Introduction

What is discrete differential geometry. A new field of *discrete differential geometry* is presently emerging on the border between differential and discrete geometry; see, for instance, the recent book Bobenko-Schröder-Sullivan-Ziegler (2008). Whereas classical differential geometry investigates smooth geometric shapes (such as surfaces), and discrete geometry studies geometric shapes with finite number of elements (such as polyhedra), discrete differential geometry aims at the development of discrete equivalents of notions and methods of smooth surface theory. The latter appears as a limit of refinement of the discretization. Current interest in this field derives not only from its importance in pure mathematics but also from its relevance for other fields: see the lecture course on discrete differential geometry in computer graphics by Desbrun-Grinspun-Schröder (2005), the recent book on architectural geometry by Pottmann-Asperl-Hofer-Kilian (2007), and the mathematical video on polyhedral meshes and their role in geometry, numerics and computer graphics by Janzen-Polthier (2007).

For a given smooth geometry one can suggest many different discretizations with the same continuous limit. Which is the best one? From the theoretical point of view, one would strive to preserve fundamental properties of the smooth theory. For applications the requirements of a good discretization are different: one aims at the best approximation of a smooth shape, on the one hand, and at on the other hand, its representation by a discrete shape with as few elements as possible. Although these criteria are different, it turns out that intelligent theoretical discretizations are distinguished also by their good performance in applications. We mention here as an example the discrete Laplace operator on simplicial surfaces ("cotan formula") introduced by Pinkall-Polthier (1993) in their investigation of discrete minimal surfaces, which turned out to be extremely important in geometry processing where it found numerous applications, e.g., Desbrun-Meyer-Alliez (2002), Botsch-Kobbelt (2004), to name but two. Another example is the theory of discrete minimal surfaces by Bobenko-Hoffmann-Springborn (2006), which turned out to have striking convergence properties: these discrete surfaces approximate their smooth analogs with all derivatives.

A straightforward way to discretize differential geometry would be to take its analytic description in terms of differential equations and to apply standard methods of numerical analysis. Such a discretization makes smooth problems amenable to numerical methods. Discrete differential geometry does not proceed in this way. Its main message is:

Discretize the whole theory, not just the equations.

This means that one should develop a discrete theory which respects fundamental aspects of the smooth one; which of the properties are to be taken into account is a nontrivial problem. The discrete geometric theory turns out to be as rich as its smooth counterpart, if not even richer. In particular, there are many famous existence theorems at the core of the classical theory. Proper discretizations open a way to make them constructive. For now, the statement about the richness of discrete differential geometry might seem exaggerated, as the number of supporting examples is restricted (although steadily growing). However, one should not forget that we are at the beginning of the development of this discipline, while classical differential geometry has been developed for centuries by the most outstanding mathematicians.

As soon as one takes advantage of the apparatus of differential equations to describe geometry, one naturally deals with parametrizations. There is a part of classical differential geometry dealing with parametrized surfaces, coordinate systems and their transformations, which is the content of the fundamental treatises by Darboux (1914-27) and Bianchi (1923). Nowadays one associates this part of differential geometry with the theory of integrable systems; see Fordy-Wood (1994), Rogers-Schief (2002). Recent progress in discrete differential geometry has led not only to the discretization of a large body of classical results, but also, somewhat unexpectedly, to a better understanding of some fundamental structures at the very basis of the classical differential geometry and of the theory of integrable systems. It is the aim of this book to provide a systematic presentation of current achievements in this field.

Returning to the analytic description of geometric objects, it is not surprising that remarkable discretizations yield remarkable discrete equations.

The main message of discrete differential geometry, addressed to the integrable systems community, becomes:

Discretize equations by discretizing the geometry.

The profundity and fruitfulness of this principle will be demonstrated throughout the book.

Integrability. We will now give a short overview of the historical development of the *integrability* aspects of discrete differential geometry. The classical period of surface theory resulted in the beginning of the 20th century in an enormous wealth of knowledge about numerous special classes of surfaces, coordinate systems and their transformations, which is summarized in extensive volumes by Darboux (1910, 1914-27), Bianchi (1923), etc. One can say that the local differential geometry of special classes of surfaces and coordinate systems has been completed during this period. Mathematicians of that era have found most (if not all) geometries of interest and knew nearly everything about their properties. It was observed that special geometries such as minimal surfaces, surfaces with constant curvature, isothermic surfaces, orthogonal and conjugate coordinate systems, Ribaucour sphere congruences, Weingarten line congruences etc. have many similar features. Among others we mention Bäcklund and Darboux type transformations with remarkable permutability properties investigated mainly by Bianchi, and the existence of special deformations within the class (associated family). Geometers realized that there should be a unifying fundamental structure behind all these common properties of quite different geometries; and they were definitely searching for this structure; see Jonas (1915) and Eisenhart (1923).

Much later, after the advent of the theory of integrable systems in the the last quarter of the 20th century, these common features were recognized as being associated with the integrability of the underlying differential equations. The theory of integrable systems (called also the theory of solitons) is a vast field in mathematical physics with a huge literature. It has applications in fields ranging from algebraic and differential geometry, enumerative topology, statistical physics, quantum groups and knot theory to nonlinear optics, hydrodynamics and cosmology.

The most famous models of this theory are the Korteweg-de Vries (KdV), the nonlinear Schrödinger and the sine-Gordon equations. The KdV equation played the most prominent role in the early stage of the theory. It was derived by Korteweg-de Vries (1895) to describe the propagation of waves in shallow water. Localized solutions of this equation called *solitons* gave the whole theory its name. The birth of the theory of solitons is associated with the famous paper by Gardner-Green-Kruskal-Miura (1967), where the inverse scattering method for the analytic treatment of the KdV equation was invented. The sine-Gordon equation is the oldest integrable equation and the most important one for geometry. It describes surfaces with constant negative Gaussian curvature and goes back at least to Bour (1862) and Bonnet (1867). Many properties of this equation which are nowadays associated with integrability were known in classical surface theory.

One can read about the basic structures of the theory of integrable systems in numerous books. We mention just a few of them: Newell (1985), Faddeev-Takhtajan (1986), Hitchin-Segal-Ward (1999), Dubrovin-Krichever-Novikov (2001).

The most commonly accepted features of integrable systems include:

> In the theory of solitons nonlinear integrable equations are usually represented as a compatibility condition of a linear system called the *zero curvature representation* (also known as Lax or Zakharov-Shabat representations). Various analytic methods of investigation of soliton equations (like the inverse scattering method, algebro-geometric integration, asymptotic analysis, etc.) are based on this representation.
>
> Another indispensable feature of integrable systems is that they possess *Bäcklund-Darboux transformations*. These special transformations are often used to generate new solutions from the known ones.
>
> It is a characteristic feature of soliton (integrable) partial differential equations that they appear not separately but are always organized in *hierarchies of commuting flows* .

It should be mentioned that there is no commonly accepted mathematical definition of integrability (as the title of the volume "What is integrability?", Zakharov (1991), clearly demonstrates). Different scientists suggest different properties as the defining ones. Usually, one refers to some additional structures, such as those mentioned above. In this book, we propose an algorithmic definition of integrability given in terms of the system itself.

In both areas, in differential geometry and in the theory of integrable systems, there were substantial efforts to *discretize* the fundamental structures.

In the theory of solitons the problem is to discretize an integrable differential equation preserving its integrability. Various approaches to this problem began to be discussed in the soliton literature starting from the mid-1970s. The basic idea is to discretize the zero curvature representation of the smooth system, i.e., to find proper discrete analogues of the corresponding linear problems. This idea appeared first in Ablowitz-Ladik (1975).

Its various realizations based on the bilinear method, algebro-geometric integration, integral equations, R-matrices, and Lagrangian mechanics were developed in Hirota (1977a,b), Krichever (1978), Date-Jimbo-Miwa (1982-3), Quispel-Nijhoff-Capel-Van der Linden (1984), Faddeev-Takhtajan (1986), Moser-Veselov (1991) (here we give just a few representative references). An encyclopedic presentation of the Hamiltonian approach to the problem of integrable discretization is given in Suris (2003).

The development of this field led to a progress in various branches of mathematics. Pairs of commuting difference operators were classified in Krichever-Novikov (2003). Laplace transformations of difference operators on regular lattices were constructed in Dynnikov-Novikov (1997); see also Dynnikov-Novikov (2003) for a related development of a discrete complex analysis on triangulated manifolds. A characterization of Jacobians of algebraic curves based on algebro-geometric methods of integration of difference equations was given in Krichever (2006).

From discrete to smooth. In differential geometry the original idea of an intelligent discretization was to find a simple explanation of sophisticated properties of smooth geometric objects. This was the main motivation for the early work in this field documented in Sauer (1937, 1970) and Wunderlich (1951). The modern period began with the works by Bobenko-Pinkall (1996a,b) and by Doliwa-Santini (1997), where the relation to the theory of integrable systems was established. During the next decade this area witnessed a rapid development reflected in numerous publications. In particular, joint efforts of the main contributors to this field resulted in the books Bobenko-Seiler (1999) and Bobenko-Schröder-Sullivan-Ziegler (2008). The present book gives a comprehensive presentation of the results of discrete differential geometry of parametrized surfaces and coordinate systems along with its relation to integrable systems. We leave the detailed bibliographical remarks to the notes at the end of individual chapters of the book.

Discrete differential geometry deals with multidimensional discrete nets (i.e., maps from the regular cubic lattice $\mathbb{Z}^m$ into $\mathbb{R}^N$ or some other suitable space) specified by certain geometric properties. In this setting, discrete surfaces appear as two-dimensional layers of multidimensional discrete nets, and their transformations correspond to shifts in the transversal lattice directions. A characteristic feature of the theory is that all lattice directions are considered on an equal footing with respect to the defining geometric properties. Due to this symmetry, discrete surfaces and their transformations become indistinguishable. We associate such a situation with the *multidimensional consistency* (of geometric properties, and of the equations which serve for their analytic description). In each case, multidimensional consistency, and therefore the existence and construction of multidimensional discrete nets, is seen to rely on some incidence theorems of elementary geometry.

Conceptually, one can think of passing to a continuous limit by refining the mesh size in some of the lattice directions. In these directions the net converges to smooth surfaces whereas those directions that remain discrete correspond to transformations of the surfaces (see Figure 0.1). Differential geometric properties of special classes of surfaces and their transformations arise in this way from (and find their simple explanation in) the elementary geometric properties of the original multidimensional discrete nets. In particular, difficult classical theorems about the permutability of Bäcklund-Darboux type transformations (Bianchi permutability) for various geometries follow directly from the symmetry of the underlying discrete nets, and are therefore built in to the very core of the theory. Thus the transition from differential geometry to elementary geometry via discretization (or, in the opposite direction, the derivation of differential geometry from the discrete differential geometry) leads to enormous conceptual simplifications, and the true roots of the classical theory of special classes of surfaces are found in various incidence theorems of elementary geometry. In the classical differential geometry these elementary roots remain hidden. The limiting process taking the discrete master theory to the classical one is inevitably accompanied by a break of the symmetry among the lattice directions, which always leads to structural complications.

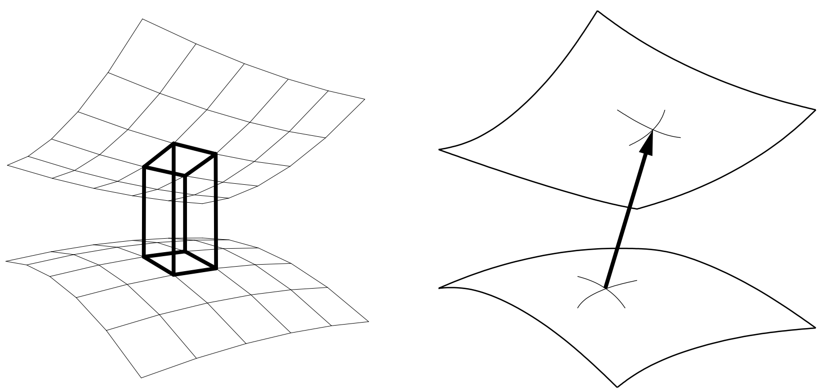

Figure 0.1. From the discrete master theory to the classical theory: surfaces and their transformations appear by refining two of three net directions.

Finding simple discrete explanations for complicated differential-geometric theories is not the only outcome of this development. It is well known that differential equations which analytically describe interesting special classes of surfaces are integrable (in the sense of the theory of integrable systems), and conversely, many interesting integrable systems admit a differential-geometric interpretation. Having identified the roots of integrable differential geometry in the multidimensional consistency of discrete nets, one is led to a new (geometric) understanding of integrability itself. First of all, we adopt the point of view that the central role in this theory is played by *discrete integrable systems*. In particular, a great variety of integrable differential equations can be derived from several fundamental discrete systems by performing different continuous limits. Further, and more importantly, we arrive at the idea that the multidimensional consistency of discrete equations may serve as a constructive and almost algorithmic *definition* of their integrability. This idea was introduced in Bobenko-Suris (2002a) (and independently in Nijhoff (2002)). This definition of integrability captures enough structure to guarantee such traditional attributes of integrable equations as zero curvature representations and Bäcklund-Darboux transformations (which, in turn, serve as the basis for applying analytic methods such as inverse scattering, finite gap integration, Riemann-Hilbert problems, etc.). A continuous counterpart (and consequence) of multidimensional consistency is the well-known fact that integrable systems never appear alone but are organized into hierarchies of commuting flows.

This conceptual view of discrete differential geometry as the basis of the theory of surfaces and their transformations as well as of the theory of integrable systems is schematically represented in Figure 0.2.

This general picture looks very natural, and there is a common belief that the smooth theories can be obtained in a limit from the corresponding discrete ones. This belief is supported by formal similarities of the corresponding difference and differential equations. However one should not underestimate the difficulty of the convergence theorems required for a rigorous justification of this philosophy.

Solutions to similar problems are substantial in various areas of differential geometry. Classical examples to be mentioned here are the fundamental results of Alexandrov and Pogorelov on the metric geometry of polyhedra and convex surfaces (see Alexandrov (2005) and Pogorelov (1973)). Alexandrov's theorem states that any abstract convex polyhedral metric is uniquely realized by a convex polyhedron in Euclidean 3-space. Pogorelov proved the corresponding existence and uniqueness result for convex Riemannian metrics by approximating smooth surfaces by polyhedra. Another example is Thurston's approximation of conformal mappings by circle packings (see Thurston (1985)). The theory of circle packings (see the book by Stephenson (2005)) is treated as discrete complex analysis. At the core of this theory is the Koebe-Andreev-Thurston theorem which states that any simplicial decomposition of a sphere can be uniquely (up to Möbius transformations)

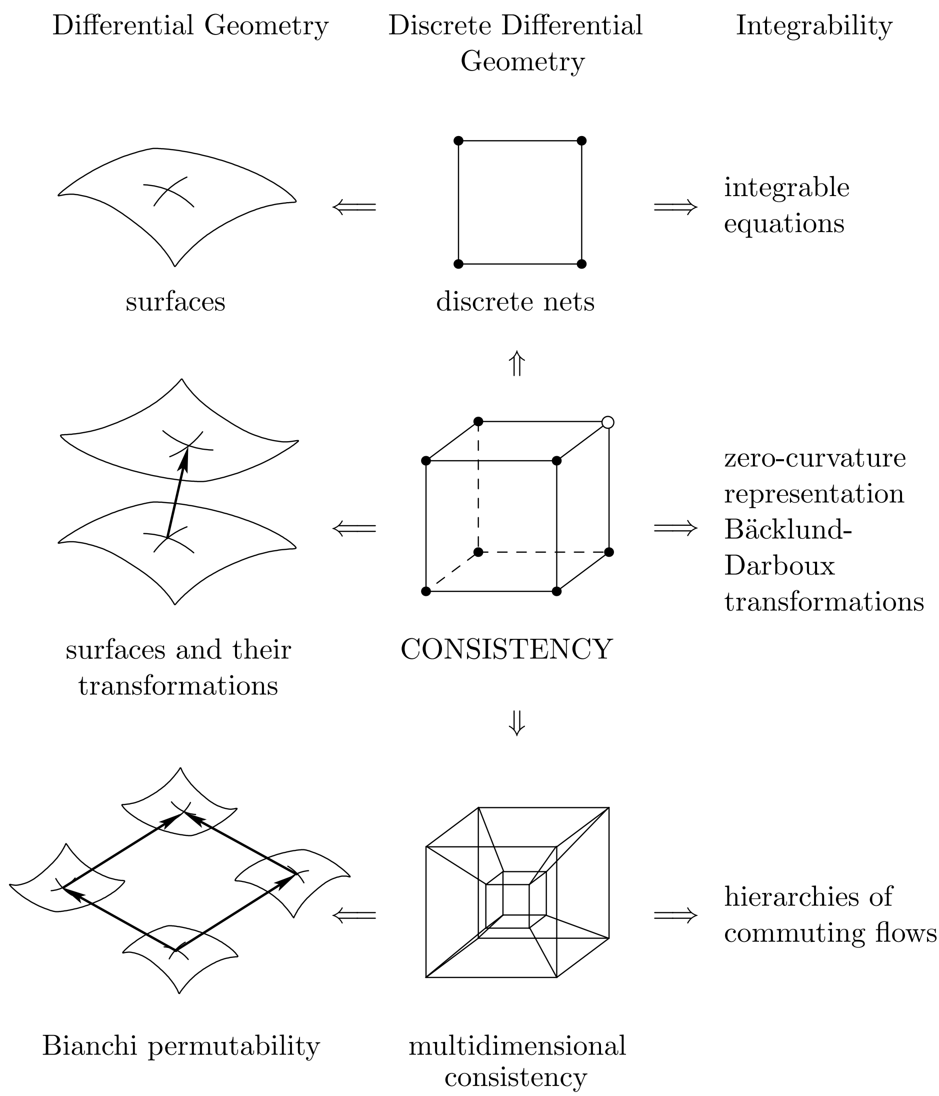

Figure 0.2. The consistency principle of discrete differential geometry as conceptual basis of the differential geometry of special surfaces and of integrability.

realized by a circle packing. According to Rodin-Sullivan (1987) the conformal Riemann map can be approximated by such circle packings (even with all the derivatives as shown by He-Schramm (1998)).

The first convergence results concerning the transition from the middle to the left column in Figure 0.2 (from discrete to smooth differential geometry) were proven in Bobenko-Matthes-Suris (2003, 2005). This turns the general philosophy of discrete differential geometry into a firmly established mathematical truth for several important classes of surfaces and coordinate systems, such as conjugate nets, orthogonal nets, including general curvature line parametrized surfaces, surfaces with constant negative Gaussian curvature, and general asymptotic line parametrized surfaces. For some other classes, such as isothermic surfaces, the convergence results are yet to be rigorously established.

The geometric way of thinking about discrete integrability has also led to novel concepts in that theory. An immanent and important feature of various surface parametrizations is the existence of distinguished points, where the combinatorics of coordinate lines changes (like umbilic points, where the combinatorics of the curvature lines is special). In the discrete setup this can be modelled by *quad-graphs*, which are cell decompositions of topological two-manifolds with quadrilateral faces; see Bobenko-Pinkall (1999). Their elementary building blocks are still quadrilaterals, but they are attached to one another in a manner which can be more complicated than in $\mathbb{Z}^2$. A systematic development of the theory of integrable systems on quad-graphs has been undertaken in Bobenko-Suris (2002a). In the framework of the multidimensional consistency, quad-graphs can be realized as quad-surfaces embedded in a higher-dimensional lattice $\mathbb{Z}^d$. This interpretation proves to be fruitful for the analytic treatment of integrable systems on quad-graphs, such as the integral representation of discrete holomorphic functions and the isomonodromic Green's function in Bobenko-Mercat-Suris (2005).

Structure of this book. The structure of this book follows the logic of this Introduction. We start in Chapter 1 with an overview of some classical results from surface theory, focusing on transformations of surfaces. The brief presentation in this chapter is oriented towards the specialists already familiar with the differential geometry of surfaces. The geometries considered include general conjugate and orthogonal nets in spaces of arbitrary dimension, Koenigs nets, asymptotic nets on general surfaces, as well as special classes of surfaces, such as isothermic ones and surfaces with constant negative Gaussian curvature. There are no proofs in this chapter. The analytic proofs are usually tedious and can be found in the original literature. The discrete approach which we develop in the subsequent chapters will lead to conceptually transparent and technically much simpler proofs.

In Chapter 2 we define and investigate discrete analogs of the most fundamental objects of projective differential geometry: conjugate, Koenigs and asymptotic nets and line congruences. For instance, discrete conjugate nets are just multidimensional nets consisting of planar quadrilaterals. Our focus is on the idea of multidimensional consistency of discrete nets and discrete line congruences.

According to Klein's Erlangen Program, the classical geometries (Euclidean, spherical, hyperbolic, Möbius, Plücker, Lie etc.) can be obtained by restricting the projective geometry to a quadric. In Chapter 3 we follow this approach and show that the nets and congruences defined in Chapter 2 can be restricted to quadrics. In this way we define and investigate discrete analogs of curvature line parametrized surfaces and orthogonal nets, and give a description of discrete asymptotic nets within the framework of Plücker line geometry.

Imposing simultaneously several constraints on (discrete) conjugate nets, one comes to special classes of surfaces. This is the subject of Chapter 4. The main examples are discrete isothermic surfaces and discrete surfaces with constant curvature. From the analytic point of view, these are represented by 2-dimensional difference equations (as opposed to the 3-dimensional equations in Chapters 2, 3).

Then in Chapter 5 we develop an approximation theory for hyperbolic difference systems, which is applied to derive the classical theory of smooth surfaces as a continuum limit of the discrete theory. We prove that the discrete nets of Chapters 2, 3, and 4 approximate the corresponding smooth geometries of Chapter 1 and simultaneously their transformations. In this setup, Bianchi's permutability theorems appear as simple corollaries.

In Chapter 6 we formulate the concept of multidimensional consistency as a defining principle of integrability. We derive basic features of integrable systems such as the zero curvature representation and Bäcklund-Darboux transformations from the consistency principle. Moreover, we obtain a complete list of 2-dimensional integrable systems. This classification is a striking application of the consistency principle.

In Chapters 7 and 8 these ideas are applied to discrete complex analysis. We study Laplace operators on graphs and discrete harmonic and holomorphic functions. Linear discrete complex analysis appears as a linearization of the theory of circle patterns. The consistency principle allows us to single out distinguished cases where we obtain more detailed analytic results (like Green's function and isomonodromic special functions).

Finally, in Chapter 9 we give for the reader's convenience a brief introduction to projective geometry and the geometries of Lie, Möbius, Laguerre and Plücker. We also include a number of classical incidence theorems relevant to discrete differential geometry.

How to read this book. Different audiences (see the Preface) should read this book differently, as suggested in Figure 0.3. Namely, Chapter 1 on classical differential geometry is addressed to specialists working in this field. It is thought to be used as a short guide in the theory of surfaces and their transformations. This is the reason why Chapter 1 does not contain proofs and exercises. Students who use this book for a graduate course and have less or no experience in differential geometry should not read this chapter and should start directly with Chapter 2 (and consult Chapter 1 at the end of the course, after mastering the discrete theory). This was the way how this course was taught in Berlin and München, with no knowledge of differential geometry required. Those interested primarily in applications of discrete differential geometry are advised to browse through Chapters 2–4 and perhaps also Chapter 5 and to pick up the problems they are particularly interested in. Almost all results are supplied with elementary geometric formulations accessible for nonspecialists. Finally, researchers with interest in the theory of integrable systems could start reading with Chapter 6 and consult the previous chapters for better understanding of the geometric origin of the consistency approach to integrability.

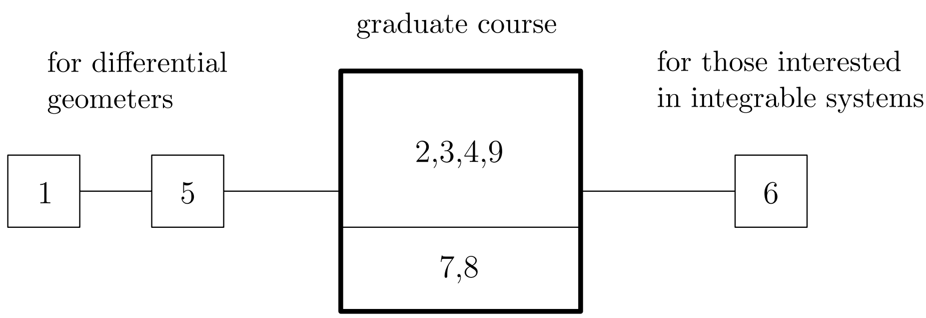

Figure 0.3. A suggestion for the focus on chapters, depending on the readers background.

Acknowledgements. Essential parts of this book are based on results obtained jointly with Vsevolod Adler, Tim Hoffmann, Daniel Matthes, Christian Mercat, Ulrich Pinkall, Helmut Pottmann, and Johannes Wallner. We warmly thank them for inspiring collaboration.

We are very grateful to Adam Doliwa, Udo Hertrich-Jeromin, Nicolai Reshetikhin, Wolfgang Schief, Peter Schröder, Boris Springborn, Sergey Tsarev, Alexander Veselov, Günter Ziegler for enjoyable and insightful discussions on discrete differential geometry which influenced the presentation in this book.

Special thanks go to Emanuel Huhnen-Venedey and Stefan Sechelmann for their help with the preparation of the manuscript and with the figures.

The support of the Deutsche Forschungsgemeinschaft (DFG) is gratefully acknowledged. During the work on this book the authors were partially supported by the DFG Research Unit "Polyhedral Surfaces" and the DFG Research Center MATHEON "Mathematics for key technologies" in Berlin.