Introduction

The author's perspective. Mathematics and music are both lifelong passions for me. For years they appeared to be independent non-intersecting interests; one did not lead me to the other, and there seemed to be no obvious use of one discipline in the application of the other. Over the years, however, I slowly came to understand that there is, at the very least, a positive, supportive coexistence between mathematics and music in my own thought processes, and that in some subtle way I was appealing to skills and instincts endemic to one subject when actively engaged with the other. In this way the relationship between mathematical reasoning and musical creativity, and the way humans grasp and appreciate both subjects, became a matter of interest that eventually resulted in a college course called Mathematics and Music, first offered in the spring of 2002 at Washington University in St. Louis, the notes of which have evolved into this book.

It has been observed that mathematics is the most abstract of the sciences, music the most abstract of the arts. Mathematics attempts to understand conceptual and logical truth and appreciates the intrinsic beauty of such. Music evokes mood and emotion by the audio medium of tones and rhythms without appealing to circumstantial means of eliciting such innate human reactions. Therefore it is not surprising that the symbiosis of the two disciplines is an age-old story. The Greek mathematician Pythagoras noted the integral relationships between frequencies of musical tones in a consonant interval; the 18th century musician J. S. Bach studied the mathematical problem of finding a practical way to tune keyboard instruments. In today's world it is not at all unusual to encounter individuals who have at least some interest in both subjects.

However, it is sometimes the case that a person with an inclination for one of these disciplines views the other with some apprehension: a mathematically inclined person may regard music with admiration but as something beyond his/her reach. Reciprocally, the musically inclined often view mathematics with a combination of fear and disdain, believing it to be unrelated to the artistic nature of a musician. Perhaps, then, it is my personal mission to attempt to remove this barrier for others, since it has never existed for me, being one who roams freely and comfortably in both worlds, going back and forth between the right and left sides of the brain with no hesitation. Thus I have come to value the ability to bring to bear the whole capacity of the mind when working in *any* creative endeavor.

Purpose of this book. This short treatise is intended to serve as a text for a freshman level college course that, among other things, addresses the issues mentioned above. The book investigates interrelationships between mathematics and music. It reviews some background concepts in both subjects as they are encountered. Along the way, the reader will hopefully augment his/her knowledge of both mathematics and music. The two will be discussed and developed side by side, their languages intermingled and unified, with the goal of breaking down the dyslexia that inhibits their mental amalgamation and encouraging the analytic, quantitative, artistic, and emotional aspects of the mind to work together in the creative process. Musical and mathematical notions are brought together, such as scales/modular arithmetic, octave identification/equivalence relation, intervals/logarithms, equal temperament/exponents, overtones/integers, tone/trigonometry, timbre/harmonic analysis, tuning/rationality. When possible, discussions of musical and mathematical notions are directly interwoven. Occasionally the discourse dwells for a while on one subject and not the other, but eventually the connection is brought to bear. Thus you will find in this treatise an integrative treatment of the two subjects.

Music concepts covered include diatonic and chromatic scales (standard and non-standard), intervals, rhythm, meter, form, melody, chords, progressions, octave equivalence, overtones, timbre, formants, equal temperament, and alternate methods of tuning. Mathematical concepts covered include integers, rational and real numbers, equivalence relations, geometric transformations, groups, rings, modular arithmetic, unique factorization, logarithms, exponentials, and periodic functions. Each of these notions enters the scene because it is involved in one way or another with a point where mathematics and music converge.

The book does not presume much background in either mathematics or music. It assumes high-school level familiarity with algebra, trigonometry, functions, and graphs. It is hoped the student has had some exposure to musical staffs, standard clefs, and key signatures, though all of these are explained in the text. Some calculus enters the picture in Chapter 10, but it is explained from first principles in an intuitive and non-rigorous way.

What is not in this book. Lots. It should be stated up front, and emphasized, that the intent of this book is *not* to teach how to create music using mathematics, nor vice versa. Also it does not seek out connections which are obscure or esoteric, possibly excepting the cursory excursion into serial music (the rationale for which, at least in part, is to ponder the arbitrariness of the twelve-tone chromatic scale). Rather, it explores the foundational commonalities between the two subjects. Connections that seem (to the author) to be distant or arguable, such as the golden ratio, are omitted or given only perfunctory mention.

Yet it should be acknowledged that there is quite a bit of material in line with the book's purpose which is not included. Much more could be said, for example, about polyrhythm, harmony, voicing, form, formants of musical instruments and human vowels, and systems of tuning. And of course there is much more that could be included if calculus were a prerequisite, such as a much deeper discussion of harmonic analysis. Also missing are the many wonderful connections between mathematics and music that could be established, and examples that could be used, involving non-Western music (scales, tuning, form, etc.). This omission owes itself to the author's inexperience in this most fascinating realm.

Overview of the chapters. The book is organized as follows:

- Chapter 1 lays out the basic mathematical and musical concepts which will be needed throughout the course: sets, equivalence relations, functions and graphs, integers, rational numbers, real numbers, pitch, clefs, notes, musical intervals, scales, and key signatures.
- Chapter 2 deals with the horizontal structure of music: note values and time signatures, as well as overall form.
- Chapter 3 discusses the vertical structure of music: chords, conventional harmony, and the numerology of chord identification.
- Musical intervals are explained as mathematical ratios in Chapter 4, and the standard keyboard intervals are introduced in this language.
- Chapter 5 lays out the mathematical underpinnings for additive measurement of musical intervals, relating this to logarithms and exponentials.
- Equal temperament (standard and non-standard) is the topic of Chapter 6, which also gives a brief introduction to twelve-tone music.

- The mathematical foundations of modular arithmetic and its relevance to music are presented in Chapter 7. This involves some basic abstract algebra, which is developed from first principles without assuming any prior knowledge of the subject.
- Chapter 8 delves further into abstract algebra, deriving properties of the integers, such as unique factorization, which are the underpinnings of certain musical phenomena.
- Chapter 9 gives a precursor of harmonics by interpreting positive integers as musical intervals and finding keyboard approximations of such intervals.
- The subject of harmonics is developed further in Chapter 10, which relates timbre to harmonics and introduces some relevant calculus concepts, giving a brief, non-rigorous introduction to continuity, periodic functions, and the basic theorem of harmonic analysis.
- Chapter 11 covers rational numbers and rational, or "just", intervals. It presents certain classical "commas", and how they arise, and it discusses some of the basic just intervals, such as the greater whole tone and the just major third. It also explains why all intervals except multi-octaves in any equal tempered scale are irrational.
- Finally, Chapter 12 describes various alternative systems of tuning that have been used which are designed to give just renditions of certain intervals. Some benefits and drawbacks of each are discussed.

Suggestions for the course. This book is meant for a one-semester course open to college students at any level. Such a course could be cross-listed as an offering in both the mathematics and music departments so as to satisfy curriculum requirements in either field. It could also be structured to fulfill a quantitative requirement in liberal arts. Since the material interrelates with and complements subjects such as calculus, music theory, and physics of sound, it could be a part of an interdisciplinary "course cluster" offered by some universities.

The course will need no formal prerequisites. Beyond the high-school level all mathematical and musical concepts are explained and developed from the ground up. As such the course will be attractive not only to students who have interests in both subjects, but also to those who are fluent with one and desire knowledge of the other, as well as to those who are familiar with neither. Thus the course can be expected to attract students at all levels of college (even graduate students), representing a wide range of majors. Accordingly, the course must accommodate the different sets of backgrounds, and the instructor must be particularly sensitive to the fact that certain material is a review to some in the class while being new to others, and that, depending on the topic, those subgroups of students can vary, even interchange. More than the usual amount of care should be taken to include all the students all the time.

Of course, the topics in the book can be used selectively, rearranged, and/or augmented at the instructor's discretion. The instructor who finds it impossible to cover all the topics in a single semester or quarter could possibly omit some of the abstract algebra in Chapters 7 and 8. However it is not advisable to avoid abstract mathematical concepts, as this is an important part of this integrative approach.

Viewing, listening to, and discussing musical examples will be an important part of the class, so the classroom should be equipped with a high-quality sound system, computer hookup, and a keyboard.

Some goals of the course are as follows:

- To explore relationships between mathematics and music.
- To develop and enhance the students' musical knowledge and creativity.
- To develop and enhance the students' skills in some basic mathematical topics and in abstract reasoning.
- To integrate the students' artistic and analytic skills.
- (if equipment is available) To introduce the computer and synthesizer as interactive tools for musical and mathematical creativity.

Regarding the last item, my suggestion is that students be given access to some computer stations that have a notation/playback software such as Finale and that the students receive some basic instruction in how to enter notes and produce playback. It is also helpful if the computer is connected via a MIDI (Musical Instrument Digital Interface) device to a tunable keyboard synthesizer, in which case the student also needs to have some instruction in how the software drives the synthesizer.

Some of the homework assignments should ask for a short composition which demonstrates a specific property or principle discussed in the course, such as a particular form, melodic symmetry, or the twelve-tone technique, which might then be turned in as a sound file along with a score and possibly an essay discussing what was done.

The course can be enhanced by a few special guest lecturers, such as a physicist who can demonstrate and discuss the acoustics of musical instruments, or a medical doctor who can explain the mechanism of the human ear. It can be quite educational and enjoyable if the entire group of students are able to attend one or more musical performances together, e.g., a symphony orchestra, a string quartet, an *a cappella* vocal ensemble, ragtime, modern jazz. This can be integrated in various ways with a number of topics in the course, such as modes, scales, form, rhythm, harmony, intonation, and timbre. The performance might be ensued in the classroom by a discussion of the role played by these various musical components, or even an analysis of some piece performed.

There is only a brief bibliography, consisting of books on my shelf which aided me in writing this book. I recommend all these sources as supplements. A lengthy bibliography on mathematics and music can be found in David J. Benson's grand treatise *Music: A Mathematical Offering* [2], which gives far more technical and in-depth coverage of nearly all the topics addressed here, plus more; it could be used as a textbook for a sequel to the course for which the present book is intended.

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