

# INVARIANT MEASURES

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## Preface

In 1940–1941 von Neumann lectured on invariant measures at the Institute for Advanced Study. This book is essentially a written version of what he said.

The lectures began with general measure theory and went on to Haar measure and some of its generalizations. Shizuo Kakutani was at the Institute that year, and he and von Neumann had many conversations on the subject. The conversations revealed facts and produced proofs—quite a bit of the content of the course, especially toward the end, was discovered just a week or two or three before it appeared on the blackboard. The original version of these notes was prepared by Paul Halmos, von Neumann’s assistant that year. Von Neumann read the handwritten version before it went to the typist, and sometimes scribbled comments on the margins. On Chapter VI, the last one, he did more than scribble—he himself wrote most of it.

The notes were typed. Two or three copies were kept in the Institute—von Neumann had one and the Institute library had another. Since then a few photocopies have been made, but until now the notes have never been published in any proper sense of the word.