb{D}'). Now it is not hard to see that in fact there exists a multi-valued harmonic conjugate v and a real number c such that any two branches of v differ by a multiple of c . If $c = 0$ we are done, so we assume that $c \neq 0$, and define $h = e^{2\pi(u+iv)/c}$. The fact that any two branches of v differ by a multiple of c shows that h is an actual single-valued holomorphic function. As in the proof of Theorem A we see that h is bounded; hence h has a removable singularity, and hence u has a removable singularity at the origin as well.

If the “informal” parts of that proof bother you don’t worry, we will give a more formal version soon. The corresponding informal proof of Theorem 20.3 is this: Although we cannot assert that there exists $\tilde{f} : \mathbb{D}' \rightarrow \Pi^+$ with $\lambda \circ \tilde{f} = f$, there does exist a “multi-valued function” \tilde{f} with this property (that is, an \tilde{f} defined in some disk contained in \mathbb{D}' which admits unrestricted continuation in \mathbb{D}'). It turns out that there exists $\phi \in \Gamma(2)$ such that any two branches of \tilde{f} differ by a power of ϕ , and, since ϕ cannot be elliptic, we know that there exists a nonconstant bounded function $h \in H(\Pi^+)$ such that $h \circ \phi = h$. It follows that $h \circ \tilde{f}$ is single-valued, and hence has a removable singularity at the origin.

It is now quite plausible, and not hard to prove using the results in the previous chapter, that in fact f has a (possibly infinite) limit at the origin, so that in particular f does not have an essential singularity (for the details here see the formal proof below).

The actual proof is going to use a covering-map argument in place of the analytic continuation [. . .]