Chapter 1

Elementary Properties of Curves of Second Degree

1.1. Definitions

If you stake a goat, it will graze the grass inside the circle that is centered at the stake and has radius the length of the rope. If you use two stakes at the ends of the rope and tie the goat using a sliding ring, the region with grazed grass will look like the one shown in Figure 1.1.

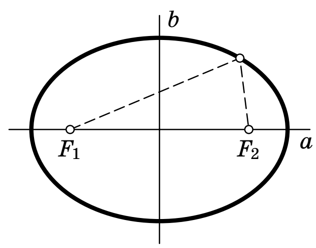

FIGURE 1.1. F_1 and F_2 are the foci; a and b are the major and the minor axes.

For all points on the boundary of that figure, the sum of the distances to the stakes equals the length of the rope. Such a curve is called an *ellipse*, and the points marked by the stakes are called the *foci*.

Clearly, an ellipse looks like an "elongated circle". It obviously has two axes of symmetry. These are the line connecting the foci and the midpoint perpendicular to the segment with endpoints at the foci. These two lines are called the *major* and the *minor axes of the ellipse*. The lengths of their parts inside the ellipse are called the lengths of the major and minor axes. The distance between the foci is called the focal distance.

It is also clear that the length of the rope holding the goat equals the length of the major axis of the elliptical boundary of the grazed region.

Intuitively it is clear that the goat can graze at any point inside the ellipse but it can never get beyond the ellipse. But a purely mathematical reformulation of this is no longer so obvious.

Exercise 1. Prove that the sum of the distances from any point inside the ellipse to the foci is less—and from any point outside the ellipse is greater—than the length of the major axis.

Solution. Denote by F_1 and F_2 the foci of the ellipse, and by X a point. Let Y be the intersection of the ray F_1X and the ellipse. Assume first that X is inside the ellipse. By the triangle inequality, $F_2X < XY + YF_2$, and hence $F_1X + XF_2 < F_1X + XY + YF_2 = F_1Y + F_2Y$ (Figure 1.2).

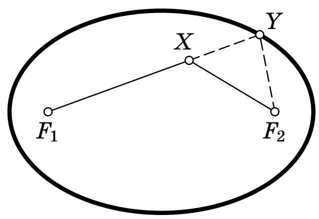

FIGURE 1.2

But $F_1Y + F_2Y$ equals the length of the rope, i.e., the major axis of the ellipse. Using a similar argument when X is outside the ellipse, we have $F_2Y < XY + XF_2$. Therefore $F_1X + XF_2 = F_1Y + YX + XF_2 > F_1Y + F_2Y$.

Ellipses often arise in mechanics. For example, a planet orbiting the Sun moves along an ellipse with the Sun at one of its foci (Kepler's Law).

An ellipse is an example of a *curve of second degree* or a *conic*. Other examples of such curves are *parabolas* and *hyperbolas*.

A *hyperbola* is the set of points for which the absolute value of the difference between the distances to two fixed points, called the *foci*, is constant.

A hyperbola consists of two branches the ends of which approach two lines called the *asymptotes of the hyperbola* (Figure 1.3). A hyperbola with perpendicular asymptotes is said to be *equilateral*.

The line passing through the foci of a hyperbola is an axis of symmetry and is called the *real axis*. The perpendicular line passing through the midpoint between the foci is also an axis of symmetry and is called the *imaginary axis* of the hyperbola.

If a comet is passing by the Sun and the gravitational force exerted by the Sun is too small to keep the comet within the solar system, then its trajectory will be an arc of a hyperbola whose focus will be at the center of the Sun.

A *parabola* is the set of points whose distances to some fixed point and line are constant. That point and line are called, respectively, the *focus* and the *directrix* of the parabola. The line perpendicular to the directrix and passing through the focus is called the *axis of the parabola* (Figure 1.4).

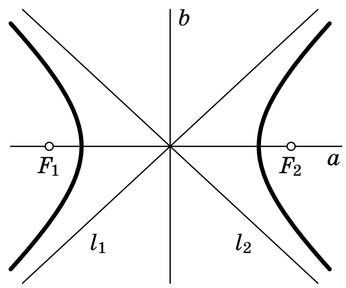

FIGURE 1.3. F_1 and F_2 are the foci, a and b are the real and imaginary axes, and l_1 and l_2 are the asymptotes.

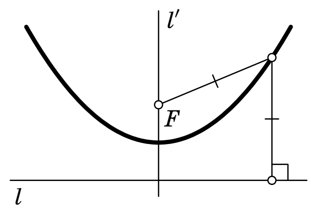

FIGURE 1.4. F is the focus; l and l' are the directrix and the axis of the parabola.

Clearly, it is an axis of symmetry of the parabola.

We remark that a stone thrown at an angle to the horizon will move along a parabola.

In a way, from the geometric point of view, there is only one parabola (just as there is only one circle). More precisely, all the parabolas are similar, i.e., they can be transformed into one another by rotational homotheties.

Consider a family of ellipses with focus at a fixed point and passing through another given point. We send the other focus to infinity along some direction. Then those ellipses will tend to a parabola with the same focus and axis parallel to the chosen direction. A similar experiment works for hyperbolas. Thus the parabola is a limit case of both the ellipse and the hyperbola.

Exercise 2. State and prove, for the parabola and the hyperbola, the results similar to the one in Exercise 1.

Solution. For the points inside the parabola the distance to the focus is less than the distance to the directrix, and for the points outside the parabola the opposite is true (Figure 1.5).

Let Y be the projection of X to the directrix, Z the intersection of XY with the parabola, and F the focus of the parabola. By the definition of the parabola, $FZ = ZY$. If X lies inside the parabola, then $XY = XZ + ZY$. By the triangle inequality, $FX < FZ + ZX = ZY + ZX = XY$. If X and the parabola are on different sides of the directrix, then the assertion

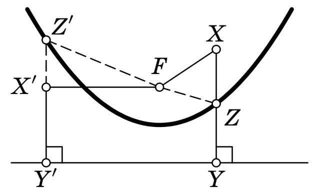

FIGURE 1.5

is obvious. Suppose X' is outside the parabola but on the same side of the directrix. Then $Z'Y' = Z'X' + X'Y'$ and, by the triangle inequality, $FX' + X'Z' > FZ' = Z'Y' = Z'X' + X'Y'$. Therefore $FX' > X'Y'$. □

In the case of a hyperbola the corresponding statement is as follows: let d be the difference of the distances from any point on the hyperbola to the foci F_1 and F_2 and let Γ be the branch of the hyperbola inside which F_1 lies. Then for the points X outside (inside) Γ the quantity $XF_2 - XF_1$ is less (greater) than d.

Suppose X lies inside Γ and let Y be the intersection of the ray F_2X and Γ. We have $F_2X = F_2Y + YX$. By the triangle inequality, $F_1X < F_1Y + YX$; therefore $F_2X - F_1X > (F_2Y + YX) - (F_1Y + YX) = F_2Y - F_1Y = d$.

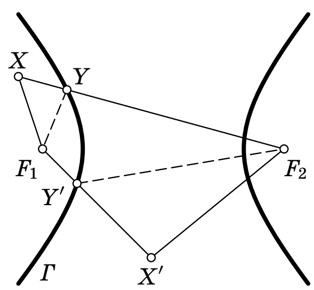

FIGURE 1.6

If X' is outside Γ, let Y' be the point of intersection of F_1X' and Γ. Then $F_1X' = F_1Y' + Y'X'$. By the triangle inequality, $F_2X' < F_2Y' + Y'X'$. Therefore $F_2X' - F_1X' < (F_2Y' + Y'X') - (F_1Y' + Y'X') = F_2Y' - F_1Y' = d$.

We remark (without a proof, for the time being) that the ellipse, the parabola and the hyperbola have the following properties: an arbitrary line intersects each of those curves in at most two points, and, given any point in the plane, there are at most two tangents from that point to the curve. These properties are obvious consequences of the results of 1.5.

Exercise 3. Find the locus of the centers of the circles tangent to two given circles.

Solution. For the sake of definiteness, consider the case when none of the circles with centers O_1, O_2 and radii r_1, r_2 contains the other. If the circle centered at O of radius r is tangent to the two circles on the outside, then

$OO_1 = r + r_1$ and $OO_2 = r + r_2$, and therefore $OO_1 - OO_2 = r_1 - r_2$, i.e., O lies on one of the branches of the hyperbola with foci O_1 and O_2. Similarly, if a circle is tangent to both circles on the inside, then its center lies on the other branch of the same hyperbola. If one of the tangencies is on the inside and the other on the outside, then the absolute value of the difference in distances OO_1 and OO_2 is equal to $r_1 + r_2$, i.e., O sweeps another hyperbola with the same foci. Similarly, if one circle is inside the other, then the desired locus consists of two ellipses with foci O_1 and O_2 and major axes $r_1 + r_2$ and $r_1 - r_2$. The case of intersecting circles is left to the reader.

1.2. Analytic definition and classification of curves of second degree

In the previous section we mentioned the fact that the ellipse, parabola, and hyperbola are particular cases of curves of degree two. Now we make this more precise by showing that, in a sense, there are no other curves of degree two.

Definition. A *curve of second degree* is a set of points whose coordinates in some (and therefore in any) Cartesian coordinate system satisfy a second order equation:

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2b_1x + 2b_2y + c = 0. \tag{1}$$

If the left-hand side of (1) is a product of two linear factors, then the curve is the union of two lines (which may coincide). In that case it is said to be *degenerate*. A curve which contains exactly one point (for example, $x^2+y^2 = 0$) is also said to be degenerate.

It is a known result from analytic geometry (see, for example, [**1**]) that for any nondegenerate curve there is a coordinate system in which its equation has a rather simple form. We now describe the main idea behind this result.

First, rotate the coordinate system through an angle ϕ. This means that, in equation (1), the coordinates x and y should be replaced by, respectively, $x\cos\phi + y\sin\phi$ and $-x\sin\phi + y\cos\phi$. Choosing an appropriate ϕ, we can make the coefficient of xy equal to zero. Next we move the origin to (x_0, y_0), i.e., we replace x by $x + x_0$ and y by $y + y_0$. By choosing an appropriate pair (x_0, y_0) we can transform (1) into one of the three canonical forms (I), (II), or (III).

A direct calculation shows that the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a \geq b > 0, \tag{I}$$

is an *ellipse* centered at the origin, with foci at $(\pm\sqrt{a^2 - b^2}, 0)$ and major and minor semi-axes (i.e., half the lengths of the corresponding axes) equal, respectively, to a and b. In the special case $a = b$, ellipse (I) is a circle.

The curve

$$\text{(II)} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad a > 0, \quad b > 0,$$

is a *hyperbola* that intersects its real axis in two points at distance $2a$ from each other. The quantities a and b are called, respectively, the real and the imaginary semi-axes of the hyperbola. The lines $x/y = \pm a/b$ are the asymptotes of the hyperbola and the points $(\pm\sqrt{a^2+b^2}, 0)$ are the foci. When $a = b$ hyperbola (II) is equilateral.

If

$$\text{(III)} \qquad y^2 = 2px, \quad p > 0,$$

the curve is a parabola, whose axis coincides with the x-axis, the focus is at $(p/2, 0)$, and the directrix is given by $x = -p/2$.

The curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$$

is called an *imaginary ellipse*; it contains no real points.

Henceforth, unless stated otherwise, a curve of degree two will always be nondegenerate and not imaginary.

Problem 1. Prove that the equation $y = 1/x$ describes a hyperbola and find its foci.

1.3. The optical property

As is known, if a ray of light is reflected in a mirror, then the reflection angle equals the incidence angle. This is related to the so-called Fermat principle, which states that the light always travels along the shortest path. We shall now prove that the path is indeed the shortest one.

Thus we have a line l and points F_1 and F_2 lying on the same side of it. We want to find a point P on the line such that the sum of the distances from P to F_1 and F_2 is minimal. Reflecting F_2 in l we have a point F_2'. Clearly, $F_2X = F_2'X$ for any point X on l. Thus we need a point P such that the sum of the distances from P to F_1 and F_2' will be the smallest possible. Clearly, the minimum is attained when P lies on the segment F_1F_2' intersecting l. Then the angles in question are obviously equal (Figure 1.7).

Exercise 1. a) When will the absolute value of the difference in distances from P to points F_1 and F_2 lying on different sides of l be maximal?

b) Given two lines l and l' and a point F not on any of those lines, find a point P on l such that the (signed) difference of distances from it to l' and F is maximal.

Solution. a) Let F_2' be the reflection of F_2 in l. Clearly, $F_2X = F_2'X$ for any point X on l. We need a point P such that the difference of distances from P to F_1 and F_2' is maximal. It follows from the triangle inequality that

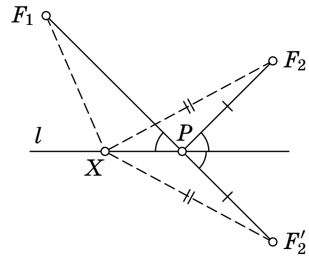

FIGURE 1.7

$|F_1P - F_2'P| < F_1F_2'$ and the maximum is attained if and only if F_1, F_2' and P lie on a straight line. Since the points F_2 and F_2' are the reflections of each other, the angles formed by the lines F_1P and F_2P with l are equal (Figure 1.8).

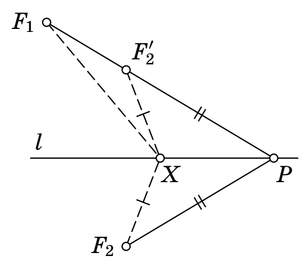

FIGURE 1.8

b) Let F' be the reflection of F in l. Of the two points F and F' choose the one whose (signed) distance to l' is minimal. Let it be F and let d be the distance from F to l'. Then for any point P on l the distance to l' is not greater than $PF + d$. Therefore the difference in question never exceeds d. On the other hand, it is exactly d when P lies on the perpendicular to l' passing through F (Figure 1.9).

We also note that if the line F_1F_2' in a) is parallel to l and the line l' in b) is perpendicular to l, then there is no maximum (it is attained at infinity).

Now we state one of the most important properties of conics, the so-called optical property.

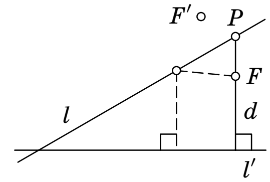

FIGURE 1.9

Theorem 1.1 (The optical property of the ellipse). *Suppose a line l is tangent to an ellipse at a point P. Then l is the bisector of the exterior angle F_1PF_2* (Figure 1.10).

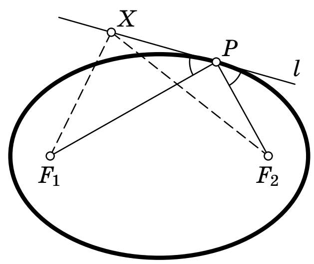

FIGURE 1.10

Proof. Let X be an arbitrary point of l different from P. Since X is outside the ellipse, we have $XF_1 + XF_2 > PF_1 + PF_2$, i.e., of all the points of l the point P has the smallest sum of the distances to F_1 and F_2. This means that the angles formed by the lines PF_1 and PF_2 with l are equal. $\square$

Exercise 2. State and prove the optical property for parabolas and hyperbolas.

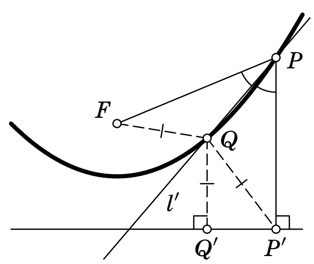

FIGURE 1.11

Solution. For parabolas the optical property is stated as follows. Suppose a line l is tangent to a parabola at a point P. Let P' be the projection of P to the directrix. Then l is the bisector of the angle FPP' (Figure 1.11).

Suppose that the bisector of the angle FPP' (call it l') intersects the parabola in yet another point, say, Q whose projection to the directrix is denoted Q'. By the definition of the parabola, $FQ = QQ'$. On the other hand, triangle FPP' is isosceles, and the bisector of the angle P is the midpoint perpendicular to FP'. Therefore for any point Q on that bisector we have $QP' = QF = QQ'$. But this is impossible because Q' is the only point on the directrix of the parabola where the distance to Q is minimal.

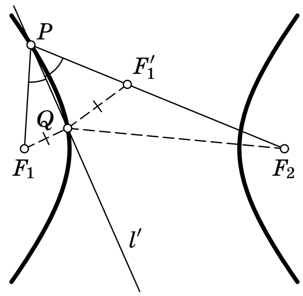

FIGURE 1.12

We now state the optical property for the hyperbola.

If a line l is tangent to a hyperbola at a point P, then l is the bisector of the angle F_1PF_2, where F_1 and F_2 are the foci of the hyperbola (Figure 1.12).

Suppose that the bisector l' of the angle F_1PF_2 intersects the hyperbola at yet another point Q (lying on the same branch with P). For convenience, assume that P lies on the branch closer to F_1. Let F_1' be the reflection of F_1 in l'. Then $F_1Q = QF_1'$, $F_1P = PF_1'$; moreover F_2, F_1' and P lie on a line. Thus, $F_2P - PF_1 = F_2Q - F_1Q$, and therefore $F_2F_1' = F_2P - PF_1' = F_2Q - QF_1'$. But, by the triangle inequality, $F_2F_1' > F_2Q - QF_1'$.

The above results can also be proved by arguments similar to the proof of the optical property of the ellipse. For that, use Exercise 1.

The optical property of the parabola was already known in ancient Greece. For example, Archimedes, by arranging copper plates into a parabolic mirror, managed to set on fire the Roman fleet laying siege to Syracuse.

Exercise 3. Consider the family of confocal conics (these are conics with the same foci). Prove that any hyperbola and any ellipse from that family intersect at right angles (the angle between two curves is by definition the angle between the tangents to them at their point of intersection; see Figure 1.13).

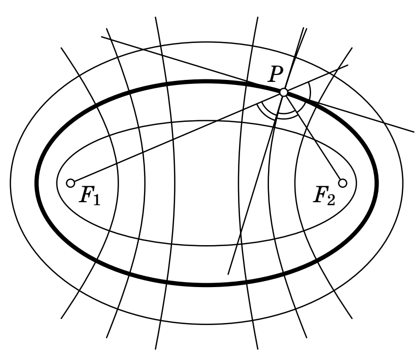

FIGURE 1.13

Solution. Suppose an ellipse and a hyperbola with foci F_1 and F_2 intersect at P. Then their tangents at that point will be the bisectors of the exterior and interior angles F_1PF_2, respectively. Therefore they are perpendicular.

Theorem 1.2. *Suppose the chord PQ contains a focus F_1 of the ellipse and R is the intersection of the tangents to the ellipse at P and Q. Then R is the center of an excircle of the triangle F_2PQ, and F_1 is the tangency point of that circle and the side PQ* (Figure 1.14).

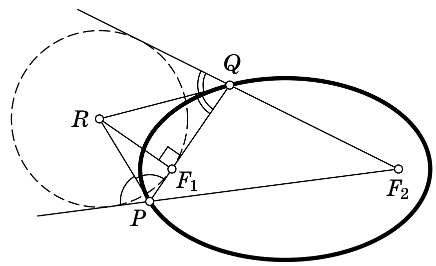

FIGURE 1.14

Proof. By the optical property, PR and QR are the bisectors of the exterior angles of the triangle F_2PQ. Therefore R is the center of an excircle. The tangency point (call it F_1') of the excircle and the corresponding side and the point F_2 cut the perimeter of the triangle into equal parts, i.e., $F_1'P+PF_2 = F_2Q+QF_1'$. But F_1 has this property and there is only one such point. Hence F_1' and F_1 coincide. □

Corollary. *The straight line connecting a focus of an ellipse and the intersection of the tangents to the ellipse at the ends of a chord containing that focus is perpendicular to the chord.*

For the hyperbola, Theorem 1.2 is also true but the excircle should be replaced by the incircle.

1.4. The isogonal property of conics

The optical property yields elementary proofs of some amazing results.

Theorem 1.3. *From any point P outside an ellipse draw two tangents to the ellipse, with tangency points X and Y. Then the angles F_1PX and F_2PY are equal (F_1 and F_2 are the foci of the ellipse).*

Proof. Let F_1', F_2' be the reflections of F_1 and F_2 in PX and PY, respectively (Figure 1.15).

Then $PF_1' = PF_1$ and $PF_2' = PF_2$. Moreover, the points F_1, Y and F_2' lie on a line (because of the optical property). The same is true for the points F_2, X and F_1'. Thus $F_2F_1' = F_2X + XF_1 = F_2Y + YF_1 = F_2'F_1$.

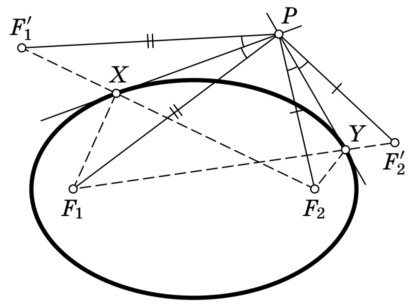

FIGURE 1.15

Thus, the triangles PF_2F_1' and PF_1F_2' are equal (having three equal sides). Therefore

$$\angle F_2PF_1 + 2\angle F_1PX = \angle F_2PF_1' = \angle F_1PF_2' = \angle F_1PF_2 + 2\angle F_2PY.$$

Hence $\angle F_1PX = \angle F_2PY$, which is the desired result.[1] □

Figure 1.16 shows that a similar property holds for the hyperbola.[2]

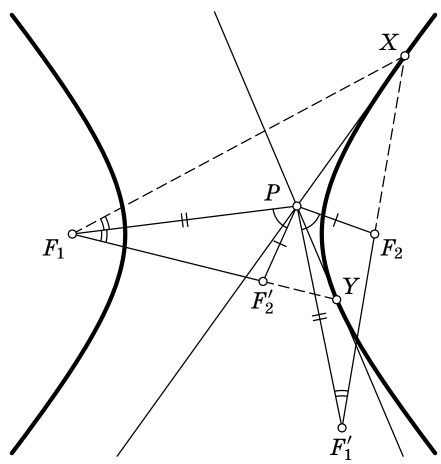

FIGURE 1.16

Suppose now that the ellipse (or hyperbola) with foci F_1 and F_2 is inscribed in triangle ABC. It follows from the above that $\angle BAF_1 = \angle CAF_2$, $\angle ABF_1 = \angle CBF_2$ and $\angle ACF_1 = \angle BCF_2$.

We shall show in 2.3 that, in a plane, for any (with rare exceptions) point X there is a unique point Y such that X and Y are the foci of a

[1]We consider the case when F_1 and F_2 are inside the angle $F_1'PF_2'$ and F_1 lies inside the angle F_2PF_1'. In the remaining cases the arguments are similar.

[2]The reader should check two cases: when the tangency points are either on different branches or on the same branch.

conic tangent to each side of a triangle. Such Y is said to be the *isogonal conjugate of* X with respect to the triangle.

The construction used in the proof of Theorem 1.3, allows one to obtain yet another interesting result. Since the triangles PF_2F_1' and $PF_2'F_1$ are equal, the angles $PF_1'F_2$ and PF_1F_2' are also equal. Therefore

$$\angle PF_1X = \angle PF_1'F_2 = \angle PF_1F_2' = \angle PF_1Y.$$

Thus we have proved the following generalization of Theorem 1.2.

Theorem 1.4. *In the notation of Theorem* 1.3, *the line* F_1P *is the bisector of the angle* XF_1Y (Figure 1.17).

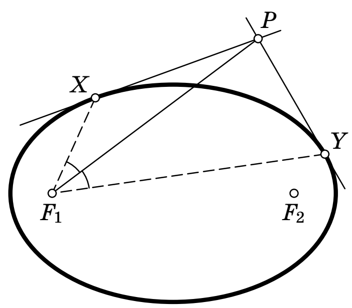

FIGURE 1.17

Theorem 1.5. *The locus of points from which a given ellipse is seen at a right angle* (*i.e., the tangents to the ellipse drawn from such a point are perpendicular*) *is a circle centered at the center of the ellipse* (Figure 1.18).

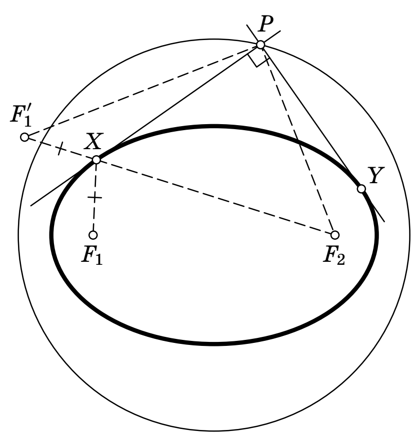

FIGURE 1.18

Proof. Let F_1 and F_2 be the foci of the ellipse and suppose that the tangents to the ellipse at X and Y intersect in P. Reflecting F_1 in PX we have a point F_1'. It follows from Theorem 1.3 that $\angle XPY = \angle F_1'PF_2$ and $F_1'F_2 = F_1X + F_2X$, i.e., the length of the segment $F_1'F_2$ equals the major axis of the ellipse (the length of the rope tying the goat). The angle $F_1'PF_2$ is right if and only if $F_1'P^2 + F_2P^2 = F_1'F_2^2$ (by the Pythagorean theorem). Therefore XPY is a right angle if and only if $F_1P^2 + F_2P^2$ equals the square of the major axis of the ellipse. But it is not difficult to see that this condition defines a circle. Indeed, suppose F_1 has Cartesian coordinates (x_1, y_1), and F_2 has coordinates (x_2, y_2). Then the coordinates of the desired points P satisfy the condition

$$(x - x_1)^2 + (y - y_1)^2 + (x - x_2)^2 + (y - y_2)^2 = C,$$

where C is the square of the major axis. But since the coefficients of x^2 and y^2 are equal (to 2) and the coefficient of xy is zero, the set of points satisfying this condition is a circle. By virtue of symmetry, its center is the midpoint of the segment F_1F_2. □

For the hyperbola such a circle does not always exist. When the angle between the asymptotes of the hyperbola is acute, the radius of the circle is imaginary. If the asymptotes are perpendicular, then the circle degenerates into the point which is the center of the hyperbola.

Example. Given points $P_1, \dots, P_n$ and numbers $k_1, \dots, k_n$ and C, the locus of points X such that $k_1XP_1^2 + \dots + k_nXP_n^2 = C$ is a circle, known as the *Fermat–Apollonius circle*. Clearly, it may have an imaginary radius (when?).

Theorem 1.6. *Suppose a string is put on an ellipse α and then pulled tight using a pencil. If the pencil is rotated about the ellipse, it will traverse another ellipse confocal with α* (Figure 1.19).

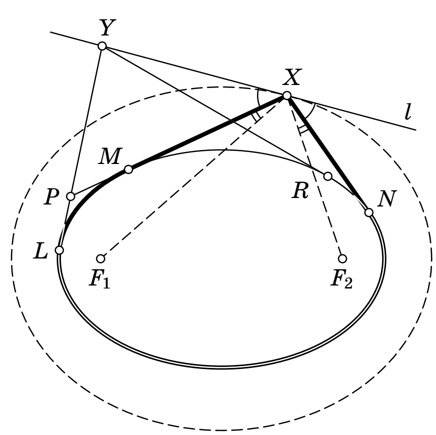

FIGURE 1.19

Proof. Clearly, the new figure (call it α_1) has a smooth boundary. We shall show that at each point X on α_1 the tangent to the new curve coincides with the bisector of the exterior angle F_1XF_2.

Let XM and XN be the tangents to α. Then $\angle F_1XN = \angle F_2XM$, and hence the bisector l of the exterior angle NXM coincides with the bisector of the exterior angle F_1XF_2. Call it l.

Let Y be an arbitrary point on l and YL and YR the tangents to α, as shown in Figure 1.19. We assume that Y lies "to the left" of X; the other case is argued similarly.

Let P be the intersection of the lines XM and YL. It is easy to see that $YN < YR + \smile RN$, and $\smile LM < LP + PM$. Moreover, since l is the exterior bisector of the angle NXP, we have $PX + XN < PY + YN$. Therefore

$$\begin{aligned} MX + XN + \smile NM &< MX + XN + \smile NL + LP + PM \\ &= PX + XN + \smile NL + LP < PY + YN + \smile NL + LP \\ &= LY + YN + \smile NL \\ &< LY + YR + \smile RN + \smile NL = LY + YR + \smile RL \end{aligned}$$

(here the arcs are meant to be the arcs under the string). Therefore Y lies outside α_1. The same is true for any point Y on l. It follows that α_1 contains a single point of l, i.e., the line is tangent. It also follows at once that the obtained curve is convex.

Thus the sum of the distances to the foci F_1 and F_2 does not change with time. Therefore the trajectory of the pencil is an ellipse.

Here is a more rigorous approach to the last claim. Suppose X is outside the ellipse. Put the pencil at X and pull the string around it and around the ellipse. Let $f(X)$ be the length of the string and $g(X) = F_1X + F_2X$ (a point is understood as a pair of its coordinates; thus both f and g depend on a pair of real numbers). One can show that those functions are continuously differentiable and that the vectors $\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ and $\operatorname{grad} g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)$ are nonzero at each point. Then, by the implicit function theorem, the curve traversed by the pencil with a string of fixed length (i.e., a level curve of f) is smooth (continuously differentiable). It now follows that the curve can be parametrized by a differentiable function $R = R(t)$ (this is again a pair of coordinate functions $x = x(t)$, $y = y(t)$) whose tangent vector is different from zero. As shown before, the tangent vector $\frac{dR}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ of the curve is tangent to a level curve of g, i.e., it is perpendicular to $\operatorname{grad} g(R)$ at $R = R(t)$. Consider the function $g(R(t))$. Its derivative is

$$\frac{dg(R(t))}{dt} = \frac{\partial g}{\partial x}\frac{dx(t)}{dt} + \frac{\partial g}{\partial y}\frac{dy(t)}{dt} \equiv 0$$

(this is the orthogonality condition mentioned above), i.e., $g(R(t))$ is constant. This means that our curve lies on an ellipse with the same foci. Since any ray starting at F_1 must contain a point on our curve, the curve coincides with the ellipse. □

Problem 2. A $2n$-gon is circumscribed about a conic with focus F. Its sides are colored in black and white in an alternating pattern. Prove that the sum of the angles at which the black sides are seen from F equals $180°$.

Problem 3. An ellipse is inscribed in a convex quadrilateral such that its foci lie on the (distinct) diagonals of the quadrilateral. Prove that the products of the opposite sides are equal.

1.5. Curves of second degree as projections of the circle

Given a circle, draw the perpendicular through its center to the plane of the circle and pick a point S on it. The lines connecting S to the points of the circle form a cone. Consider the section of the cone by a plane π intersecting all of its rulings and not perpendicular to its axis of symmetry.

Now inscribe in the cone two spheres touching π at points F_1 and F_2 (Figure 1.20).

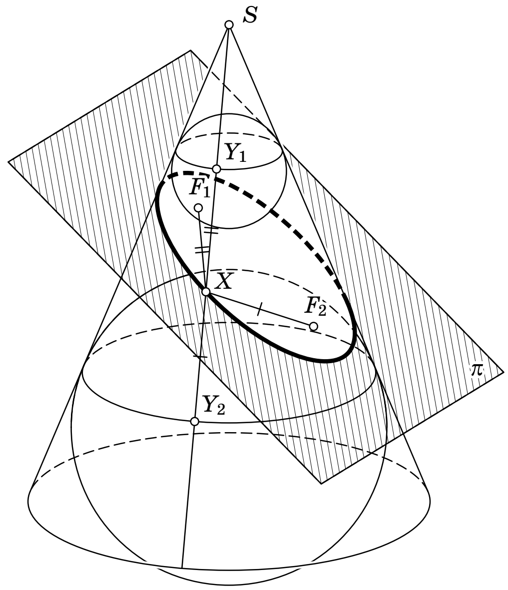

FIGURE 1.20

Let X be an arbitrary point on the intersection of the cone and the plane π. The ruling SX intersects the inscribed spheres at points Y_1 and Y_2. We have $XF_1 = XY_1$ and $XF_2 = XY_2$, since the segments of tangents to a sphere drawn from the same point are equal. Therefore $XF_1 + XF_2 = Y_1Y_2$. But Y_1Y_2 is the segment of the ruling lying between the two planes perpendicular to the axis of the cone, and its length does not depend on the choice of X. Hence the intersection of the cone with π is an ellipse. The ratio of its semiaxes depends on the tilt of the plane and, obviously, can take on any value. Therefore any ellipse can be obtained as a central projection of the circle.

A similar proof shows that if the secant plane is parallel to two rulings of the cone, then the cross-section is a hyperbola (Figure 1.21).

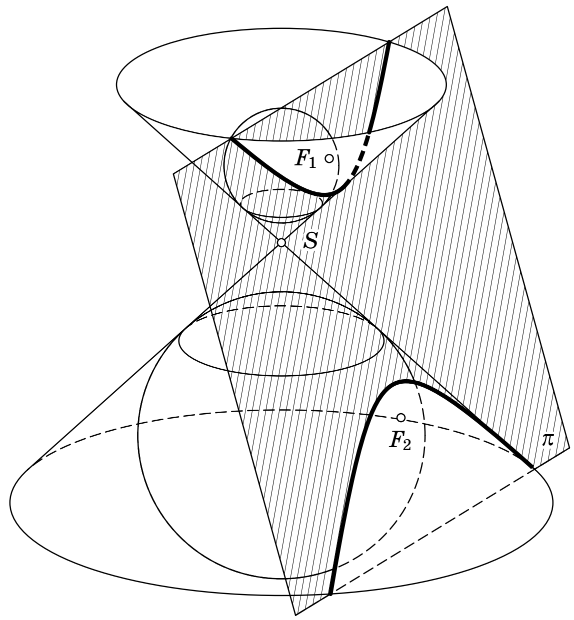

FIGURE 1.21

Finally, consider the case when the secant plane is parallel to one ruling (Figure 1.22).

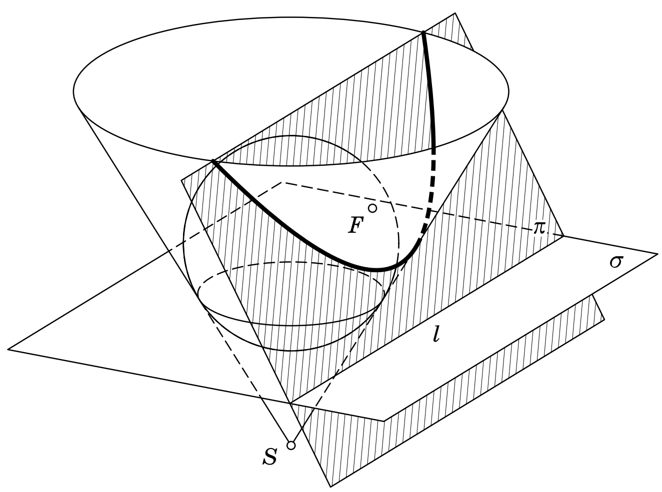

FIGURE 1.22

Inscribe in the cone the sphere tangent to π at a point F. This sphere is tangent to the cone along a circle lying in a plane σ. Let l be the line of intersection of the planes π and σ. For an arbitrary point X in the intersection of the cone and the plane π let Y be the point of intersection of the ruling SX with the plane σ and let Z be the projection of X to l. Then $XF = XY$ since the two segments are tangent to the sphere. On the other hand, Y and Z lie in σ, the angle between XY and σ is equal to the angle between a ruling and a plane perpendicular to its axis, and the angle between XZ and σ is equal to the angle between the planes π and σ. By the choice of π, those angles are equal. Hence $XY = XZ$, since these segments form equal angles with the plane σ. Therefore $XF = XZ$ and X lies on the parabola with focus F and directrix l.

Thus any nondegenerate curve of order two can be obtained as a section of the cone. Because of that, such curves are also called *conic sections* or simply *conics*.

We remark that if the cone is replaced by the cylinder, then the same argument shows that the corresponding section will be an ellipse. Accordingly, the ellipse can be obtained as a parallel projection of the circle.

Exercise 1. Find the locus of the midpoints of the chords of an ellipse which are parallel to a given direction.

Solution. Consider the ellipse as a parallel projection of a circle. Then the parallel chords of the ellipse and their midpoints correspond to parallel chords of the circle and their midpoints, the latter lying on a diameter of the circle. Therefore the locus of the midpoints of parallel chords of the ellipse is also a diameter (i.e., a chord passing through the center).

Exercise 2. Using a straightedge and a compass find the foci of a given ellipse.

Solution. Construct two parallel chords of the ellipse. By the preceding exercise, the line connecting their centers is a diameter of the ellipse. After constructing another diameter, we can find the center O of the ellipse. By the symmetry of the ellipse, a circle centered at O intersects the ellipse at four points forming a rectangle with sides parallel to the axes of the ellipse. Now the foci of the ellipse can be found as the points of intersection of the major axis and the circle centered at the end of the minor axis of radius equal to the major half-axis.

The spheres inscribed in the cone and touching the secant plane are called the *Dandelin spheres*.

1.6. The eccentricity and yet another definition of conics

The construction just described of the Dandelin spheres yields another important property of conics.

Suppose a plane π intersects all the rulings of a circular cone with vertex S. Consider a sphere inscribed in the cone and touching π at a point F_1. As in the parabola case, let σ be the plane containing the tangency points. Let l be the line of intersection of π and σ. Suppose a point X is in the intersection of the cone and the plane π. Let Y be the intersection of the line SX with σ and Z the projection of X to l. We shall show that the ratio of XY and XZ is constant, i.e., does not depend on X.

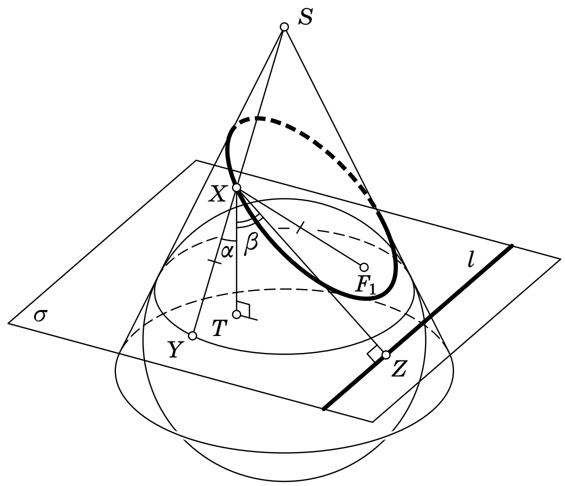

FIGURE 1.23

Let T be the projection of X to σ. The ratio of XT and XY does not depend on X and equals the cosine of the angle between a ruling of the cone and its axis (call that angle α). The ratio of XT and XZ also does not depend on X and equals the cosine of the angle between the plane π and the cone axis (call that angle β). Therefore

$$\frac{XY}{XZ} = \frac{XY}{XT} \cdot \frac{XT}{XZ} = \frac{\cos\beta}{\cos\alpha}.$$

Since XF_1 and XY are equal (as tangents to the sphere passing through X), the ratio of XF_1 and XZ is constant.

Thus for any conic there is a line l such that for any point on the conic the ratio of the distances to the focus and that line is constant. This ratio is called the *eccentricity* of the conic curve, and the lines are called the *directrices*. Both the ellipse and the hyperbola have two directrices (one for each focus).

It is easy to see that this property leads to yet another definition of curves of degree two.

A conic curve with focus F, directrix l (F not on l), and eccentricity ϵ is the set of points where the ratio of distances to F and to l equals ϵ.

If $\epsilon > 1$, then the curve is a hyperbola, if $\epsilon < 1$, it is an ellipse, and when $\epsilon = 1$, it is a parabola.

Problem 4. Prove that the asymptotes of all equilateral hyperbolas with focus F and passing through a point P are tangent to two circles (one circle for each family of the asymptotes).

1.7. Some remarkable properties of the parabola

In this section F denotes the focus of the parabola under consideration.

We begin with a lemma that we will use more than once.

Lemma 1.1. *If the focus of a parabola is reflected in a tangent, then its image will be on the directrix. That image is the projection of the point where the tangent touches the parabola* (Figure 1.24).

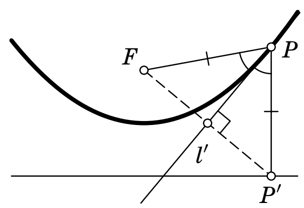

FIGURE 1.24

Proof. Suppose a line l touches the parabola at P and let P' be the projection of P to the directrix. Since the triangle FPP' is isosceles and l is the bisector of the angle P, l is an axis of symmetry of the triangle. Hence the reflection P' of F in l is on the directrix. □

Corollary. *The projections of the focus of the parabola to its tangents lie on the line tangent to the parabola at its vertex.* (Figure 1.25).

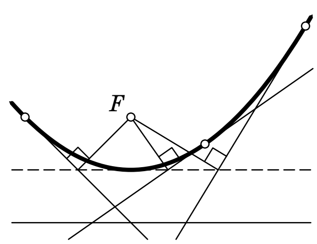

FIGURE 1.25

Lemma 1.2. *Suppose the tangents to the parabola at points X and Y intersect at a point P. Then P is the center of the circumcircle of the triangle $FX'Y'$, where X' and Y' are the projections of X and Y to the directrix of the parabola, and F is the focus of the parabola* (Figure 1.26).

Proof. By Lemma 1.1, these two tangents are midpoint perpendiculars to the segments FX' and FY'. Therefore their point of intersection is the center of the circumcircle of the triangle $FX'Y'$ (Figure 1.26). □

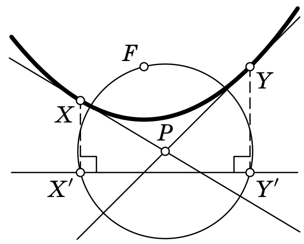

FIGURE 1.26

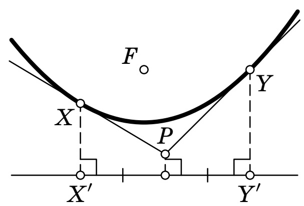

FIGURE 1.27

Corollary. *If PX and PY are tangent to the parabola, then the projection of P to the directrix is the midpoint of the segment with end-points at the projections of X and Y* (Figure 1.27)*.*

The next theorem is similar, with the parabola in place of the ellipse, to Theorems 1.2 and 1.5. What is the set of points where the parabola is seen at a right angle? The answer is given by

Theorem 1.7. *The set of points P where the parabola is seen at a right angle is the directrix of the parabola. Moreover, if PX and PY are tangent to the parabola, then XY contains F and PF is a height of the triangle PXY* (Figure 1.28).

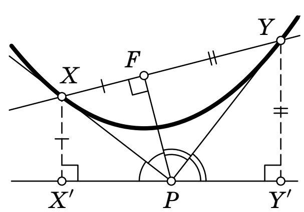

FIGURE 1.28

Proof. Suppose P lies on the directrix, and let X' and Y' be the projections of X and Y to the directrix. Then the triangles PXF and PXX' are equal (since they are symmetric with respect to PX). Hence $\angle PFX = \angle PX'X = 90°$. Similarly, $\angle PFY = \angle PY'Y = 90°$. Moreover, $\angle XPY = \frac{1}{2}(\angle FPX' + \angle FPY') = 90°$. The fact that there are no other points with this property is obvious. □

Since similar assertions are true for the remaining conics, the above theorem seems to be rather natural. However, the first part of the theorem has an unexpected generalization that holds only for parabolas. It will be used later in 3.2 in the proof of Frégier's theorem.

Theorem 1.8. *The set of points from which a parabola is seen at an angle* ϕ *or* $180° - \phi$ *is a hyperbola with focus* F *and directrix* l (Figure 1.29).

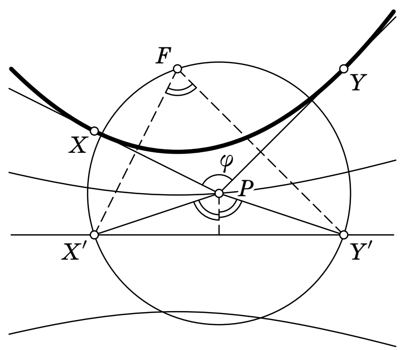

FIGURE 1.29

Proof. Indeed, suppose the tangents PX and PY to the parabola drawn from P form an angle ϕ. We first consider the case when $\phi > 90°$.

Let X' and Y' be the projections of X and Y to the directrix. Clearly, $\angle X'FY' = 180° - \phi$. By Lemma 1.2, P is the center of the circumcircle of the triangle $FX'Y'$. Therefore $\angle X'PY' = 360° - 2\phi$.

Thus the distance from P to the directrix equals $PF\,|\cos(180° - \phi)| = PF\,|\cos\phi|$ and P lies on the hyperbola whose focus and directrix coincide with the focus and directrix of the parabola, and whose eccentricity equals $|\cos\phi|$ (i.e., the angle between the asymptotes equals 2ϕ).

The same is true if the angle between the tangents is $180° - \phi$. Moreover, if the parabola lies inside an acute angle between the tangents, then P is on the "farther" from F branch of the hyperbola, and if it lies inside an obtuse angle, then P is on the "closer" branch. □

For parabolas one can also state a result similar to Theorems 1.3 and 1.4.

Theorem 1.9. *Let* PX *and* PY *be the tangents to the parabola passing through* P*, and let* l *be the line passing through* P *parallel to the axis of the parabola. Then the angle between the lines* PY *and* l *is equal to* $\angle XPF$ *and the triangles* XFP *and* PFY *are similar* (*as a consequence,* FP *is the bisector of the angle* XFY*; see Figure* 1.30).

Proof. Let X' and Y' be the projections of X and Y to the directrix. Then, by Theorem 1.2, the points F, X', and Y' lie on a circle centered at P. Hence $\angle X'Y'F = \frac{1}{2}\angle X'PF = \angle XPF$. On the other hand, the angle between PY

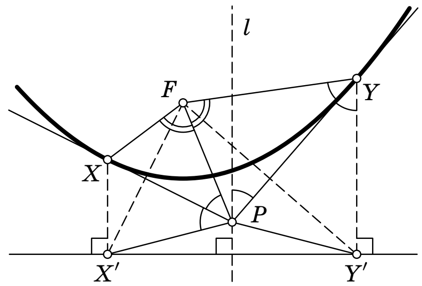

FIGURE 1.30

and l is equal to the angle between $Y'F$ and $X'Y'$ because l is perpendicular to $X'Y'$ (the directrix of the parabola) and $Y'F$ is perpendicular to PY (moreover, PY is the midpoint perpendicular to $Y'F$). This proves the first part of the theorem.

We now prove the second part. Since l is parallel to YY', the angle between PY and l is equal to the angle PYY', which, by the optical property, is equal to the angle FYP. Thus $\angle FYP = \angle XPF$. Similarly, $\angle FXP = \angle YPF$. Therefore the triangles XFP and PFY are similar. □

The next theorem is actually a consequence of Theorem 1.9. But we shall prove it using Simson's line, which will help us find even more interesting properties of the parabola.

Theorem 1.10. *Suppose a triangle ABC is circumscribed about a parabola (i.e., the lines AB, BC, CA are tangent to the parabola). Then the focus of the parabola lies on the circumcircle of the triangle ABC.*

Proof. By the Corollary of Lemma 1.1, the projections of the focus to the sides all lie on a straight line (which is parallel to the directrix and lies at half the distance from the focus). Now we can use Simson's lemma.

Lemma 1.3 (Simson). *The projections of P to the sides of a triangle ABC lie on a line if and only if P lies on the circumcircle of the triangle.*

Proof. Let P_a, P_b and P_c be the projections of P to BC, CA and AB, respectively. We consider the case shown in Figure 1.31; the remaining cases are argued similarly.

The quadrilateral PCP_bP_a is inscribed, hence $\angle PP_bP_a = \angle PCP_a$. Similarly, $\angle PP_bP_c = \angle PAP_c$. The points P_a, P_b and P_c lie on a line if and only if $\angle PP_bP_c = \angle PP_bP_a$ or, equivalently, $\angle PAP_c = \angle PCP_a$. But this means that P lies on the circumcircle of the triangle ABC. The remaining cases are argued similarly.

An identical argument proves the converse. If P lies on the circumcircle of a triangle ABC, then $\angle PAB = \angle PCP_a = \angle PP_bP_a$ (the latter holds since P, C, P_a and P_b lie on a circle). Similarly, $\angle PAB = \angle PP_bP_c$. Therefore P_a, P_b and P_c lie on a straight line. □

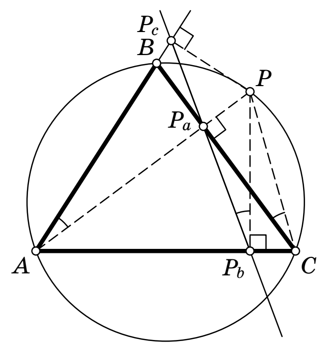

FIGURE 1.31

This proves Theorem 1.10. □

The line just described is called *Simson's line* of P.

Thus with each point on the circumcircle of a triangle ABC we can associate a unique parabola tangent to the sides of the triangle. More precisely, take an arbitrary point P on the circumcircle of the triangle ABC and reflect it in the sides of the triangle. We obtain points P_A, P_B and P_C, lying on a line. The parabola with focus at P and directrix P_AP_C is tangent to all the sides of the triangle (for example, it will touch BC at the point of intersection of BC and the perpendicular to P_AP_C; see Figure 1.32).

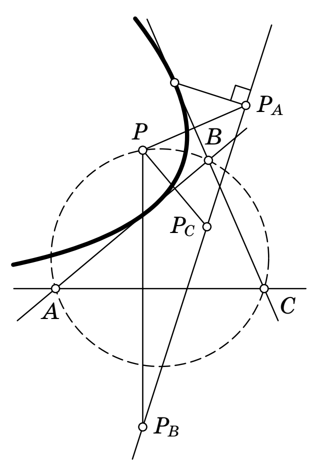

FIGURE 1.32

Simson's line has some interesting properties.

Lemma 1.4. *Suppose a point P lies on the circumcircle of a triangle ABC. Choose a point B' on the circumcircle such that the line PB' is perpendicular to AC. Then BB' is parallel to Simson's line of P* (Figure 1.33).

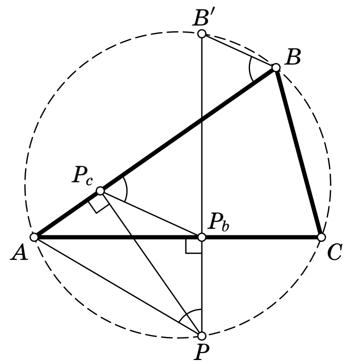

FIGURE 1.33

Proof. Consider the case shown in Figure 1.33; the remaining cases are argued similarly. Let P_c and P_b be the projections of P to the sides AB and AC, respectively. Then $\angle ABB' = \angle APB'$ as the angles subtending the arc AB'. Since quadrilateral AP_cP_bP is inscribed (AP is a diameter of its circumcircle) and the sum of the opposite angles of an inscribed quadrilateral equals 180°, we have $\angle APB' = \angle APP_b = 180^\circ - \angle AP_cP_b = \angle BP_cP_b$. Therefore P_bP_c is parallel to BB'. □

Corollary 1. *When the point P moves along the circle, Simson's line rotates in the opposite direction with velocity one half the rate of change of the arc PA.*

Corollary 2. *Simson's line of P relative to a triangle ABC cuts the segment PH (where H is the orthocenter of the triangle ABC) in half* (Figure 1.34).

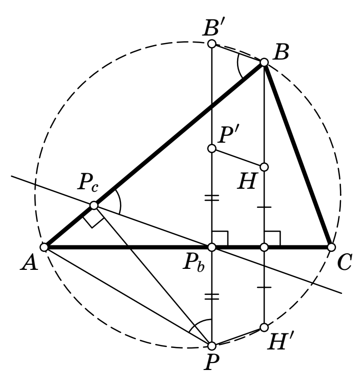

FIGURE 1.34

Proof. It is easy to see that $\angle AHC = 180^\circ - \angle ABC$, and therefore the reflection H' of H in AC lies on the circumcircle of the triangle ABC. Since the lines PB' and BH' are perpendicular to AC, the quadrilateral $PB'BH'$ is a trapezoid; being inscribed, it must be equilateral. Therefore the reflection of PH' in AC (which is a line parallel to the axes of symmetry of the trapezoid) is parallel to BB'. Therefore $P'H$ is parallel to BB', and therefore to Simson's line of P (here P' is the reflection of P in AC). Since P_b (the projection of P to AC) is the midpoint of PP', Simson's line is a midline of the triangle HPP' and therefore cuts HP in half. □

Corollary 2 together with Theorem 1.10 imply the following beautiful result.

Theorem 1.11. *The orthocenter of a triangle circumscribed about a parabola lies on the directrix* (Figure 1.35).

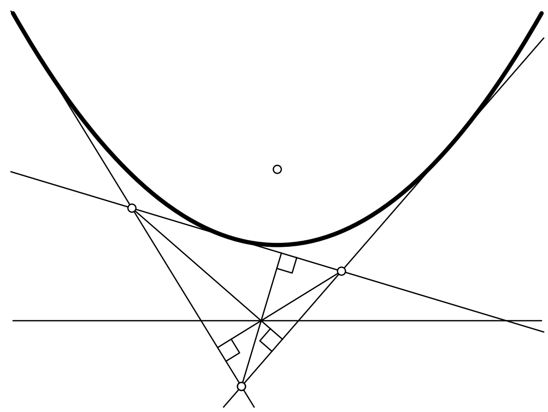

FIGURE 1.35

Problem 5. Suppose a point X moves along a parabola, the normal to the parabola at X (i.e., the perpendicular to the tangent) intersects its axis at a point Y, and Z is the projection of X to the axis. Prove that the length of the segment ZY does not change.

Problem 6. Two travelers move along two straight roads with constant speeds. Prove that the line connecting them is always tangent to some parabola (the roads are not parallel and the travelers pass the intersection at different times).

Problem 7. A parabola is inscribed in an angle PAQ. Find the locus of the midpoints of the segments cut out by the sides of the angle on the tangents to the parabola.