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ESSAYS ON NUMBERS AND FIGURES

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Contents

Foreword

This book contains twenty essays, each of which deals with a separate mathematical topic. It may be a brilliant mathematical statement with an interesting proof; or a simple, but effective method of problem solving; or an interesting property of polynomials; or it may refer to exceptional points of the triangle.

In its time, when I first came across each of these topics, I was seriously impressed and was led to reflect about them. I later returned, more than once, to many of them, and each time found something new to think about.

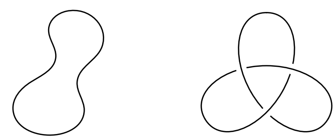
The twenty essays are, for the most part, independent of each other. The only essential exception is the last essay about cubic curves related to a triangle. This essay is based on a previous one, the one about isogonal conjugation with respect to a triangle. But the topic of cubic curves related to triangles is too difficult to be developed without preliminaries anyway.

I have lectured on all these topics to high school students in Moscow, Tel-Aviv, Haifa, and Cheliabinsk.

CHAPTER 5

Can Any Knot Be Unraveled?

A knot may be imagined as a string whose ends are joined together after the string has been tangled in some way. The simplest examples of knots are shown in Figure 1. The knot shown in Figure 1(b) is known as the *trefoil*.



(a) (b)

FIGURE 1

After the ends of the string have been joined, one may try to untangle it or tangle it up further, of course without tearing the string. If you try to "unravel" the trefoil (i.e., to transform it into the knot shown in Figure 1(a)), you will soon see that this cannot be done. But how does one prove that no succession of tangling and untangling manipulations will unravel the trefoil? The first proofs of this fact were based on quite complicated techniques from algebraic topology. It is only recently, in the eighties, that elementary proofs of the fact that the trefoil (and some other knots) cannot be unraveled appeared. Such a proof will now be described.

Any knot can be represented by its projection on some plane, showing which branches are higher and which are lower by interrupting the line depicting the lower branch. Such a picture is called a *knot diagram*; an example of a knot diagram for the trefoil is Figure 1(b).

Any knot diagram consists of several arcs; for instance, the diagram in Figure 1(b) consists of three arcs. Consider all possible colorings of a knot diagram in three colors, each arc being painted in one color. Let us call a coloring *correct* if at each crossing the three arcs either all have the same color, or have three different colors (i.e., two arcs of the same color and a third one of a different color cannot meet at a crossing).

THEOREM. *The number of correct colorings of any knot diagram does not change when we tangle or untangle the knot.*

PROOF. We shall use without proof the following rather obvious statement: the transformation of a knot diagram when we tangle or untangle it reduces to carrying out a succession of operations of the three types shown on Figure 2. (A rigorous proof of this statement can be obtained by replacing the knot by a closed polygonal line and following what happens to its projection in the tangling untangling process.)

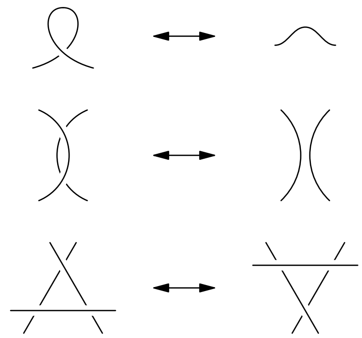


Figure 2

We must check that under the operations shown above each correct coloring becomes a well-defined correct coloring of the transformed diagram. Let us explain what we have in mind. Suppose a correct coloring of the diagram is given. Consider the diagram obtained from the given one by one of the three operations indicated above. This operation is performed within a circular disk D. Let us leave the coloring outside the the disk D. The arcs of the transformed diagram intersecting the boundary of the disk will be colored in some way. We must verify that this coloring can be extended to a correct coloring of all the arcs inside the disk D and that this coloring is unique.

For colorings in one color this is obvious. For all essentially distinct correct colorings in more than one color, the colorings of the corresponding transformed diagrams are shown in Figure 3. The proof of the theorem is complete. □

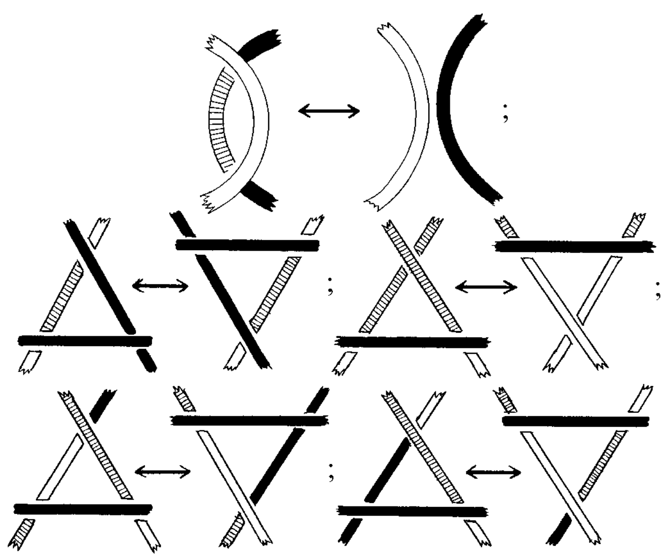


Figure 3

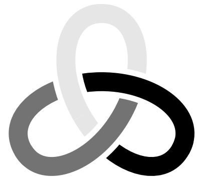


Figure 4

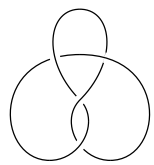


Figure 5

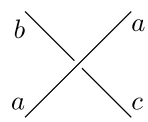


Figure 6

Now we can prove that the trefoil cannot be unraveled. Figure 4 shows a correct coloring of a diagram of the trefoil in which all three colors are used. But the unknotted circle shown in Figure 1(a) can only be colored in one color.

Unfortunately, by using correct colorings of knot diagrams in three colors, it is not always possible to establish that a knot cannot be unraveled. For example, the *figure eight knot*, shown in Figure 5 cannot be unraveled although all of its diagrams possess only monochromatic colorings in three colors, just as the diagrams of the unknotted circle.

In order to prove that the figure eight knot cannot be unraveled, one can use correct colorings in five colors. It is then convenient to regard the colors as remainders under division by 5. We shall say that a coloring of a knot diagram in five colors is *correct* if at each crossing we have the relation

$$b + c \equiv 2a \pmod 5$$

(see Figure 6); this expression means that the numbers $b + c$ and $2a$ have the same remainder under division by 5. In a similar way one can define a correct coloring of a knot diagram in n colors.

By looking at Figure 7 it is easy to check that the number of correct colorings of a knot diagram in n colors does not change in the process of tangling and untangling a knot.

Figure 8 shows that the diagram of the figure eight knot has a multicolor correct coloring in five colors. Hence the figure eight knot cannot be unraveled.

It is not hard to check that all the correct colorings of the trefoil in five colors are monochromatic, so that the trefoil and the eight are different knots (they cannot be transformed into each other).

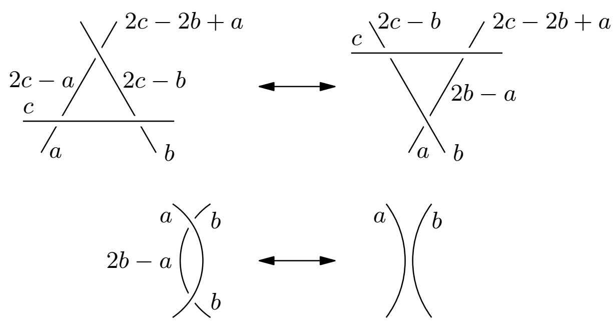


FIGURE 7

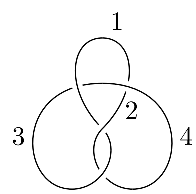


FIGURE 8

CHAPTER 12

Intersection Points of the Diagonals of Regular Polygons

Quite a few problems about triangles with integer angles are known. Here are two examples of such problems.

PROBLEM 1. In an isoceles triangle ABC with base BC the angle at A equals $80°$. A point M is chosen inside the triangle so that $\angle MBC = 30°$ and $\angle MCB = 10°$ (Figure 1(a)). Prove that $\angle AMC = 70°$.

PROBLEM 2. In an isoceles triangle ABC with base AC the angle at B equals $20°$. Points D and E are chosen on the sides BC and AB respectively so that $\angle DAC = 60°$ and $\angle ECA = 50°$ (Figure 1(b)). Prove that $\angle ADE = 30°$.

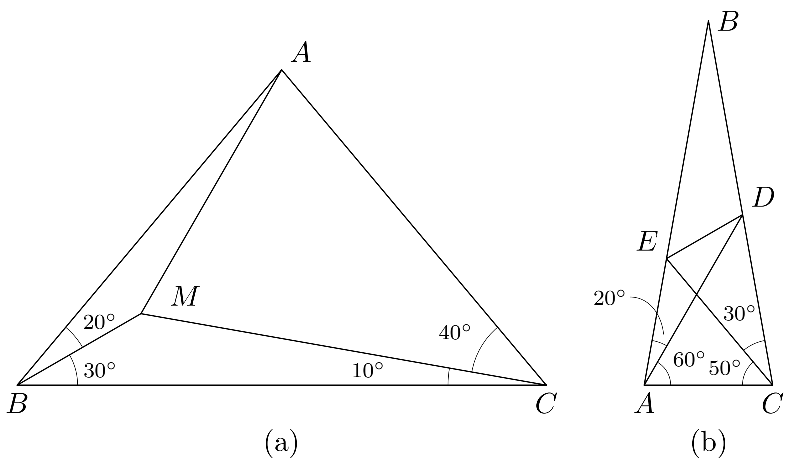


FIGURE 1

Problems of this type are usually related with intersection points of triples of diagonals of a regular polygon, in the present case, of a regular 18-gon.

Let us look at Figure 2. This figure shows that Problem 1 is equivalent to the following statement: *in a regular polygon of 18 sides the diagonals A_1A_{13}, A_3A_{14}, and A_6A_{15} intersect at one point.* Indeed, if these diagonals intersect at some point M, then

$$\angle A_1MA_6 = \frac{1}{2}(\smile A_1A_6 + \smile A_{13}A_{15}) = 50° + 20° = 70°.$$

It is also clear that the angles of triangle $A_1A_6A_{14}$ are $80°$, $40°$, $40°$ and $\angle MA_{14}A_6 = 30°$, $\angle MA_6A_{14} = 10°$.

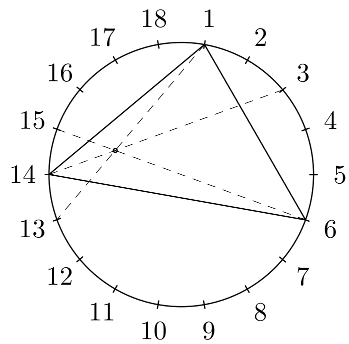


FIGURE 2

As to Problem 2, it is equivalent to the following statement: in a regular 18-gon the diagonals A_1A_{14}, A_7A_{16} and $A_{11}A_{17}$ intersect at one point (Figure 3).

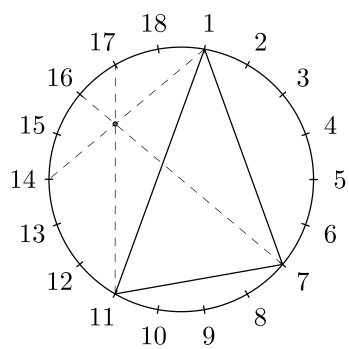


FIGURE 3

But Problem 2 can be solved by using a different triple of intersecting diagonals, namely A_1A_{13}, A_3A_{14} and A_6A_{15} (Figure 4). In the role of triangle ABC we take $A_{14}OA_{15}$. The diagonals A_1A_{13} and A_9A_{15} are symmetric with respect to the diagonal A_5A_{14}, hence both diagonals intersect the diameter at one point.

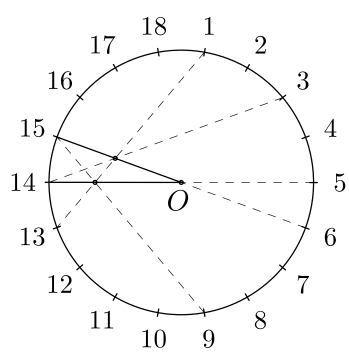


FIGURE 4

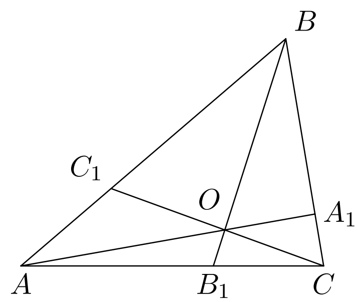


FIGURE 5

But we have not yet proved that the diagonals in Figures 2–4 do in fact intersect at one point. It is convenient to verify that triples of diagonals intersect at one point by using the following statement.

THEOREM. *The points A_1, B_1, C_1 are chosen on the sides of triangle ABC (A_1 on BC, etc.). The segments AA_1, BB_1 and CC_1 intersect at one point if and only if*

$$\frac{\sin BAA_1}{\sin CAA_1} \cdot \frac{\sin ACC_1}{\sin BCC_1} \cdot \frac{\sin CBB_1}{\sin ABB_1} = 1.$$

PROOF. First let us assume that the segments AA_1, BB_1 and CC_1 intersect at the point O (Figure 5). Then

$$2S_{AOB} : 2S_{AOC} = (AB \cdot AO \sin BAO) : (AC \cdot AO \sin CAO).$$

Therefore,

$$\begin{aligned} 1 &= \frac{S_{AOB}}{S_{AOC}} \cdot \frac{S_{COA}}{S_{COB}} \cdot \frac{S_{BOC}}{S_{BOA}} \\ &= \left(\frac{AB}{AC} \cdot \frac{CA}{CB} \cdot \frac{BC}{BA}\right) \frac{\sin BAO}{\sin CAO} \cdot \frac{\sin ACO}{\sin BCO} \cdot \frac{\sin CBO}{\sin ABO}, \end{aligned}$$

i.e.,

$$\frac{\sin BAA_1}{\sin CAA_1} \cdot \frac{\sin ACC_1}{\sin BCC_1} \cdot \frac{\sin CBB_1}{\sin ABB_1} = 1.$$

Now assume that the equation indicated in the theorem holds for the points A_1, B_1 and C_1. Let O be the intersection point of the segments AA_1 and BB_1. We must prove that the segment CC_1 passes through O as well. In other words, if C_1' is the intersection point of the lines CO and AB, then $C_1' = C_1$. The segments AA_1, BB_1 and CC_1' intersect at one point, so, as we have just proved,

$$\frac{\sin BAA_1}{\sin CAA_1} \cdot \frac{\sin CBB_1}{\sin ABB_1} \cdot \frac{\sin ACC_1'}{\sin BCC_1'} = 1.$$

Comparing this formula with the condition of the theorem, we obtain

$$\sin ACC_1 : \sin BCC_1 = \sin ACC_1' : \sin BCC_1'.$$

It remains to prove that when the point X moves along the segment AB, the value $\sin ACX : \sin BCX$ changes monotonically. The angles $\angle ACX$ and $\angle BCX$ themselves do change monotonically, but their sines are not necessarily monotone in the case of an obtuse angle C. But this is no tragedy. In any triangle there is an acute angle, and from the very beginning we could have taken such an angle for C. This completes the proof of the theorem. □

Now the verification of the fact that the triple of diagonals shown in Figure 2 intersect at one point reduces to proving the following identity.

$$\frac{\sin 10^\circ}{\sin 70^\circ} \cdot \frac{\sin 30^\circ}{\sin 20^\circ} \cdot \frac{\sin 40^\circ}{\sin 10^\circ} = 1.$$

This is not hard to prove:

$$\sin 30^\circ \sin 40^\circ = \frac{1}{2}\sin 40^\circ = \sin 20^\circ \cos 20^\circ = \sin 20^\circ \sin 70^\circ.$$

The triple of diagonals shown in Figure 3 corresponds to the identity

$$\sin 20^\circ \sin 40^\circ \sin 20^\circ = \sin 30^\circ \sin 60^\circ \sin 10^\circ.$$

There are three more identities that yield triples of intersecting diagonals:

$$\sin 10^\circ \sin 20^\circ \sin 80^\circ = \sin 20^\circ \sin 20^\circ \sin 30^\circ,$$
$$\sin 20^\circ \sin 30^\circ \sin 30^\circ = \sin 10^\circ \sin 40^\circ \sin 50^\circ,$$
$$\sin 10^\circ \sin 20^\circ \sin 30^\circ = \sin 10^\circ \sin 10^\circ \sin 100^\circ.$$

Their verification is left to the reader.

Note that the change of order of the factors in these identities leads to completely different triples of intersecting diagonals.

Our interest in the regular polygon of 18 sides rather than some other regular polygon is related to the fact that triangles with angles that are multiples of 10° lead to that particular polygon. Among all the regular polygons with less than 18 vertices, interesting families of diagonals appear only in the 12-gon. For example, the diagonals A_1A_5, A_2A_6, A_3A_8, and A_4A_{11} of the regular 12-gon intersect at one point (Figure 6). This statement is equivalent to the following well-known problem.

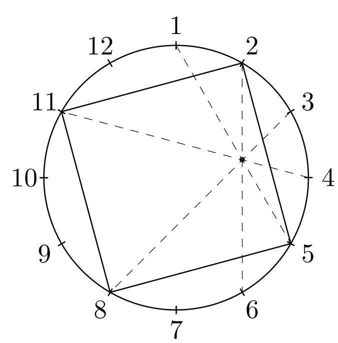


FIGURE 6

PROBLEM 3. The point P is chosen inside square $ABCD$ so that triangle ABP is equilateral. Prove that $\angle PCD = 15^\circ$.

Exercises. 1. Triangle ABC with angles $\angle A = 50^\circ$, $\angle B = 60^\circ$, $\angle C = 70^\circ$ is given.

a) Points D and E on sides BA and AC are chosen so that $\angle DCB = \angle EBC = 40^\circ$. Prove that $\angle AED = 30^\circ$.

b) Points D and E on sides BA and AC are chosen so that $\angle DCA = 50^\circ$ and $\angle EAC = 40^\circ$. Prove that $\angle AED = 30^\circ$.

2. In triangle ABC the angles A, B, and C equal $14°$, $62°$, and $104°$. The points D and E are chosen on sides AC and AB respectively so that $\angle DBC = 50°$ and $\angle ECB = 94°$. Prove that $\angle CED = 34°$.

3. Prove that the diagonals A_1A_{n+2}, $A_{2n-1}A_3$,and $A_{2n}A_5$ of a regular $2n$-gon intersect at one point.

4. Prove that the diagonals A_1A_7, A_3A_{11}, and A_5A_{21} of a regular 24-gon intersect at a point lying on the diameter A_4A_{16}.

5. Prove that in a regular 30-gon the seven diagonals

$$A_1A_{13},\ A_2A_{17},\ A_3A_{21},\ A_4A_{24},\ A_5A_{26},\ A_8A_{29},\ A_{10}A_{30}$$

intersect at one point.