CHAPTER 1

Introduction

1.1. Subject

The study of the three-body problem began with Isaac Newton and his attempt, in the third volume of the *Principia*, to test his law of universal gravitation by using it to produce an accurate quantitative description of the orbital motion of the Moon. Newton's efforts met with mixed success and the problem soon gained a reputation for being extremely intractable, thereby attracting the attention of many of the world's most gifted mathematicians. Despite this attention, most of the early progress was achieved by means of approximate solutions, and almost a century had passed from the publication of the *Principia* before the first non-trivial solutions in closed-form were discovered: the now well-known particular solutions of Euler and Lagrange. A further century then elapsed before, in the late 1870s, the American mathematical astronomer George William Hill constructed, as a basis for his new lunar theory, the next family of periodic solutions: the variational orbits. Hill's outstanding work was initially slow to receive acclaim, but in the decades that followed it would inspire a number of mathematicians and astronomers to actively research the periodic solutions of the three-body problem. The most notable of these was Henri Poincaré, who, in 1890, devoted a large part of his prize-winning *Acta Mathematica* memoir to the subject and who then continued throughout the 1890s to extend and elaborate the ideas contained in that essay.

By 1920, the celestial mechanics landscape had changed considerably. A general solution to the three-body problem, albeit one of no practical use, had been found and Newton's law had been superseded by Einstein's theory of general relativity. During the following decade, thanks to events such as the discovery of galaxies beyond our own Milky Way, the popularity of the many new branches of astronomy soared, much to the detriment of the older disciplines of astrometry and celestial mechanics. Other scientists and mathematicians were enticed to exciting new academic fields, such as quantum mechanics or dynamical systems theory. The result was that celestial mechanics, and research on the three-body problem, entered into a period of stagnation from which it did not emerge until the mid-1950s and the dawn of the Space Age.

A period in the history of celestial mechanics which, until now, has received little attention is that between the publication of Poincaré's seminal work in the 1890s and the decline of celestial mechanics in the 1920s. During this period—a period which may be referred to as the end of the 'golden age' of celestial mechanics—a number of researchers were actively investigating the periodic orbits of the three-body problem. Prominent amongst these were the three mathematical astronomers George Darwin, Elis Strömgren and, the main protagonist of this book, Forest Ray Moulton.

During the first decades of the 20th century, F. R. Moulton was one of the world's leading mathematical astronomers and, without doubt, the leading mathematical astronomer in the United States.[1] Moulton is today remembered as the author of several introductory books on astronomy, in particular his celebrated text on celestial mechanics; for his role in the formulation of the Chamberlin–Moulton planetesimal hypothesis; and for his work on ballistics during World War I.[2] It was in connection with his wartime work on ballistics that he developed the popular method of numerical integration which now bears his name.[3] However, for most of his 30-year career at the University of Chicago, it was the three-body problem which held his interest. His research on its periodic solutions began with his 1899 PhD thesis on oscillating satellites and culminated over 20 years later with the publication of his *magnum opus*, the book *Periodic Orbits* [1920].

Whilst detailed accounts are available both of Moulton's work on ballistics and on his contribution to the planetesimal hypothesis, little to date has been published on his research on the periodic orbits of the three-body problem.[4] One of my main aims in writing this book has been to amend that situation. It is also my hope that, by throwing more light on the research of a figure such as Moulton, who, although important in his day, is certainly far less well known than a mathematician such as Poincaré, this book will help to fill a gap in the history of celestial mechanics. Of the three mathematical astronomers mentioned above, Moulton soon became a clear focus for my study. Not only does the period of Moulton's active research coincide almost exactly with the period that I referred to above as the end of the golden age of celestial mechanics, but there was also the puzzle of explaining his 500-page tome on periodic orbits. *Periodic Orbits* was published in 1920, more than ten years after it was originally conceived, and contains essentially a collection of articles describing the research performed by Moulton and his doctoral students over the previous 20 years. The book's presence immediately raises a number of questions: What was the motivation for the research it contains? Did Moulton have a clear goal in mind and, if so, to what extent was it achieved? Why was there such a long delay in the book's publication and how was his research received by his contemporaries? Why did the book mark the end of his research programme on periodic orbits? As well as providing answers to these questions, I will put Moulton's research into context by looking at the key influences on his particular style of celestial mechanics and at the influence that his work, in turn, had on future generations of researchers.

[1] Szebehely and Mark [2004, pp. 11–12] include Moulton in their 'list of major contributors to celestial mechanics'.

[2] In the preface to his own introductory text on celestial mechanics, Danby [1962, p. vii] wrote: "No text of this nature can fail to be influenced by Moulton's classic, and if this present text resembles that of Moulton, it is not a coincidence".

[3] The Adams–Moulton method is essentially a modification of the multistep method of numerical integration developed by John Couch Adams. Moulton downplayed this aspect of his work, laying far more weight on the contribution he made to the subject by proving both the existence of a solution and the convergence of his method; see Gluchoff [2011, p. 20].

[4] For accounts on Moulton's work on ballistics and the planetesimal hypothesis, see Gluchoff [2011] and Brush [1978] respectively. To date, perhaps the only author to have given more than a superficial commentary on Moulton's research on periodic orbits is Victor Szebehely. This is found in his well-known book on the three-body problem [1967a] together with a few research papers.

As the title of this book indicates, what motivated much of Moulton's research was his quest for a new lunar theory. Although as early as 1902 we can see evidence of his interest in this subject, there is only a single published paper in which he properly addresses it. By the time a modified version of this paper was included in *Periodic Orbits*, the section on the lunar theory had been reduced to occupy just a single page. Moulton's true motivations and the nature of his ambitious project only become apparent through an examination of his correspondence. His letters reveal how he believed that the theory of periodic orbits was ushering in a new epoch in celestial mechanics and show his plans for using it to create a new and superior lunar theory. They also show the transformation of his book *Periodic Orbits*, from one which would showcase his lunar theory into one which would conclude with a synthesis of the periodic orbits, and document the book's long and troubled road to publication.

After a detailed historical review of the subject of periodic orbits, concentrating on the research of Moulton's immediate predecessors, I will describe Moulton's own research and look at the progress he made with his lunar theory. I will conclude with an examination of the contents of *Periodic Orbits* and a discussion of its reception and Moulton's legacy.

1.2. Organization

In Chapter 2, I provide the motivation for the study of the three-body problem and show the latter's intimate connection with the lunar theory. We look at the work performed by Newton and several famous 18th century mathematicians to account for the various lunar inequalities, the most troublesome of these being the rotation of the line of apsides. The chapter concludes with the discovery by Euler and Lagrange of the first periodic solutions of the three-body problem.

In Moulton's *Periodic Orbits*, the first three paragraphs address the work of Hill, Poincaré and Darwin respectively.[5] Each of these three figures from the generation prior to Moulton's made important contributions to the theory of periodic orbits and clearly influenced Moulton's research. To their contributions I devote the following three chapters of this book.

Hill was an inspirational figure for applied mathematics in America at the end of the 19th century, kindling interest in periodic orbits with his discovery of the variational orbit. In Chapter 3, I describe this discovery and its innovative use as the intermediate orbit in his lunar theory. The paper that then followed is generally considered to contain Hill's most brilliant work: his calculation, to an unprecedented degree of accuracy, of the principal part of the motion of the lunar perigee [1886]. I outline this paper and also sketch how Hill's lunar theory was completed many years later by Darwin's student Ernest Brown.

By far the greatest influence on Moulton came from Poincaré. It was Poincaré whose methods Moulton extended and applied to more practical problems, and whose drive to rigorize celestial mechanics Moulton shared. Inspired by Hill's discovery, Poincaré developed an interest in periodic solutions and became perhaps the first person to recognize fully their importance for an understanding of the three-body problem. In Chapter 4, I explain what makes the periodic solutions so special and take a look at some of the key mathematical ideas in Poincaré's *Les Méthodes*

[5]The same three names also appear in the preface to the first edition of Moulton's *An Introduction to Celestial Mechanics* [1902b], as well as in many of his research papers.

Nouvelles de la Mécanique Céleste, ideas which are a great help in understanding Moulton's research and many of the results presented in later chapters.

In the late 1890s, Darwin studied a particular realization of the restricted three-body problem. He discovered, by means of painstaking numerical processes, many concrete examples from several new families of periodic orbits and his results simultaneously raised a number of theoretical questions. Moulton later derived literal expressions for many of these families and made use of Darwin's numerical data for testing his results. In Chapter 5, I present Darwin's research on periodic orbits and the interpretation of his results both by Poincaré and by Darwin's student Sidney Hough.

The next two chapters introduce Moulton (Chapter 6) and his research (Chapter 7). Chapter 6 begins with a summary of Moulton's main scientific achievements and his qualities as an educator and administrator, before taking a look at his philosophical leanings, in particular his devotion to science and his support for the conventionalist views of Poincaré. After saying a few words about his recreational activities and character, I then trace Moulton's personal trajectory from his humble beginnings in a frontier settlement to his career as a professor at the University of Chicago. In the final sections of this chapter, I describe his disputes with other astronomers and briefly summarize the work of his students (further details are provided in Appendix A). The following chapter, Chapter 7, is important for an understanding of Moulton's research. In it we look at Moulton's particular style of celestial mechanics, connecting his methods to Poincaré's research and discussing the theoretical advances which he made.

Chapter 8 addresses the first type of periodic orbit studied by Moulton, the orbits Darwin had named the oscillating satellites. After summarizing the results of his predecessors, I describe the research that Moulton performed for his PhD, in which he obtains analytic expressions for the collinear oscillating satellites and discovers the first examples of three-dimensional periodic orbits. In the remainder of this chapter, I describe the treatment, by one of his students, of the equilateral triangle oscillating satellites and Moulton's extension of his own research to the much more difficult case of the elliptical restricted three-body problem.

In 1906 Moulton wrote to Robert Simpson Woodward, President of the Carnegie Institution of Washington, seeking financial assistance for his project to develop a new lunar theory. By means of Moulton's correspondence with Woodward, Chapter 9 tells the story of these plans for a new lunar theory and looks at the progress which Moulton made before the project ran into difficulties. Moulton's only published paper on the subject is examined in detail.

In 1908 Moulton announced his plan to publish a book the following year on periodic orbits and his new lunar theory. The book was later split into two volumes, the first volume being published as *Periodic Orbits* some 12 years later. The story of this book, including the difficulties which Moulton encountered and the extensions which he made to his original research, is the subject of Chapter 10. Following the book's publication, Moulton's research activities were dedicated exclusively to the completion of his wartime work on exterior ballistics, and in January 1927, with this research completed and published in his *New Methods in Exterior Ballistics* [1926], he resigned from the University of Chicago. The chapter concludes with a discussion of the reasons for Moulton's resignation.

Moulton's mathematical research ended when he left the University. However, his work on periodic orbits has continued to attract attention and to be used as a valuable reference. In the final chapter, we turn to the reception of Moulton's *Periodic Orbits*, beginning with a discussion of the criticism it received from Strömgren and Wintner, and the praise that followed years later from Szebehely, subsequent to his vindication of certain of Moulton's findings. I then provide a brief survey of the progress that has been made over the past century on the three-body problem, before concluding with an examination of Moulton's legacy for modern-day researchers in this field.

1.3. Conventions

The research discussed in this book comes from works written by many different authors and these works employ a diversity of mathematical notations. In an attempt to keep my notation consistent, I have chosen to adopt that which was typically used by Moulton. Examples of this are his use of $\sqrt{-1}$ instead of the symbol i (or i) and the shorthand notation

$$\frac{dx_i}{dt} = f_i(x_j; t),$$

for a system of n differential equations in the n independent variables x_1, x_2, …, x_n.

Even though the early chapters of this book address research performed prior to Moulton, I have often included commentaries made by him many years later. My hope is that these will help to provide further insight into his ideas and to connect the topics being discussed with Moulton's own research.

All translations are my own, unless otherwise stated.

CHAPTER 8

Oscillating Satellites

8.1. Collinear oscillating satellites

8.1.1. Introduction. In 1899, Moulton was awarded his PhD for a thesis titled *Periodic Oscillating Satellites* (see Chapter 6). The following year, Moore presented a paper by Moulton at the Summer Meeting of the AMS.[1] From the title of this paper, *Oscillating Satellites*, it seems highly likely that the subject matter was the same as Moulton's thesis. The work then remained unpublished for a further 20 years until it finally appeared as Chapters 5 and 6 of his book *Periodic Orbits*.[2] From these chapters, we discover that the oscillating satellites under study in his thesis were those which revolve in periodic orbits around the collinear equilibrium points of the restricted three-body problem. For short, I will refer to bodies in such orbits as *collinear oscillating satellites*; similarly, those with periodic orbits around the equilateral triangle equilibrium points will be called *equilateral triangle oscillating satellites*.

Darwin's *Periodic Orbits* provided Moulton with a reference for much of his work and his research on oscillating satellites is no exception. In a 1903 progress report to Hale, Moulton describes his work on oscillating satellites as treating certain classes of Darwin's orbits by the "more powerful methods of recent mathematics".[3] From this letter, we also learn that it was his desire to include a numerical comparison of his results with the orbits found by Darwin which was responsible for the initial delay in the paper's publication; Moore had wished to see the paper published in the *Transactions of the AMS*.

However, Darwin was not the first to discover oscillating satellites. Back in 1899, when Moulton wrote his thesis, there already existed an extensive literature on the oscillating satellites of the restricted three-body problem. Despite the fact that the collinear equilibrium points are unstable, Gyldén [1884] had shown that, for suitably chosen initial conditions, elliptical orbits about them were still possible. These oscillating satellites are said to be infinitesimal since the analysis is based on the linearized equations of motion.[4] At the end of his paper, Gyldén had suggested that the existence of these orbits might lead one to suppose that a swarm of particles

[1]Moulton did not travel to this meeting, which was held in New York [Cole, 1900, p. 12].

[2]We know this from the footnotes in Moulton [1920, p. 151 and p. 199], which identify these chapters with his 1900 paper, and from the fact that Moulton [1920, p. iv] does not include these chapters in his list of those that had already been published. Szebehely refers to these chapters as containing the "activities of the Moulton school" [1967a, p. 299], which seems rather unfair given that they were performed single-handedly by Moulton whilst he was still a student.

[3]Moulton to Hale, 9 March 1903, The University of Chicago Library.

[4]For the equilateral triangle equilibrium points, two families of infinitesimal oscillating satellites exist provided that the mass ratio is such that the points are linearly stable (see Section 5.2). Brown [1911b] was later to prove that when the mass ratio is such that the condition of linear stability is not satisfied, then only oscillating satellites of finite dimensions exist.

might accumulate about the collinear equilibrium points of the Earth–Sun system, and this in turn might provide an explanation for the source of zodiacal light in the antisolar direction known as the gegenschein. In 1900, unaware of Gyldén's hypothesis, Moulton independently arrived at the same conclusion and published his theory in a paper titled *A meteoric theory of the Gegenschein* [1900c].[5] However, observations from space have since dispelled this theory and it is today generally accepted that the the gegenschein is due to a backscattering of sunlight from interplanetary dust.[6] Despite this, there is still a possibility that there may be a higher concentration of particles in orbit around these points.[7]

A decade after Gyldén's discovery, numerical methods were used to discover finite oscillating satellites about the collinear equilibrium points. As described in Section 5.4, Burrau [1894a] (and Thiele) found oscillating satellites for the Copenhagen problem and later Darwin [1897] found oscillating satellites for a mass ratio of ten to one. Perchot and Mascart [1895] were the first to provide an analytical treatment of the problem. In their paper they first use Poincaré's method to prove the existence of Burrau's oscillating satellites and then the method of undetermined coefficients to find analytical expressions for these orbits.[8] However, according to Moulton, their existence proof is flawed and their construction of the solution fails at the point at which they stopped.[9]

The basic procedure which Moulton followed in his thesis is the same as that followed by Perchot and Mascart. That is, he also proves the existence of oscillating satellites by applying Poincaré's method to the infinitesimal oscillating satellites obtained from the linearized problem, and he also uses the method of undetermined coefficients to construct the solution. As noted in Chapter 7, this is the procedure which Moulton would use repeatedly throughout all his research on periodic orbits. However, whereas Perchot and Mascart's analysis was limited to the Copenhagen problem, Moulton's analysis is general and applies to primaries of any mass ratio. Moreover, by not constraining the infinitesimal body to move within the plane of the primaries—i.e. considering the non-planar restricted three-body problem—he

[5] In a note at the end of [1900c], added whilst the paper was at the publisher's, Moulton gives credit to Gyldén for suggesting the theory and explains how he came to have overlooked Gyldén's paper.

[6] In March 1972, observations made from NASA's Pioneer 10 probe found that, even at a distance of over 9 million kilometres from the Earth (which is 6 times that to the L_2 equilibrium point), the direction of the gegenschein remained antisolar [Hanner and Weinberg, 1973; O'Meara, 2000]. Note, L_2 is today the accepted designation of the collinear equilibrium point in the direction away from the primary body (here the antisolar direction), but other conventions for numbering the five equilibrium points may be encountered, e.g. Szebehely [1967a, p. 133] and Roy [2005, p. 117].

[7] In mid-2014 it was reported that the European Space Agency's Gaia spacecraft, which is in orbit about the L_2 equilibrium point, was being bombarded by 50–500 times the number of micrometeoroids that one would expect (see New Scientist, 26 July 2014). However, following a close examination of the spacecraft's angular momentum profile, these 'hits' are now attributed to sudden, minute distortions of the spacecraft's structure.

[8] Perchot and Mascart [1895, pp. 349–351] also claim to prove the non-existence of oscillating satellites for the equilateral triangle equilibrium points. In fact, what they show is that, for the Copenhagen problem, the infinitesimal oscillating satellites—which they used as the generating solutions for the case of the collinear points—do not exist about the equilateral triangle points.

[9] "The analysis of this paper is apparently vitiated by assuming that the equation which defines the solutions is solvable by series in the square root of one of its parameters. The practical construction of the solutions in the derived form can not be carried out" [Moulton, 1913a, p. 442]. See also Moulton [1920, p. 152].

finds the "first examples of three dimensional periodic orbits".[10] Moulton notes that it would be "practically impossible to discover three-dimensional orbits by numerical processes" [1920, p. 152].

In Chapters 5 and 6 of *Periodic Orbits*, Moulton gives two alternative methods for treating the oscillating satellites. The first he describes as being more convenient for the particular problem, whereas the second he considers as being the more useful technique in general. Both of these chapters deal only with orbits about the collinear equilibrium points, although it would seem from the abstract of Moulton's 1900 talk that he had also at the time of his PhD considered those about the equilateral triangle points [Cole, 1900, p. 12]. In *Periodic Orbits* the oscillating satellites about the equilateral triangle points are covered in Chapter 9 by Moulton's student Thomas Buck (see Section 8.2).

8.1.2. First method. The first method is described in Chapter 5 of *Periodic Orbits*. Moulton begins by deriving the differential equations which give the small displacements, x', y', z', in a synodic reference frame, of the infinitesimal body from an equilibrium point

$$\begin{aligned} \frac{d^2x'}{dt^2} - 2\,\frac{dy'}{dt} &= \quad P_1\left(x',\, y'^2,\, z'^2\right), \\ \frac{d^2y'}{dt^2} + 2\,\frac{dx'}{dt} &= y'\,P_2\left(x',\, y'^2,\, z'^2\right), \\ \frac{d^2z'}{dt^2} \qquad\quad &= z'\,P_3\left(x',\, y'^2,\, z'^2\right), \end{aligned} \tag{8.1}$$

where P_1, P_2 and P_3 are power series in x', y'^2 and z'^2.

After determining, for each of the three equilibrium points, the regions of convergence of the series P_1, P_2 and P_3,[11] he introduces two parameters, ϵ' and δ, and transforms to new dependent variables x, y and z and to a new independent variable τ:

$$x' = x\epsilon', \quad y' = y\epsilon', \quad z' = z\epsilon' \quad (\epsilon' \neq 0), \quad t - t_0 = (1+\delta)\tau. \tag{8.2}$$

It will be seen that the parameter ϵ' is used to control the dimensions of the periodic orbits and the parameter δ allows their periods to vary. Although ϵ' must be non-zero in the physical problem, Moulton makes, what he admits may seem like a trivial generalization, and replaces ϵ' throughout by a new parameter ϵ, which he allows to also take the value zero. In Moulton's words, "when $\epsilon \neq \epsilon'$, the differential equations belong to a purely mathematical problem; but when $\epsilon = \epsilon'$ they belong to the physical problem."

He then shows that for the case where $\delta = \epsilon = 0$ the general solution of the differential equations is

$$\begin{aligned} x &= K_1 e^{\sigma\sqrt{-1}\tau} + K_2 e^{-\sigma\sqrt{-1}\tau} + K_3 e^{\rho\tau} + K_4 e^{-\rho\tau}, \\ y &= n\sqrt{-1}[K_1 e^{\sigma\sqrt{-1}\tau} - K_2 e^{-\sigma\sqrt{-1}\tau}] + m[K_3 e^{\rho\tau} - K_4 e^{-\rho\tau}], \\ z &= c_1 \cos\sqrt{A}\tau + c_2 \sin\sqrt{A}\tau. \end{aligned} \tag{8.3}$$

[10]Moulton to Woodward, 30 August 1906, Carnegie Institution of Washington.

[11]Shirmin [1974] explains why the regions of convergence found by Moulton are erroneous and shows how to correct the analysis. However, as the regions of convergence are not used in the remainder of the chapter, Moulton's mistake is fortunately of little consequence.

In these equations A is a positive constant, determined from the masses of the finite bodies and the equilibrium point in question,[12] $\pm\rho$ and $\pm\sigma\sqrt{-1}$ are the real and imaginary roots of a certain biquadratic whose coefficients are functions of the constant A, and $m = \frac{\rho^2-1-2A}{2\rho}$, $n = \frac{\sigma^2+1+2A}{2\sigma}$.[13] The remaining four constants, $K_{i=1,\ldots,4}$, are arbitrary and Moulton shows that, by assigning them suitable values, three distinct periodic solutions may be produced. These solutions represent three families of infinitesimal oscillating satellites and each is to be used as the generating solution for a corresponding class of finite oscillating satellites.

Before analytically continuing these solutions to the case where $\epsilon \neq 0$, Moulton puts the differential equations into normal form (i.e. a diagonal system of first-order differential equations). Heeding the form of the general solution of the linearized differential equations (which are identical to (8.3) with an additional $1+\delta$ factor appearing in the exponents), Moulton uses the linear transformations

$$\begin{aligned} x &= (u_1+u_2)+(u_3+u_4), \\ \frac{dx}{d\tau} &= \sigma(1+\delta)\sqrt{-1}(u_1-u_2)+\rho(1+\delta)(u_3-u_4), \\ y &= n\sqrt{-1}(u_1-u_2)+m(u_3-u_4), \\ \frac{dy}{d\tau} &= -n\sigma(1+\delta)(u_1+u_2)+m\rho(1+\delta)(u_3+u_4), \end{aligned} \tag{8.4}$$

to transform x, $\frac{dx}{d\tau}$, y, $\frac{dy}{d\tau}$ to four new independent variables u_1, u_2, u_3 and u_4.[14] Initial conditions

$$u_i = a_i + \alpha_i \quad (i=1,\ldots,4), \quad z = 0,^{15} \quad \frac{dz}{d\tau} = c+\gamma,$$

are assigned such that the a_i and c give, for $\epsilon = 0$, one of the generating solutions, and the Cauchy–Poincaré theorem (see p. 46) is invoked so that the general solution may be written as power series in δ, ϵ and the increments, $\alpha_{i=1,\ldots,4}$ and γ, to the initial values of the dependent variables. For sufficiently small values of these parameters, the series converge over a range of τ corresponding to the period of the generating solution.

Applying conditions to ensure that any solutions are also periodic for $\epsilon \neq 0$,[16] results in a system of equations, analytic in the above parameters. Moulton uses the theory of analytic implicit functions, as described in the first chapter of *Periodic Orbits* (and Chapter 6 of *Differential Equations*), to prove that this system can be solved to give the α_i, γ and δ as convergent power series in ϵ. Upon having obtained such solutions, it would then be merely a matter of back-substitution to obtain the literal series in ϵ for the periodic orbits. So the proof of the existence of these solutions proves the existence of the oscillating satellites.

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[12]More precisely, $A = \frac{1-\mu}{[(\xi_0+\mu)^2]^{\frac{3}{2}}} + \frac{\mu}{[(\xi_0-1+\mu)^2]^{\frac{3}{2}}}$, where μ and $1-\mu$ are the two finite masses located on the ξ-axis at $1-\mu$ and $-\mu$ respectively and ξ_0 is the ξ-coordinate of the equilibrium point in question.

[13]The biquadratic in question is $\lambda^4 + (2-A)\lambda^2 + (1-A)(1+2A) = 0$.

[14]In [1930], Aurel Wintner questioned the validity of Moulton's existence proof, precisely on account of his use of this complex linear transformation. Unfortunately, similar transformations were also used in Moulton's 'second method' and in the work of some of his students. More will be said on this subject in Section 11.4.

[15]That is, t_0 in (8.2) is chosen so that $z = 0$ when $\tau = 0$.

[16]The period in τ (not t) is chosen to be equal to that of the generating solution.

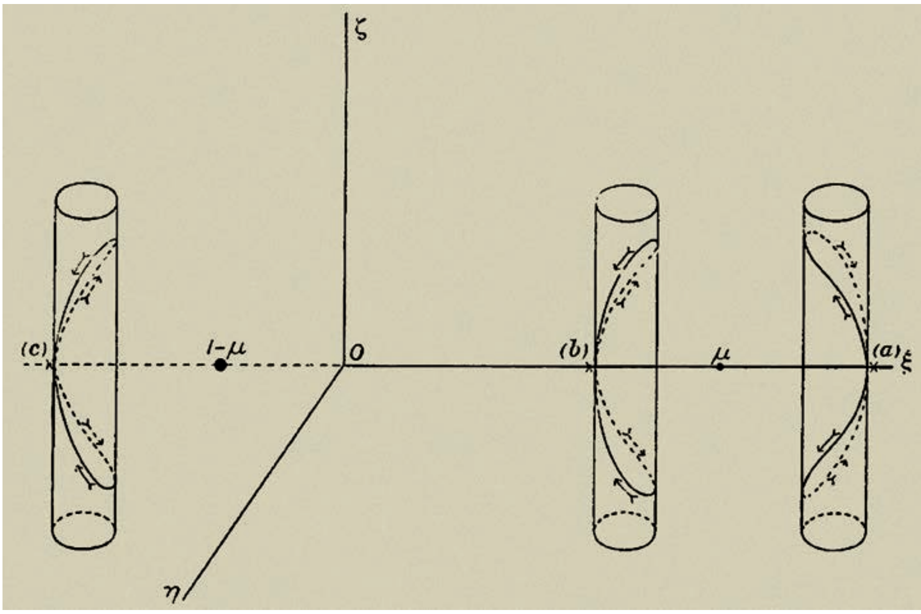


FIGURE 8.1. Orbits of Class A

Having shown that the solutions may be expressed as literal series in ϵ, Moulton finds the coefficients of these series by the method of undetermined coefficients. He determines the coefficients of the terms in ϵ and ϵ^2 explicitly and uses induction to show that the process may be continued as far as one desires.

8.1.3. Results. Moulton succeeds in proving the existence of two classes of oscillating satellite. For both types he demonstrates how to compute the solution as power series in the parameter ϵ'.

The first generating solution which he uses to find a family of periodic orbits is

$$x = y = 0, \quad z = \frac{c}{\sqrt{A}} \sin \sqrt{A}\tau,$$

where c is an arbitrary (real) constant. Having shown that each orbit has a period equal to the synodic period (i.e. it is re-entrant after one synodic revolution) and crosses the x'-axis perpendicularly, Moulton then shows that the terms of the solution in ϵ'^n are homogeneous functions of degree n in c, so that ϵ' and c are equivalent to a single parameter. He therefore sets c equal to unity so that the dimensions of the orbits are determined solely by the parameter ϵ'. By projecting the first few terms of the literal series he has found on to the $x'y'$-, $y'z'$- and $x'z'$-planes, it is an easy matter to infer the shape of these orbits, which he likens to the handles of ice-tongs (see Figure 8.1). Moulton names these orbits *Orbits of Class A* and finds that they are retrograde. As Moulton claims,[17] these appear to be the first examples of three-dimensional periodic orbits.[18]

[17] Moulton to Woodward, 30 August 1906, Carnegie Institution of Washington.

[18] The first published examples appear to be those in Moulton's paper on the oscillating satellites of the elliptical restricted three-body problem (see Section 8.4).

In a similar manner, *Orbits of Class B* are constructed by means of the generating solution

$$x = a\cos\sigma\tau, \quad y = -na\sin\sigma\tau, \quad z = 0,$$

where a is a (real) arbitrary constant. This generating solution is precisely the family of elliptical infinitesimal oscillating satellites found by Gyldén [1884]. The corresponding finite oscillating satellites are also found to lie in the $x'y'$-plane, to be symmetrical with respect to the x'-axis and to once again have a period equal to the synodic period. The terms of the solution in ϵ'^n are found to be homogeneous in a^n, so that in this case ϵ' and a are equivalent and Moulton is able to simplify by setting a equal to unity. The motion is shown to be retrograde and for small values of ϵ' the orbits are approximately elliptical. The oscillating satellites which Darwin [1899] discovered belong to this class and Moulton computes the first few terms of their expansion as literal series in ϵ'.

Moulton also attempts to prove the existence of solutions corresponding to the generating solution

$$x = a\cos\sigma\tau + b\sin\sigma\tau, \quad y = -na\sin\sigma\tau + nb\cos\sigma\tau,^{19} \quad z = \frac{c}{\sqrt{A}}\sin\sqrt{A}\tau,$$

where σ and $\sqrt{A}$ are commensurable, which he proves to be possible [1920, pp. 160–161]. Such solutions he calls *Orbits of Class C*. However, the complexity of the problem forces him to abandon his attempt to determine whether there exist orbits of Class C which are distinct from those of the other two classes [1920, p. 198].[20]

Today Moulton's orbits of Class A and Class B are known as the vertical and planar *Lyapunov orbits* respectively, since their existence is guaranteed by the *Lyapunov centre theorem*.[21] The vertical Lyapunov orbits about the central equilibrium point are also sometimes referred to as the generalized Sitnikov family of orbits, e.g. Llibre et al. [1999], since they coincide with the solutions of the circular Sitnikov problem (i.e. the MacMillan problem) when $\mu = \frac{1}{2}$ (see p. 159). In the late 1960s, Moulton's orbits of Class C were also proved to exist and they have since been put to use on several space observatory missions. These orbits are today known as the *halo orbits* and they are described in more detail in Section 11.6.

8.1.4. Second method. In Chapter 6 of *Periodic Orbits* Moulton presents the original method he used to treat the problem. He describes one of the principal features of the physical problem as being that the orbits of the oscillating satellites form a continuous series of varying dimensions, beginning with orbits of zero dimensions located at the equilibrium points. And as their dimensions vary so do their periods. In the previous method, Moulton used the parameter ϵ' to vary the dimensions of the orbits and then introduced a second parameter δ to reflect the associated variation in their periods. δ was then found as a function of ϵ. In this method he instead develops the solutions in terms of the single parameter λ,

[19] Moulton's generating solution contains a typing error in the expression for y [1920, p. 161].

[20] During his analysis for this generating solution, he does however succeed in completing his proof that all the orbits of Class A and Class B re-enter after one revolution—earlier he had found it necessary to impose the condition that σ and $\sqrt{A}$ be incommensurable.

[21] See Meyer et al. [2009, pp. 219–220] for both a statement of the Lyapunov centre theorem and its application to the collinear and the equilateral triangle equilibrium points of the restricted three-body problem.

which he introduces by means of the substitution $\mu = \mu_0 + \lambda$.[22] The parameter μ_0 determines the period of the orbits and, during the development of the solutions in terms of λ, it is held fixed. Thus, a family of solutions is obtained—all having the same period—but with dimensions which vary with λ. However, only one of these solutions, that for which $\lambda = \mu - \mu_0$, belongs to the physical problem and provided that μ_0 is taken sufficiently close to μ the series for this solution will converge. Finally, allowing the parameter μ_0 to vary, it is seen that the family of solutions belonging to the physical problem has been obtained.[23]

Moulton finds the coordinates of the periodic orbits to be expandable as power series in $\lambda^{\frac{1}{2}}$, where the coefficients are either all of period $2\pi/\sqrt{A_0}$ or all of period $2\pi/\sigma_0$ and where A_0 and σ_0 correspond to the A and σ defined above with μ replaced by μ_0. Upon substituting $\lambda = \mu - \mu_0$ in these two solutions, he retrieves the orbits of Class A and Class B respectively. Once again he shows how each class of orbit may be constructed directly. (No mention is made of the possibility of the Class C orbits during his presentation of the second method.)

8.2. Equilateral triangle oscillating satellites

In 1909, two of Moulton's students were awarded doctorates for successfully applying his 'first method' to other problems. One of these was Thomas Buck, whose thesis demonstrated the existence of equilateral triangle oscillating satellites for the restricted three-body problem. Like Moulton, Buck considered the non-planar problem and found examples of both planar and three-dimensional oscillating satellites.[24]

As is well known, for $0 < \mu < \mu_0 = 0.0385$ (Routh's value), the roots of the characteristic equation for the linearized differential equations are purely imaginary, $\pm\sigma_1\sqrt{-1}$ and $\pm\sigma_2\sqrt{-1}$ say, where the values of σ_1 and σ_2 depend only on μ. Corresponding to these are two families of elliptical orbits, typically referred to nowadays as the infinitesimal long-period and short-period orbits. Using each of these in turn as the generating solution, Buck found the corresponding families of finite oscillating satellite orbits, of periods $2\pi/\sigma_1$ and $2\pi/\sigma_2$, which are today known as the finite long-period and short-period orbits. He illustrated these orbits, for $\mu = 0.01$ and $\epsilon = 0.001$, using the first two terms of his power series solutions (Figure 8.2). (The generating solutions are drawn with dashed lines and the more eccentric orbits are the long-period orbits.)

Similarly, there is a third generating solution (for $0 < \mu \leq 0.5$) corresponding to sinusoidal motion in the z-direction and Buck used this to construct a family of three-dimensional oscillating satellites, all of period 2π, which resemble those found

[22] The substitution is only made where μ occurs explicitly in the force function (which includes terms giving both the gravitational and centrifugal forces). Elsewhere μ retains its original numerical value.

[23] Equivalently, replacing μ_0 by $\mu - \lambda$, the family of solutions belonging to the physical problem is obtained with respect to the parameter λ [Moulton, 1920, p. 505].

[24] The three families of oscillating satellites found by Buck are mentioned in the abstract of Moulton's 1900 talk [Cole, 1900, p. 12], so it is conceivable that Moulton made a start on this problem himself. An independent proof of the existence of the two planar families was given by the astronomer William Greaves [1922], whilst an Isaac Newton student at the University of Cambridge. Greaves [1922, p. 149] notes that the work was completed in late 1920, just as Buck's paper was being published in Moulton [1920, ch. 9].

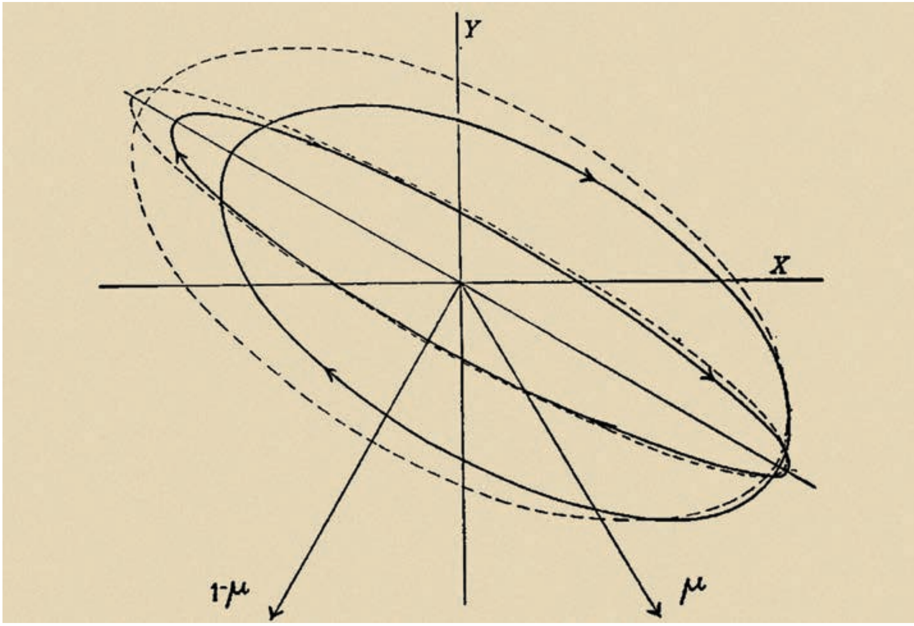


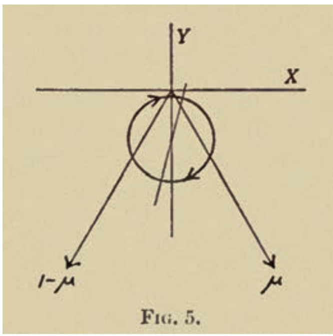
FIGURE 8.2. In-plane equilateral triangle oscillating satellites

by Moulton for the collinear points (i.e. the 'ice-tong handle' orbits).[25] Figure 8.3 shows Buck's illustration of a member of this family for $\mu = 0.01$ and $\epsilon = 0.5$, once again using just the first two terms of his power series solution. (The projection on the xy-plane shown in the left-hand side plot is an ellipse of small eccentricity; the line through this ellipse indicates its major axis.)

In his work on the collinear oscillating satellites, Moulton demonstrated the existence of a third generating solution by showing that there are values of μ for which the periods of the first two generating solutions are commensurable. In a similar way, Buck shows that for particular values of μ the three generating solutions mentioned above can be combined to produce four more generating solutions. Despite several pages of analysis, he also fails to produce finite periodic orbits from these hybrid generating solutions and decides that the existence of such orbits is improbable. The commensurability of the periods of the infinitesimal oscillating satellites did turn out to be important, however. In [1935], Pedersen showed that when the period of the infinitesimal long-period orbits is an integer multiple of that of the infinitesimal short-period orbits, the family of finite long-period orbits fails to exist.[26] So for these critical values of the mass parameter, the generation of a family of finite long-period orbits by the analytic continuation of the infinitesimal long-period orbits must fail. As for the finite oscillating satellites about the collinear equilibrium points found by Moulton, the finite oscillating satellites

[25]Following it to its termination at the other equilateral triangle point, Zagouras [1985] would later discover that this family bifurcates with one of the families of three-dimensional orbits found by Moulton.

[26]By means of the third-order theory employed in this paper, Pedersen also finds how the period of these orbits varies with their size, an effect which, as Szebehely [1967a, p. 270] notes, is far from insignificant.



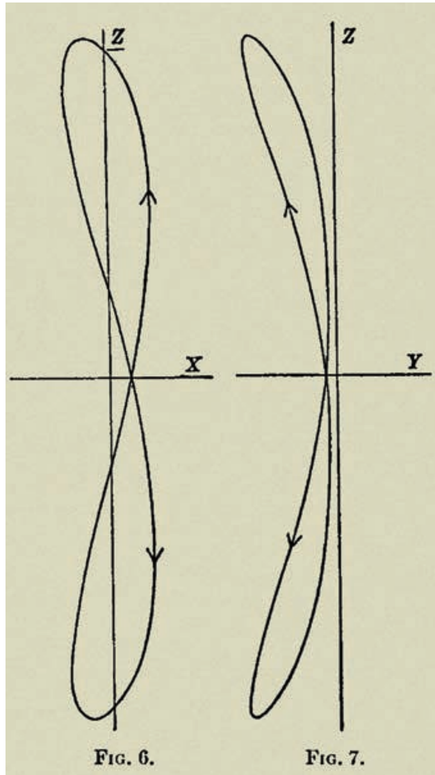


FIGURE 8.3. Three-dimensional equilateral triangle oscillating satellites

found here by Buck are examples of Lyapunov orbits. The critical values of the mass parameter are precisely those at which the Lyapunov centre theorem fails to guarantee the existence of the family of finite long-period orbits [see Meyer et al., 2009, pp. 219–220].[27]

The discovery of numerous celestial bodies in orbits about the equilateral triangle points of our solar system has helped to generate and maintain additional interest in these particular oscillating satellites. The first examples were found for the Sun–Jupiter system and these asteroids, being named after figures from the mythological Trojan war, became collectively known as the *Trojan asteroids*. Since then, Trojans have been discovered for several planetary (and lunar) systems and in 2010 the first example of an Earth Trojan, 2010 TK_7, was found [Connors et al., 2011]. The 'tadpole orbit' of this Trojan asteroid closely resembles the long-period, large-amplitude oscillating satellites found by Brown [1911a] for the Sun–Jupiter system.[28] As 2010 TK_7 revolves in its tadpole-shaped orbit about the leading equilateral triangle point, L_4, it describes an almost circular arc about the Sun, with the 'head' of the tadpole near the Earth and the 'tail' extending almost to the

[27] Szebehely [1967a, p. 299] notes that these critical values also follow from Horn's work and refers to the (Poincaré–)Lyapunov centre theorem as the Siegel–Liapunov theorem.

[28] Brown was led to search for such large-amplitude oscillating satellites following the discovery by L. J. Linders of a Trojan with a significant (17°) motion of the heliocentric vector [1911a, p. 438].

collinear equilibrium point, L_3, on the opposite side of the Sun. Each revolution takes approximately 400 years. It is believed from computer simulations that its close approaches to L_3 may occasionally cause the body to make a transition to a tadpole orbit about the other equilateral triangle point or to a 'horseshoe orbit'.[29]

For the case where $\mu > \mu_0$, infinitesimal oscillating satellites do not exist and so the method just described cannot be used. However, as demonstrated by Brown in [1911b], for $\mu = \mu_0 + \delta$ with $\delta > 0$ sufficiently small, finite long- and short-period oscillating satellites continue to exist provided the value of the orbital parameter ϵ is sufficiently large. For each value of δ there is a limiting (minimum) value of ϵ, ϵ_0 say, at which the long- and the short-period orbits coincide. (Their periods then equal $2\pi\sqrt{2}$, the same as that of the infinitesimal oscillating satellites when they coincide at $\mu = \mu_0$.) Brown referred to these orbits as *limiting orbits* and their existence was later established by Pedersen [1933]. Figure 8.4, taken from Pedersen's paper, shows the variation of ϵ_0 with δ. Corresponding to each point

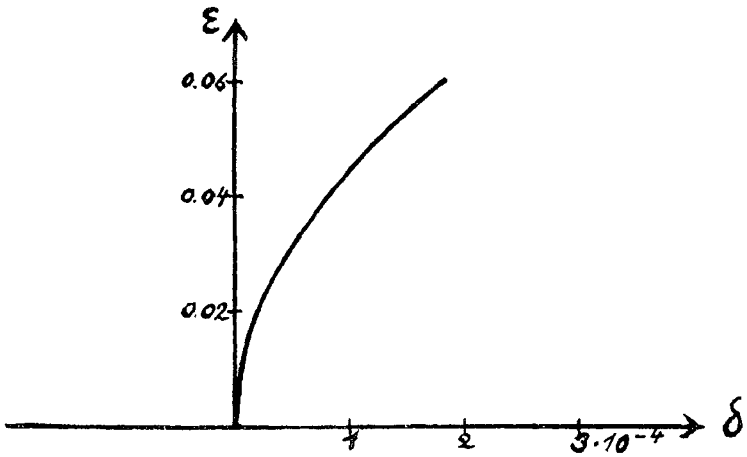


FIGURE 8.4. Pedersen's plot showing the variation of ϵ_0 with δ [1933, Fig. 2].[30]

above the curve there exists both a long- and a short-period orbit; for points on the curve there is a single limiting orbit; and for points below the curve there are no finite oscillating satellites. In [1966], Deprit made use of d'Alembert series to order 14 to compute limiting orbits up to about $\mu = 0.044$ ($\delta = 0.005$), but was unable to go any further with his method, leaving open the question of the existence of finite oscillating satellites for larger values of μ.[31]

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[29] Horseshoe orbits were also conjectured by Brown in the same paper; several examples of objects in such orbits for the Sun–Earth system have now been discovered.

[30] By permission of Oxford University Press on behalf of the Royal Astronomical Society.

[31] A d'Alembert series is the name given, apparently by Brown, to a particular form of trigonometric series in which each coefficient is a power series (in a second parameter) with the property that the parity of the terms is odd for the odd harmonics and even for the even harmonics [see Brouwer and Clemence, 1961, p. 79].

8.3. Extension to the general case of three finite masses

The other student to obtain a doctorate in 1909 for an application of Moulton's 'first method' was Herbert Buchanan. Buchanan [1923, 1925] extended Moulton's investigation of the collinear oscillating satellites to the case where all three masses are finite. Once again the analysis revolves around a discussion of seven cases, corresponding to the commensurability or otherwise of the three principal frequencies of motion of the infinitesimal oscillating satellites. In a later paper [1928], Buchanan performed a similar extension of Buck's work on the equilateral triangle oscillating satellites.

8.4. Extension to the elliptical restricted three-body problem

In his [1913a], Moulton extends his PhD research on the collinear oscillating satellites to the case where the finite masses revolve in elliptical orbits, i.e. to the non-planar elliptical restricted three-body problem. This problem is far more difficult than that in which the finite bodies move in circular orbits, and his 'first method' is no longer applicable. In this paper, Moulton employs his 'second method' and, whereas in the earlier research on oscillating satellites (both by Moulton and his students) the method of undetermined coefficients produced linear differential equations with constant coefficients, here the coefficients are periodic and the theory described in Section 7.3.2 is required. The oscillating satellites which he finds no longer form a geometrically continuous series in which their periods vary continuously with the linear dimensions of the orbit. Both the orbits and their periods vary discontinuously and for each period (which must be a multiple of that of the finite bodies) there are now two geometrically distinct solutions. In Chapter 7 of *Periodic Orbits*, Moulton gives another presentation of this research—entering into greater detail in places (as the proofs of a number of results were omitted from the original paper) and making several amendments.

As usual, canonical units are employed, the finite bodies are assumed to be of mass μ and $1-\mu$ ($\mu \leq 1/2$) and a reference frame is chosen which rotates with their mean motion. In this problem the finite bodies no longer remain at fixed locations on the ξ-axis, but make small periodic excursions in the $\xi\eta$-plane, which Moulton expresses in terms of the power series in the eccentricity e for two-body elliptic motion (see Section 7.4.1).[32] Next, Moulton finds expressions (power series in the eccentricity) for the coordinates ξ_0, η_0, ζ_0 of the infinitesimal mass when it is moving according to one of Lagrange's three collinear solutions. In these solutions the three bodies move in similar ellipses and the determination involves Lagrange's quintic equation (see, for example, Moulton [1914a, pp. 309–318]).

The next step is to investigate the oscillations about these solutions. This is achieved by putting

$$\xi = \xi_0 + x, \quad \eta = \eta_0 + y, \quad \zeta = 0 + z,$$

and expanding as power series in x, y and z. In line with Moulton's 'second method' (see Section 8.1), the parameter μ is then generalized by means of the substitution

[32] The axes and the time origin are chosen such that the finite bodies are at their periapses and lie on the ξ-axis at $t = 0$.

$\mu = \mu_0 + \lambda$.[33] The differential equations which result are:

$$
\begin{aligned}
&\frac{d^2x}{dt^2} - 2\frac{dy}{dt} - [1 + 2A + 6Ae\cos t - 3Ae^2(1 - 5\cos 2t) + \dots]x \\
&\qquad - [\quad 6Ae\sin t + \tfrac{51}{4}Ae^2\sin 2t \quad + \dots]y = X, \\
&\frac{d^2y}{dt^2} + 2\frac{dx}{dt} - [\quad 6Ae\sin t + \tfrac{51}{4}Ae^2\sin 2t \quad + \dots]x \\
&\qquad - [1 - A - 3Ae\cos t + \tfrac{3}{2}Ae^2(3 - 7\cos 2t) + \dots]y = Y, \\
&\frac{d^2z}{dt^2} \qquad + [\quad A + 3Ae\cos t + \tfrac{3}{2}Ae^2(1 + 3\cos 2t) + \dots]z = Z,
\end{aligned}
\tag{8.5}
$$

where A is a constant and X, Y and Z are power series in x, y, z and λ, with coefficients which are power series in e.

As usual, to make the orbit symmetric, the initial conditions are chosen to correspond to a symmetric conjunction. For the normal restricted three-body problem it is sufficient that the infinitesimal body starts at an orthogonal crossing of the x-axis (i.e. ξ-axis). For the elliptical problem, to ensure that the finite bodies are initially moving orthogonally to this axis, it is also necessary that these bodies be started at their apses (periapses or apoapses). Therefore, Moulton starts with the finite bodies at periapses and with the initial conditions:

$$x(0) = \alpha, \quad x'(0) = 0, \quad y(0) = 0, \quad y'(0) = \beta, \quad z(0) = 0, \quad z'(0) = \gamma.$$

Having chosen the initial conditions to make the orbit symmetrical, the solution of (8.5) may be expanded as power series in the four parameters α, β, γ and λ.[34] By the Cauchy–Poincaré theorem, for any $0 \le t \le T$, these series will converge provided that the moduli of the parameters are sufficiently small. To prove the existence of periodic solutions and to find their properties, Moulton now solves the differential equations corresponding to the terms of different degrees in α, β, γ and λ, beginning with that corresponding to the terms of first degree. All the differential equations which result are of the special type considered in Moulton and MacMillan [1911] (see Section 7.3.2). That is, they are linear differential equations with periodic coefficients and each of the coefficients is a power series in e such that the term independent of e is constant. The differential equations of the terms of the first degree are homogeneous and those of the terms of higher degrees differ only in the presence of additional non-homogeneous terms.[35]

Considering the differential equations for the terms of the first degree,[36] Moulton remarks that the form of the solution depends upon the nature of the roots of the characteristic equation corresponding to $e = 0$ (see Section 7.3.2). For $e = 0$, the third equation of (8.5) is independent of the other two and its characteristic equation has roots $\pm\omega_0\sqrt{-1}$ (where $\omega_0 = \sqrt{A}$). Moulton shows the roots of the characteristic equation for the other two differential equations to be $\pm\sigma_0\sqrt{-1}$ and

[33] This is another example in which Moulton makes use of his artifice, only making this substitution where μ occurs explicitly, not where it occurs implicitly in ξ_0 and η_0.

[34] In *Periodic Orbits*, Moulton takes a slightly different approach and allows the initial conditions to be entirely general at this stage.

[35] As noted in Section 7.3.2, the non-homogeneous terms are found to be products of periodic terms and exponential terms; a case which is considered both in Moulton and MacMillan [1911] and in *Periodic Orbits*.

[36] The complementary functions are the same for the terms of all degrees.

$\pm\rho_0$, where σ_0 and ρ_0 are real. Assuming the roots to be distinct and not to differ by an imaginary integer,[37] the form of the solution is known to be

$$
\begin{aligned}
x_1 &= a_1 e^{\sigma\sqrt{-1}t} u_1(t) + a_2 e^{-\sigma\sqrt{-1}t} u_2(t) + a_3 e^{\rho t} u_3(t) + a_4 e^{-\rho t} u_4(t),\\
y_1 &= a_1 e^{\sigma\sqrt{-1}t} v_1(t) + a_2 e^{-\sigma\sqrt{-1}t} v_2(t) + a_3 e^{\rho t} v_3(t) + a_4 e^{-\rho t} v_4(t),\\
z_1 &= c_1 e^{\omega\sqrt{-1}t} w_1(t) + c_2 e^{-\omega\sqrt{-1}t} w_2(t),
\end{aligned}
$$

where the u_i, v_i, w_i, σ, ρ and ω are all power series in e.[38] The a_1, a_2, a_3, a_4, c_1 and c_2 are arbitrary constants, which will assume an importance once the periodicity conditions are applied. Moulton derives the properties of the solutions for the terms to the third order, as certain of these are needed in his existence proof, but he does not compute the series explicitly.

Having found the form of the symmetric solutions, Moulton next imposes the periodicity conditions. Since (8.5) involve t explicitly and the terms are 2π-periodic, it follows that any periodic solutions will have periods which are integer multiples of 2π (see Chapter 4, footnote 55). Letting the period be $T = 2\pi n$, a total of six conditions are imposed on the parameters a_1, a_2, a_3, a_4, c_1, c_2 and λ. Three of these ensure a second symmetric conjunction at $t = T/2$ and three follow from the symmetry of the solution. The resulting system of six equations, analytic in these parameters, is satisfied identically by the trivial solution $a_1 = \cdots = a_4 = c_1 = c_2 = 0$. For there to exist another (i.e. non-trivial) solution for a_1, a_2, a_3, a_4, c_1 and c_2, which vanishes with λ, the implicit function theorem tells us that the Jacobian determinant with respect to these six parameters must also vanish with λ. Moulton shows that this in turn implies that either σ or ω—which are power series in e that reduce to σ_0 and $\sqrt{A}$ respectively when e is zero—must be rational numbers. More precisely, $\sigma = \frac{N_1}{n}$ or $\omega = \frac{N_2}{n}$, where N_1 and N_2 are integers and $T = 2\pi n$.

In the remainder of the paper, Moulton proves the existence of both three-dimensional and two-dimensional periodic solutions and shows how they may be constructed. The three-dimensional solutions are shown to be of the form

$$
x = \sum_{j=1}^{\infty} x_{2j}\lambda^j, \quad y = \sum_{j=1}^{\infty} y_{2j}\lambda^j, \quad z = \sum_{j=1}^{\infty} z_{2j-1}\lambda^{\frac{2j-1}{2}}.
$$

For each period $T = 2\pi n$, there are two geometrically distinct periodic orbits, corresponding to the sign of $\lambda^{\frac{1}{2}}$. (Since different apses may be chosen for the time origin, there are in fact $2n$ distinct solutions.)

The two-dimensional solutions are shown to be of the form

$$
x = \sum_{j=1}^{\infty} x_j\lambda^{\frac{j}{2}}, \quad y = \sum_{j=1}^{\infty} y_j\lambda^{\frac{j}{2}}.
$$

For $T = 2\pi n$, Moulton finds the orbit for $+\lambda^{\frac{1}{2}}$ to be geometrically distinct from

[37] Moulton remarks that exceptional values of μ and μ_0 do exist for which certain roots differ by an imaginary integer.

[38] All coefficients of the series u_i, v_i and w_i are 2π-periodic except those of zero-degree in e which are constants. σ and ρ reduce to σ_0 and ρ_0 when $e = 0$; similarly for ω. If required, the coefficients may be found by the method of undetermined coefficients.

that for $-\lambda^{\frac{1}{2}}$ when n is odd, but to be identical when n is even. However, due to the presence of two other geometrically identical periodic orbits which cross the x-axis perpendicularly at the apoapsides, it turns out that there are, once again, two geometrically distinct periodic orbits for the case where n is even.