Translator's Introduction

Topology before Poincaré

Without much exaggeration, it can be said that only *one* important topological concept came to light before Poincaré. This was the Euler characteristic of surfaces, whose name stems from the paper of Euler (1752) on what we now call the *Euler polyhedron formula*. When writing in English, one usually expresses the formula as

$$V - E + F = 2,$$

where

$$V = \text{number of vertices,}$$
$$E = \text{number of edges,}$$
$$F = \text{number of faces,}$$

of a convex polyhedron or, more generally, of a subdivided surface homeomorphic to the two-dimensional sphere $\mathbb{S}^2$. In Poincaré (for example, in §16 of his *Analysis situs* paper) one finds the French version

$$S - A + F = 2,$$

where S stands *sommets* and A for *arêtes*.

The formula is usually proved by showing that the quantity $V - E + F$ remains invariant under all possible changes from one subdivision to another. It follows that $V - E + F$ is an invariant of any surface, not necessarily homeomorphic to $\mathbb{S}^2$. This invariant is now called the *Euler characteristic*. Thus $\mathbb{S}^2$ has Euler characteristic 2, whereas the torus has Euler characteristic 0.

Between the 1820s and 1880s, several different lines of research were found to converge to the Euler characteristic.

(1) The classification of polyhedra, following Euler (and even before him, Descartes). Here the terms *edges* and *faces* have their traditional meaning in Euclidean geometry.

(2) The classification of surfaces of constant curvature, where the "edges" are now geodesic segments. Here one finds that surfaces of positive Euler characteristic have positive curvature, those of zero Euler characteristic have zero curvature, and those of negative Euler characteristic have negative curvature.

(3) More generally, the *average* curvature of a smooth surface is positive for positive Euler characteristic, zero for zero Euler characteristic, and negative for negative Euler characteristic. This follows from the *Gauss–Bonnet theorem* of Gauss (1827) and Bonnet (1848).

(4) The study of algebraic curves, revolutionized by Riemann (1851) when he modelled each complex algebraic curve by a surface—its *Riemann surface*.

Under this interpretation, a number that Abel (1841) called the *genus* of an algebraic curve turns out to depend on the Euler characteristic of its Riemann surface. Genus g is related to Euler characteristic χ by

$$\chi = 2 - 2g.$$

Moreover, the genus g has a simple geometric interpretation as the number of *holes* in the surface. Thus $\mathbb{S}^2$ has genus 0 and the torus has genus 1.

(5) The topological classification of surfaces, by Möbius (1863). Möbius studied closed surfaces in $\mathbb{R}^3$ by slicing them into simple pieces by parallel planes. By this method he found that every such surface is homeomorphic to a standard surface with g holes. Thus closed surfaces in $\mathbb{R}^3$—that is, all orientable surfaces—are classified by their genus, and hence by their Euler characteristic. (Despite his discovery of the nonorientable surface that bears his name, Möbius did not classify nonorientable surfaces. This was done by Dyck (1888).)

(6) The study of *pits, peaks, and passes* on surfaces in $\mathbb{R}^3$ by Cayley (1859) and Maxwell (1870). A family of parallel planes in $\mathbb{R}^3$ intersects a surface S in curves we may view as curves of *constant height* (contour lines) on S. If the planes are taken to be in general position, and the surface is smooth, then S has only finitely many pits, peaks, and passes relative to the height function. It turns out that

$$\text{number of peaks} - \text{number of passes} + \text{number of pits}$$

is precisely the Euler characteristic of S.

All of these ideas admit generalizations to higher dimensions, but the only substantial step towards topology in arbitrary dimensions before Poincaré was that of Betti (1871). Betti was inspired by Riemann's concept of *connectivity of surfaces* to define connectivity numbers, now known as *Betti numbers* $P_1, P_2, \ldots$, in all dimensions. The connectivity number of a surface S may be defined as the maximum number of disjoint closed curves that can be drawn on S without separating it. This number P_1 is equal to the genus of S, hence it is just the Euler characteristic in disguise.

For a three-dimensional manifold M one can also consider the maximum number P_2 of disjoint closed surfaces in M that fail to separate M as the *two-dimensional connectivity number* of M. However, the idea of separation fails to explain the *one-dimensional connectivity* of M, since no finite set of curves can separate M. Instead, one takes the maximum number of curves that can lie in M without forming the *boundary of a surface* in M. (For a surface M, this maximum is the same as Riemann's connectivity number.) Betti defined P_m similarly, in a manifold M of arbitrary dimension, as the maximum number of m-dimensional pieces of M that do not form the boundary of a connected $(m+1)$-dimensional piece of M. Thus Betti brought the concept of *boundary* into topology in order to generalize Riemann's concept of connectivity.

This was Poincaré's starting point, but he went much further, as we will see.

Poincaré before topology

In the introduction to his first major topology paper, the *Analysis situs*, Poincaré (1895) announced his goal of creating of creating an n-dimensional geometry. As he memorably put it:

> ...geometry is the art of reasoning well from badly drawn figures; however, these figures, if they are not to deceive us, must satisfy certain conditions; the proportions may be grossly altered, but the relative positions of the different parts must not be upset.

Because "positions must not be upset," Poincaré sought what Leibniz called *Analysis situs*, a geometry of position, or what we now call topology. He cited as precedents the work of Riemann and Betti, and his own experience with differential equations, celestial mechanics, and discontinuous groups. Of these, I believe the most influential was the last, which stems from his work (and the related work of Klein) on Fuchsian functions in early 1880s.

Poincaré's major papers on Fuchsian functions may be found translated into English in Poincaré (1985). The ideas relevant to topology may be summarized as follows.

One considers a group Γ of translations of the plane which is *fixed-point-free* (that is, nonidentity group elements move every point) and *discontinuous* (that is, there is a nonzero lower bound to the distance that each point is moved by the nonidentity elements). A special case is where the plane is $\mathbb{C}$ and Γ is generated by two Euclidean translations in different directions. Generally the *plane* is the *hyperbolic plane* $\mathbb{H}^2$, which may be modeled by either the upper half plane of $\mathbb{C}$ or the open unit disk $\{z : |z| < 1\}$.

In either case, Γ has a *fundamental domain* $\mathcal{D}$ which is a polygon, and the plane is filled without overlapping by its translates $\gamma\mathcal{D}$ for $\gamma \in \Gamma$. In the special case where the plane is $\mathbb{C}$, the fundamental domain can be taken to be a parallelogram, the translations of which in the two directions fill $\mathbb{C}$. In the hyperbolic case, $\mathcal{D}$ is a polygon with $4g$ sides for some $g \geq 2$, and Γ is generated by $2g$ elements, each of which translates $\mathcal{D}$ to a polygon with just one side in common with $\mathcal{D}$.

It follows that the quotient $\mathbb{C}/\Gamma$ in the special case is a torus (obtained by identifying opposite sides of the fundamental parallelogram), while in the hyperbolic case the quotient $\mathbb{H}^2/\Gamma$ is a surface of genus $g \geq 2$, obtained by identifying sides of the fundamental $4g$-gon in certain pairs.[1]

Each pair of identified sides come together on the quotient surface as a closed curve. For example, the identified sides of a fundamental parallelogram become two closed curves a and b on the torus $\mathbb{C}/\Gamma$, as shown in Figure 1. The curve a

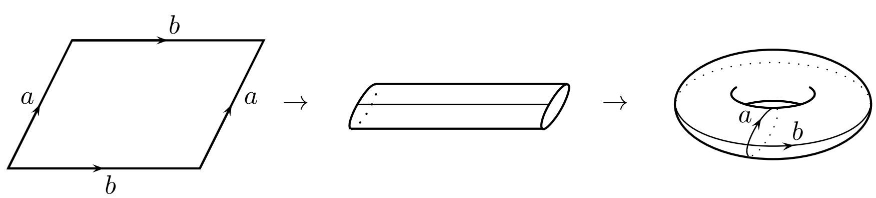

FIGURE 1. Constructing a torus from a fundamental parallelogram.

corresponds to the translation of $\mathbb{C}$ with direction and length of the sides marked a of the parallelogram, and curve b similarly corresponds to the translation with

[1]The connection with Fuchsian functions, often mentioned by Poincaré but not very relevant to topology, is that there are functions f that are *periodic* with respect to substitutions from the group Γ; that is, $f(\gamma(z)) = f(z)$ for all $\gamma \in \Gamma$. In the special case of the torus, these functions are the famous elliptic functions.

direction and length of the sides marked b. Thus one is led to think of a *group of curves* on the torus, isomorphic to the group of translations of $\mathbb{C}$ generated by the translations a and b.

This group is what Poincaré later called the *fundamental group*, and we can see why he viewed it as a group of *substitutions*—in this case, translations of $\mathbb{C}$—rather than as a group of (homotopy classes of) closed curves with fixed origin on the torus, as we now do. Indeed, Poincaré in the 1880s much preferred to work with fundamental polygons in the plane, and it was Klein (1882) who realized that insight could be gained by looking at the quotient surface instead. In particular, Klein used the Möbius classification of surfaces into canonical forms to find canonical *defining relations* for Fuchsian groups.

Here is how we find the defining relation in the case of the torus (in which case one relation suffices). Clearly, if we perform the translations a, b, a^{-1}, b^{-1} of $\mathbb{C}$ in succession, where a^{-1} and b^{-1} denote the inverses of a and b, respectively, the whole plane arrives back at its starting position. We write this relation symbolically as

$$aba^{-1}b^{-1} = 1,$$

where 1 denotes the identity translation. On the torus surface, $aba^{-1}b^{-1}$ denotes a closed curve that bounds a parallelogram (reversing the process shown in Figure 1), and hence this curve is contractible to a point. This is the *topological* interpretation of the relation

$$aba^{-1}b^{-1} = 1.$$

With either interpretation it is quite easy to show that *all* relations between a and b follow from the single relation $aba^{-1}b^{-1} = 1$. This is why we call $aba^{-1}b^{-1} = 1$ the *defining* relation of the torus group.

In a similar way, we find the defining relation of any surface group by cutting the surface along closed curves so as to produce a polygon. Equating the sequence of edges in the boundary of this polygon to 1 then gives a valid relation, and again it is not hard to show that the relation thus obtained is a defining relation. The curves most commonly used for the surface S_g of genus g are called $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$, and cutting S_g along them produces a polygon with boundary

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}.$$

Consequently, the group of S_g may be generated by elements $a_1, b_1, \ldots, a_g, b_g$ and it has defining relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1.$$

Figure 2 shows the curves a_1, b_1, a_2, b_2 on the surface S_2, and the resulting polygon.

The *Analysis situs* paper

Poincaré set the agenda for his 1895 *Analysis situs* paper with a short announcement, Poincaré (1892), a translation of which is also included in this volume. In it he raises the question whether the Betti numbers suffice to determine the topological type of a manifold, and introduces the fundamental group to further illuminate this question. He gives a family of three-dimensional manifolds, obtained as quotients of $\mathbb{R}^3$ by certain groups with a cube as fundamental region, and shows that certain of these manifolds have the same Betti numbers but *different* fundamental

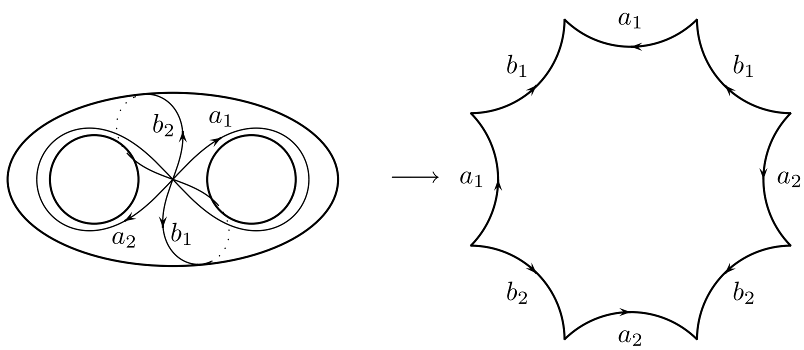

FIGURE 2. Genus 2 surface and its fundamental polygon

groups. It follows, assuming that the fundamental group is a topological invariant, that the Betti numbers do not suffice to distinguish three-dimensional manifolds.

In *Analysis situs*, Poincaré develops these ideas in several directions.

(1) He attempts to provide a new foundation for the Betti numbers in a rudimentary *homology theory*, which introduces the idea of computing with topological objects (in particular, adding, subtracting, testing for linear independence). As Scholz (1980), p. 300, puts it:

> The first phase of algebraic topology, inaugurated by Poincaré, is characterized by the fact that its algebraic relations and operations always deal with *topological objects* (submanifolds).

(2) Using his homology theory, he discovers a *duality theorem* for the Betti numbers of an n-dimensional manifold:

$$P_m = P_{n-m} \quad \text{for} \quad m = 1, 2, \ldots, n-1.$$

In words: "the Betti numbers equidistant from the ends are equal." He later called this the *fundamental theorem* for Betti numbers (p. 96).[2]

(3) He generalizes the Euler polyhedron formula to arbitrary dimensions and situates it in his homology theory.

(4) He constructs several three-dimensional manifolds by identifying faces of polyhedra, observing that this leads to natural presentations of their fundamental groups by generators and relations.

(5) Recognizing that the fundamental group first becomes important for three-dimensional manifolds, Poincaré asks whether it suffices to distinguish between them. He is not able to answer this question.

Analysis situs is rightly regarded as the origin of algebraic topology, because of Poincaré's construction of homology theory and the fundamental group. The fundamental group is the more striking of the two, because it is a blatantly abstract structure and generally noncommutative, yet surprisingly easy to grasp via generators and relations. Homology theory reveals an algebraic structure behind the bare Betti numbers and Euler characteristics of Poincaré's predecessors, but it is not easy to say what this structure really is. Indeed, Poincaré did not realize that the Betti numbers are only part of the story, and he had to write Supplements 1

[2]All page numbers referenced in Poincaré's topology papers are the page numbers in *this* volume, not the page numbers in the original papers.

and 2 to *Analysis situs* before the so-called torsion coefficients came to light. And it was only in 1925 that Emmy Noether discovered the homology *groups*, which we now view as the proper home of the Betti and torsion numbers. She announced this discovery in Noether (1926).[3]

Thus, along with great breakthroughs, there is also confusion in *Analysis situs*. The confusion extends to the very subject matter of algebraic topology, the manifolds (or "varieties" as Poincaré calls them). His definitions suggest that he is generally thinking of differentiable manifolds, but most of his three-dimensional examples are defined combinatorially, by identifying faces of polyhedra, without checking their differentiability. His definition of Betti numbers needs revision, as he discovers in Supplements 1 and 2, and imprecise arguments are frequently used.

Another source of confusion concerns *simply connected manifolds*, and it ultimately led to the famous Poincaré conjecture in Supplement 5. In *Analysis situs*, §14, he defines a manifold to be *simply connected* if its fundamental group is trivial. It follows easily that a sphere of any dimension is simply connected, but Poincaré sometimes forgets that the converse is not obvious. On occasion, he assumes that any simply connected manifold is homeomorphic to a sphere (for example, on p.111), and on other occasions he even assumes (wrongly) that a region with trivial *homology* is simply connected (for example, on p. 39). These errors have been flagged by footnotes, some by the original editors of Volume VI of Poincaré's *Œuvres*, René Garnier and Jean Leray (the actual author of these footnotes is not identified), and some by myself.

Actually, it is not surprising that Poincaré made mistakes, given the novelty and subtle nature of the subject, and his style of work. Darboux (1952), p. lvi, describes Poincaré's working method as follows:

> Whenever asked to resolve a difficulty, his response came with the speed of an arrow. When he wrote a memoir, he drafted it all in one go, with only a few erasures, and did not return to what he had written.

It is perhaps wise to read Poincaré's memoirs in the same style: try to take in their general sweep without lingering too long over gaps and errors.

The five supplements

Poincaré (1899) wrote a *Complément a l'analysis situs* in response to the criticism of Heegaard (1898). Heegaard had become interested in three-dimensional manifolds, and he found an example where Poincaré's definition of Betti numbers comes into conflict with his duality theorem. To save the theorem, Poincaré revised his homology theory in the *Complément*, moving towards a more combinatorial theory in which manifolds are assumed to have a polyhedral structure and computing Betti numbers from the incidence matrices of this structure. He also arrived at a clearer explanation of the duality theorem in terms of the dual ("reciprocal") subdivision of a polyhedron, in which cells of dimension m in the original polyhedron correspond to cells of dimension $n-m$ in its dual. He concluded the *Complement* with a (rather unconvincing) attempt to prove that every differentiable manifold

[3] To be fair to Poincaré, he came close to discovering the first homology group H_1 as the abelianization of the fundamental group π_1 in *Analysis situs*, §13. There he considered the homologies obtained by allowing the generators of π_1 to commute and observed that "knowledge of these homologies immediately yields the Betti number P_1."

has a polyhedral subdivision. This theorem was first proved rigorously by Cairns (1934).

Poincaré may have thought that the *Complément* would complete his *Analysis situs* paper, but four more "complements" were to follow, as further gaps and loose ends came to light. For this reason, I have chosen to use the word "supplement" rather than "complement" (as Poincaré himself did on occasion).

In the second supplement, Poincaré (1900) dug more deeply into the problems of his original homology theory, uncovering the existence of *torsion*, and expanding his technique for computing Betti numbers to one that also computes torsion coefficients. He motivated his choice of the word "torsion" by showing that torsion occurs only in manifolds, such as the Möbius band, that are nonorientable and hence "twisted onto themselves" in some way (p. 134). When Emmy Noether built the Betti numbers and torsion numbers into the homology groups in 1926, the word "torsion" took up residence in algebra, much to the mystification of group theory students who were not informed of its origin in topology.

Having now attained some mastery of homology theory, Poincaré was emboldened to conjecture (p. 134) that *the three-dimensional sphere is the only closed three-dimensional manifold with trivial Betti and torsion numbers.* This was his first (and incorrect) version of the Poincaré conjecture.

The third and fourth supplements (Poincaré (1902a), Poincaré (1902b)) hark back to the first major application of Betti numbers to classical mathematics, the work of Picard (1889) on the connectivity of algebraic surfaces. An algebraic surface (or "algebraic function of two variables" as Picard called it) is taken to have complex values of the variables, hence it has four real dimensions. By a mixture of analytic and topological arguments, Picard succeeded in finding the first Betti number P_1 of algebraic surfaces, but he had less success in finding P_2. Invoking his new homology theory, Poincaré pushed on to P_2 in his fourth supplement (as far he needed to go, since $P_3 = P_1$, by Poincaré duality). Like Picard, Poincaré also appealed to results from analysis, in his case referring to his work on Fuchsian functions and non-Euclidean geometry from the early 1880s. An exposition of Poincaré's argument (in German) may be found in Scholz (1980), pp. 365–371.

The return to non-Euclidean geometry paid off unexpectedly in the fifth supplement, Poincaré (1904), with an interesting geometric algorithm (p. 200) to decide whether a curve on a surface is homotopic to a simple curve. Poincaré's result is that, in the case of genus greater than 1 where the surface can be given a non-Euclidean metric, a homotopy class contains a simple curve if and only if the *geodesic representative* is simple. Informally speaking, one can decide whether a curve κ is homotopic to simple curve by "stretching κ tight" on the surface and observing whether the stretched form of κ is simple. It seems likely that Poincaré's application of non-Euclidean geometry to surface topology inspired the later work of Dehn and Nielsen between 1910 and the 1940s, and the work of Thurston in the 1970s.

In the fifth supplement, the result on simple curves is just part of a rather meandering investigation of curves on surfaces, and their role in the construction of three-dimensional manifolds (*Heegaard diagrams*). In the final pages of the paper this investigation leads to a spectacular discovery: the *Poincaré homology sphere*. By pasting together two handlebodies of genus 2, H_1 and H_2 say, so that certain carefully chosen curves on H_1 become identified with canonical disk-spanning curves on H_2, Poincaré obtains a three-dimensional manifold V whose fundamental group

$\pi_1(V)$ he can write down in terms of generators and relations. To the reader's astonishment, the presentation of $\pi_1(V)$ implies relations that hold in the icosahedral group, so $\pi_1(V)$ *is nontrivial*. On the other hand, by allowing the generators of $\pi_1(V)$ to commute one finds (in our language) that $H_1(V) = 0$, so that V *is a closed three-dimensional manifold with trivial homology but nontrivial fundamental group* (hence V is *not* simply connected.)

The Poincaré homology sphere therefore refutes the conjecture made at the end of the second supplement, and it prompts the revised Poincaré conjecture (now known to be correct): *the three-sphere is the only closed three-dimensional manifold with trivial fundamental group.*

Poincaré prudently concludes the fifth supplement by remarking that investigation of the revised conjecture "would carry us too far away."

The Poincaré conjecture

In the *Analysis situs* and its five supplements, Poincaré opened up a vast new area of mathematics. It is not surprising that he left it incompletely explored. Among the most important gaps in his coverage were:

(1) The topological invariance of dimension, first proved by Brouwer (1911).
(2) The topological invariance of the Betti and torsion numbers, first proved by Alexander (1915).
(3) The existence of nonhomeomorphic three-dimensional manifolds with the same fundamental group, first proved by Alexander (1919).
(4) The existence of a polyhedral structure on every differentiable manifold, first proved by Cairns (1934).
(5) The existence of topological manifolds *without* a polyhedral structure, first proved by Kirby and Siebenmann around 1970, and published in Kirby and Siebenmann (1977).

But the deepest of the unsolved problems left by Poincaré was one he first thought was trivial—the Poincaré conjecture.

In the beginning, there was no conjecture, because Poincaré thought it obvious that a simply connected closed manifold was homeomorphic to a sphere. In the second supplement he came up with a sharper claim that was less obvious, hence in his view worth conjecturing: a closed manifold with trivial *homology* is homeomorphic to a sphere. But on occasions thereafter he forgot that there is a difference between trivial homology and trivial fundamental group. Finally, the discovery of Poincaré homology sphere in Supplement 5 opened his eyes to the real problem, and the Poincaré conjecture as we know it today was born: *a closed* 3*-manifold with trivial fundamental group is homeomorphic to the* 3*-sphere.*

The existence of homology spheres shows that three dimensions are more complicated than two, but just *how much* more complicated they are was not immediately clear. Further results on three-dimensional manifolds came with glacial slowness, and they often revealed new complications. Dehn (1910) found infinitely many homology spheres, and Whitehead (1935) found an *open* three-dimensional manifold that is simply connected but not homeomorphic to $\mathbb{R}^3$.

In the 1950s and 1960s there was at last some good news about three-dimensional manifolds: for example, they all have a polyhedral structure (Moise (1952)). The news did not include a proof of the Poincaré conjecture, however. Instead, progress on the conjecture came in higher dimensions, with a proof by Smale (1961) of the analogous conjecture for the n-dimensional sphere $\mathbb{S}^n$ for $n \geq 5$. Unfortunately, while three dimensions are harder than two, five are *easier* than three in some respects. So Smale's proof did not throw much light on the classical Poincaré conjecture, or on the analogous conjecture for $\mathbb{S}^4$ either.

The analogue of the Poincaré conjecture for four-dimensional manifolds was finally proved by Freedman (1982). Freedman's proof was a *tour de force* that simultaneously solved several longstanding problems about four-dimensional manifolds. That his approach worked at all was a surprise to many of his colleagues, and finding a similar approach to the classical Poincaré conjecture seemed out of the question.

Indeed, an entirely new approach to the Poincaré conjecture had already been taking shape in the hands of William Thurston in the late 1970s. Thurston, like Poincaré and Dehn, was interested in *geometric* realizations of manifolds, exemplified by the surfaces of constant curvature that realize all the topological forms of closed surfaces. He conjectured that all 3-manifolds may be realized in a similar, though more complicated, way. Instead of the three 2-dimensional geometries of constant curvature, one has eight "homogeneous" 3-dimensional geometries. (The eight geometries were discovered by Bianchi (1898), and rediscovered by Thurston.) And instead of a single geometry for each 3-manifold M one has a *decomposition* of M into finitely many pieces, each carrying one of the eight geometries.

Thurston's *geometrization conjecture* states that each closed and connected 3-manifold is homeomorphic to one with such a decomposition. The Poincaré conjecture follows from a special case of the geometrization conjecture for manifolds of positive curvature. For more details on the evolution of the Poincaré conjecture up to this point, see Milnor (2003).

Thurston was able to prove many cases of his geometrization conjecture, but geometrization seemed to run out of steam in the early 1980s. This was not entirely disappointing to some topologists, who still hoped for a proof of the Poincaré conjecture by purely topological methods. However, *more* geometry was to come, not less, and *differential* geometry at that. It was not enough to consider manifolds with "homogeneous" geometry; one had to consider manifolds with *arbitrary* smooth geometry, and to let the geometry "flow" towards homogeneity.

The idea of "flowing towards homogeneity" was initiated by Hamilton (1982), using what is called the *Ricci curvature flow*. Hamilton was able to show that the Ricci curvature flow works in many cases, but he was stymied by the formation of singularities in the general case. The difficulties were brilliantly overcome by Grigory Perelman in 2003. Perelman published his proof only in outline, in three papers posted on the Internet in 2002 and 2003, but experts later found that these papers contained all the ideas necessary to construct a complete proof of the geometrisation conjecture. Perelman himself, apparently sure that he would be vindicated, published nothing further and seems to have gone into seclusion.

For a very thorough and detailed account of Perelman's proof of the Poincaré conjecture, see Morgan and Tian (2007).

Comments on terminology and notation

Poincaré's topology papers pose an unusual problem for the translator, inasmuch as they contain numerous errors, both large and small, and misleading notation. My policy (which is probably not entirely consistent) has been to make only small changes where they help the modern reader—such as correcting obvious typographical errors—but to leave serious errors untouched except for footnotes pointing them out.

The most serious errors must be retained because they were a key stimulus to the development of Poincaré's thought in topology. As mentioned above, some of the five supplements exist only because of mistakes in the *Analysis situs* paper. It is more debatable whether one should retain annoying notation, such as the + sign Poincaré uses to denote the (generally noncommutative) group operation in the fundamental group, or the $\equiv$ sign and the word "congruence" he uses for the (asymmetric) boundary relation. I have opted to retain these, partly to assist readers who wish to compare the translation with the original papers, and also because they may be a clue to what Poincaré was thinking when he first applied algebra to topology.

I have also retained the word "conjugate" that Poincaré uses for the paired sides of a polyhedron, or the paired sides of a polygon, that are to be identified to form a manifold. One can replace "conjugate" by "identified" in many cases, but sometimes "paired" is better, so I thought it safest not to meddle. Fortunately, Poincaré does not use the word "conjugate" in the group-theoretic sense, even though the *concept* of conjugacy in group theory briefly arises.

On the other hand, I consistently use the word "manifold" where Poincaré uses "variety," and I call manifolds "orientable" where he calls them "two-sided" and "nonorientable" where he calls them "one-sided." The word "variety" always suggests algebraic geometry today, whereas Poincaré is really thinking about the topology of manifolds (even though many of them are in fact algebraic varieties), so "manifold" is the right word for the modern reader. The words "two-sided" and "one-sided" are less misleading than "variety," but Poincaré also uses them in a second sense, to describe separating and nonseparating curves on a surface, which have "two sides" and "one side," respectively. Calling manifolds "orientable" and "nonorientable" therefore removes a possible source of confusion.

Acknowledgements

I have drawn on the work of Sarkaria (1999), which gives a detailed summary, with some comments and corrections, of Poincaré's papers in topology. The commentary of Dieudonné (1989) on Poincaré's *Analysis situs* and its first two supplements is also useful, as is Chapter VII of Scholz (1980).

I translated *Analysis situs* and the the first, second, and fifth supplements in the 1970s, when I was first learning topology. At the time, I did not think there would be an opportunity to publish these papers, so I did not bother to translate the remaining two supplements, which were further from my interests at the time. Thirty years later, I was pleasantly surprised to be contacted by Andrew Ranicki and Cameron Gordon about their classic papers project. With their encouragement, I translated the two missing supplements and edited the whole sequence into roughly the form you see today. I am delighted that Poincaré's topological work is finally appearing in English, and I thank Andrew and Cameron for making this possible.

The publication of this book in the AMS/LMS History of Mathematics series is a great bonus, given that I did not originally expect to publish these translations at all. I am grateful to Rob Kirby for suggesting to me that the AMS might be interested, and to Ina Mette for enthusiastically taking up the suggestion. Finally, I thank Jennifer Wright Sharp for her extraordinary editing, which made an enormous number of improvements to the text.

John Stillwell
Monash University and University of San Francisco
July 2010