Preface to the Second Edition

My exposition in the first edition of *The Life and Times of the Central Limit Theorem* concludes with what has been termed the classical period in the life of the Central Limit Theorem, the work of Lyapunov. The modern era in its life begins with Lindeberg, [**1**, pp. 1–23], [**2**, pp. 221–225]. For discussion of the modern era I recommend the papers of Feller [**1**, pp. 800–832] and Le Cam [**1**, pp. 78–92], comments on Le Cam's paper by Trotter [**1**, pp. 92–93], Doob [**1**, pp. 93–94], Pollard [**1**, pp. 94–95], and a rejoinder by Le Cam [**2**, pp. 95–96]. These papers are reprinted in the second part of this second edition.

For those who teach a course in probability whose objective is to prove the Central Limit Theorem of interest in the aforementioned exchange is commentary on the characteristic function approach employed by Lyapunov versus the "very simple" proof, as Le Cam [**1**] describes it, given by Lindeberg.[1] Lindeberg's approach is discussed in books by Billingsley [**1**], Breiman [**1**], Krickeberg [**1**], and Thomasian [**1**].

The revisions to the first edition of *The Life and Times of the Central Limit Theorem* consist of corrections of errata that came to my attention, refinements prompted by review comments—Seneta [**1**], Chang [**1**], Stigler [**1**], Peak [**1**], Clarke [**1**], Eisenhart [**4**]—the inclusion of two key papers by Lyapunov (which are not readily accessible), and the aforementioned papers and comments by Feller, Le Cam, Trotter, Doob, and Pollard.

I should like to express my appreciation to Professor Peter Duren of the University of Michigan for encouraging me to revisit the history of the Central Limit Theorem and recommending that it be made available to contemporary generations.

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[1] Simple, like beauty, is in the eyes of the beholder. Uspensky [**1**, p. 284] comments: "Here we shall follow Liapounoff [Lyapunov]; for his method of proof has the advantage of simplicity even compared with more recent proofs, of which that given by J.W. Lindeberg deserves special mention."

Preface to the First Edition

A number of random phenomena are found to exhibit the following structure. The outcome of a process is subject to the action of a very large number of independently acting random factors, each of which has only a slight effect on the course of the process as a whole. The independently acting random factors can be identified with random variables $X_1, X_2, \ldots, X_n$ and their cumulative effect can be identified with the sum $S_n = X_1 + X_2 + \cdots + X_n$. The problem that arises is to determine the probabilistic behavior of S_n (that is, the distribution function of S_n), where very little is known about the behavior of $X_1, X_2, \ldots, X_n$. Describe the behavior of the process as a whole when very little is known about the numerous independently acting random factors which determine the behavior of the process as a whole; a problem which at first sight seems of such overwhelming difficulty as to be immune to the most powerful weapons found in the arsenal of mathematical analysis. Yet, clues were uncovered and mathematical methods were developed. It is most remarkable that the distribution function of S_n can be shown to be approximately normal, if it is assumed that the random factors described by $X_1, X_2, \ldots, X_n$ are numerous, mutually independent, and the effect of each factor on the cumulative sum S_n is very slight. The Central Limit Theorem (in the classical sense) is the generic name of a class of theorems which give, in precise mathematical terms, conditions under which the distribution function of a suitably standardized sum of independent random variables is approximately normal. This theorem is one of the most remarkable results in all of mathematics and is a dominating personality in the world of probability and statistics.

In this monograph I trace the fascinating history of the Central Limit Theorem from its origins to the early twentieth century. In doing so I have kept in mind the needs of nonspecialists in probability and statistics and I have attempted to present an account which would be accessible and interesting to students, teachers of secondary and college mathematics, and professionals in fields which make use of probabilistic and statistical methods. Of course, specialists in probability and statistics are cordially invited to partake as well.

The emergence of the Central Limit Theorem in an abstract mathematical form was a syncretic process in which there were fused such ingredients as the development of the hypothesis of elementary errors as a foundation for the view that the probability law governing the error distribution arising from a measurement process is normal, investigations connected with determining the probability that the arithmetic mean of the errors of observation arising from a measurement process is contained within given limits, Laplace's formulation and study of the problem of determining the probability that the mean inclination of the orbits of a given number of comets to a given plane is between given limits, de Moivre's pioneering work on the approximation of binomial sums by integrals of e^{-t^2}, and the development of the use of integrals of e^{-t^2} as an approximation tool. Chapters 1–6 trace the development and interaction of these factors.

Chebyshev ushered in the mathematically abstract phase in the life of the Central Limit Theorem. His formulation of an "abstract" Central Limit Theorem and its proof by his ingenious method of moments are discussed in Chapter 7. In the concluding Chapter 8 the brilliant work of Chebyshev's pupils Alexander Mikhailovich Lyapunov and Andrei Andreevich Markov, who ventured forth in strikingly different directions, is examined. Markov further developed and refined Chebyshev's method of moments as his mathematical instrument while Lyapunov overcame grave analytic difficulties to establish a Central Limit Theorem of great generality by fashioning an approach from the tools of classical analysis. The work of these distinguished scholars closed the classical period in the life of the Central Limit Theorem and is the stopping point of this study.

The references in the text are related to the bibliography in such a way that David [**1**, p. 110], for example, constitutes a reference to page 110 of David's work *Games, Gods and Gambling*, which appears in the bibliography as the first title listed under the name of David.

Professors Warren Hirsch and Stephen Willoughby (New York University) read certain parts of an early draft of the manuscript and I am most grateful for their valuable comments. I should also like to express my gratitude to Dr. Churchill Eisenhart (National Bureau of Standards), who provided me with some extremely difficult to obtain references, and to Mr. Henry Birnbaum, Director of Libraries at Pace University, and his staff, who offered me the facilities of the Pace University Library and their friendly help. For permission to use portraits from Professor David Eugene Smith's portfolio of famous mathematicians I am indebted to Mr. Kenneth A. Lohf, Librarian for Rare Books and Manuscripts, Columbia University.

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