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Preface

This book contains an account of the history of the representation theory of finite groups, presented through an analysis of the published work of the four principal contributors to the theory in its formative stages: Ferdinand Georg Frobenius, William Burnside, Issai Schur, and Richard Brauer. The impetus for the project comes from my collaboration with the late Irving Reiner on a series of expository books on representation theory, and a gradually awakening interest on the part of both of us in the history of the subject.

The articles surveyed in the book were published mainly in the last two decades of the nineteenth century, and in the first half of the twentieth century. An introductory chapter contains an outline of some research on algebra and number theory involving characters of finite abelian groups, which appeared earlier in the nineteenth century, and was part of the tradition inherited by Frobenius and his successors. Later chapters contain reports on the main papers of the principals, and related work done at about the same time by others: Jacques Deruyts, Theodor Molien, Élie Cartan, Alfred Young, Alfred Loewy, Heinrich Maschke, Emmy Noether, Emil Artin, Tadasi Nakayama, Hsio-Fu Tuan, and Michio Suzuki, in more or less chronological order.

For the most important papers, enough of the mathematics is included to make it possible for a reader to follow the thread of the argument in detail. While many of the proofs of the main theorems are given today using new methods, the first proofs, obtained a century ago, are still interesting, and complete the historical picture by showing exactly how the creators of the theory used the mathematics that was available to them at the time. While the steps taken from one point to the next in the presentation of the mathematics are set down much as they appear in the original papers, I have taken the liberty of rewriting the arguments in modern mathematical terminology, and have sometimes given modern versions of parts of them.

The mathematical parts of the book are intended to be accessible to students as well as professional mathematicians and others with a basic knowledge of algebra. Background material needed to understand the papers is summarized in Chapter I, §1, and throughout the book as needed.

By concentrating on the mathematics in the original papers, I have not given as complete a survey of the historical setting of the mathematical discoveries as a reader might desire. Fortunately a full historical account of the development of the theory of representations of finite groups and associative algebras has been given in a series of articles by Thomas Hawkins, and frequent references to them are included.

The creators of representation theory all had long and productive mathematical careers, and made important contributions to other areas of mathematics besides the representation theory of finite groups. Each of them deserves a more complete mathematical biography than it has been possible for me to include here. To this end, however, sections containing biographical sketches, excerpts from correspondence, together with information about their teaching and relationships with their contemporaries, have been included in some of the chapters. Many quotations from the work under discussion have been inserted; they contain statements of some of the main results, remarks about motivation for the research, and comments on its relation to other contemporary investigations. All four of the principals are counterexamples to the myth that first-rate mathematics is done only by persons in their youth. Now that the collected works of Frobenius, Schur, and Brauer have been published, hopefully soon to be followed by the publication of the collected papers of Burnside, the full range of their mathematical work is readily available.

There are several reasons for undertaking an historical study of this particular branch of mathematics. We have passed the centennial of the publication of the first papers on characters and representations of finite groups, by Frobenius, in 1896-97. The principal contributors at the early stages, Frobenius, Burnside, Schur, and Brauer, each made discoveries which had a profound influence on the direction taken in research in algebra and number theory throughout most of the twentieth century. Their work deserves to be better known, along with something about their personal circumstances. The subject is flourishing today, with applications to other parts of mathematics as well as physics and chemistry. Its current vigor reinforces Schur's enthusiastic statement about it, in his inaugural lecture following his election to the Berlin Academy of Sciences in 1922: "What fascinated me so extraordinarily in these investigations was the fact that here, in the midst of a standstill that prevailed in other areas in the theory of forms, there arose a new and fertile chapter of algebra which is also important for geometry and analysis and which is distinguished by great beauty and perfection." The origins of this well-established area of modern mathematics, however, bear little resemblance to its content and methodology at the present time, so that it is a matter of historical interest, and mathematical interest as well, to follow the early steps in the development of the subject from this point of view.

We include a few remarks that are intended to guide the reader towards an historical overview, based on the survey to follow, of the first half century of the representation theory of finite groups. The prehistory, encompassing developments in nineteenth century mathematics which led to group representation theory, is summarized in Chapter I, Chapter V (on the work of Jacques Deruyts), and in the articles of Hawkins [**172**], [**173**], and [**174**]. For the history itself, we make two suggestions.

The first suggestion is to follow the changes in approach to the subject at different stages in the development, and the reasons for them. First one has Frobenius's theory of characters and his factorization of the group determinant (see Chapter II). This was followed by the classification of matrix representations of finite groups, first by Burnside (see Chapter III), and then by Schur, using a different method (see Chapter IV). Finally there was the absorption of the representation theory of finite groups over arbitrary fields in the theory of representations of nonsemisimple algebras, by Brauer, Noether, and Nakayama (Chapters VI and VII).

The second suggestion is to follow the problems, conjectures, and applications considered by Frobenius, Burnside, Schur, and Brauer. It will be seen that substantial parts of the theoretical development came about as a result of an attempt to solve a problem, or to make an application. We cite a few examples. Frobenius invented characters in connection with the problem of factoring the group determinant. He applied his theory of induced characters to the calculation of characters of the symmetric group, and to determining the structure of Frobenius groups (see Chapter II). Burnside raised the problem of existence of nonabelian finite simple groups of a given order, and in particular whether a nonabelian simple group of odd order can exist, or whether a nonabelian simple group whose order contains two distinct prime factors can exist. He solved the second problem with his p^aq^b-Theorem, while the first problem was solved more than fifty years later by Walter Feit and John Thompson (see Chapter III). Maschke and Burnside considered the splitting field problem, and solved it in special cases. Schur settled it for solvable groups using his theory of the Schur index (see Chapter IV). Schur classified the polynomial representations of the general linear group, using, among other things, Frobenius's work on the characters of the symmetric group (see Chapter V). Brauer, in collaboration with Hasse and Noether, proved Dickson's conjecture concerning division algebras over algebraic number fields (see Chapter VI). Brauer proved Artin's conjecture concerning L-series with general group characters, as a consequence of what is now called the Brauer Induction Theorem. He found several other applications of the Induction Theorem, including a solution of the splitting field problem in the general case.

The importance of challenging problems for the health of a branch of mathematics has long been recognized. In his lecture *Mathematical Problems*,[1] David Hilbert said, "As long as a branch of science offers an abundance of problems, so long is it alive; a lack of problems foreshadows extinction or the cessation of independent development." The representation theory of finite groups has always had an abundance of challenging problems. The fact that Frobenius, Burnside, Schur, and Brauer not only raised many of them, but had the strength and resourcefulness to solve some of them, is a measure of their greatness.

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[1]Bull. Amer. Math. Soc. **8** (1902), 437-479, translated by Mary Winston Newson from the original, which appeared in the Göttinger Nachrichten, 1900, 253-297.

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