Chapter 10

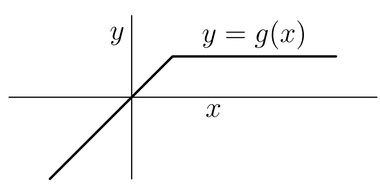
Timbre and Periodic Functions

Timbre. The term *timbre* refers to the quality or distinguishing properties of a musical tone other than its pitch, i.e., that which enables one to distinguish between a violin, a trombone, a flute, the vowel ō, or the vowel ē, even though the tones have the same pitch. In order to address this phenomenon we need to discuss a few more concepts relating to functions and graphs.

Piecewise Definitions and Continuity. A function can be defined in piecewise fashion, for example,

$$g(x) = \begin{cases} x, & \text{for } x \leq 1, \\ 1, & \text{for } x > 1, \end{cases}$$

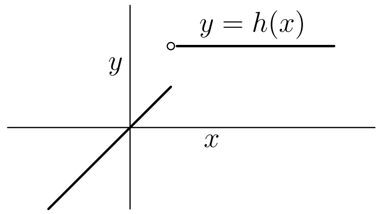
whose graph is:



or

$$h(x) = \begin{cases} x, & \text{for } x \leq 1, \\ 2, & \text{for } x > 1, \end{cases}$$

whose graph is:



Note the "jump" that appears in the graph of $y = h(x)$ at $x = 1$. This is an example of a *discontinuity,* i.e., the situation at a point $x = a$ at which the function fails to be continuous, as per the following definition.

DEFINITION. *A function $y = f(x)$ is defined to be continuous at $x = a$ if given any $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.*

This says that $f(x)$ will be arbitrarily close to $f(a)$ when x is sufficiently close to a. In the example $h(x)$ above, note that, for $a = 1$ and $\epsilon = 1/2$, there does not exist $\delta > 0$ such that if x lies within δ of 1, then $h(x)$ will lie within $1/2$ of $h(1) = 1$; for as x approaches 1 from above, all values of $h(x)$ are 2.

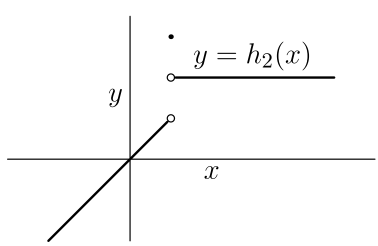
The function

$$h_1(x) = \begin{cases} x, & \text{for } x < 1, \\ 2, & \text{for } x \geq 1, \end{cases}$$

has the same graph as $h(x)$ except at $x = 1$. We could assign $f(1)$ to be some other number, as in

$$h_2(x) = \begin{cases} x, & \text{for } x < 1, \\ 3, & \text{for } x = 1, \\ 2, & \text{for } x > 1, \end{cases}$$

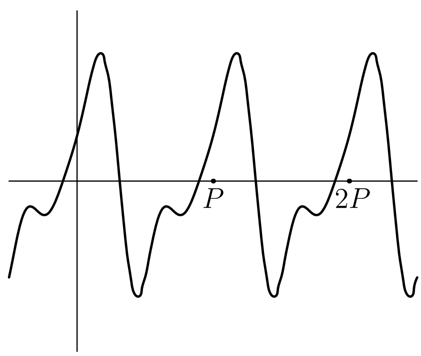
which has the graph



which again has a discontinuity at $x = 1$. It is not hard to prove that there is, in fact, no way to reassign $h(1)$, leaving all other values of h unchanged, in such a way that h is continuous at $x = 1$.

A rough interpretation of a discontinuity is a "jump" in the graph. (This is not precise mathematical terminology, but it serves us pretty well intuitively.) A function which is continuous on an interval I is one whose graph has no "jumps" for any $x \in I$.

Periodic Functions. A function $f(x)$ whose domain is all of $\mathbb{R}$ is called *periodic* if there is a positive number P such that for all $x \in \mathbb{R}$, $f(x+P) = f(x)$. This means that the behavior of the function is completely determined by its behavior on the half-open interval $[0, P)$ (or on any half-open interval of length P).



The number P is called the *period* of the function.

Example. The functions $y = \sin x$ and $y = \cos x$ are periodic of period 2π.

Any function $f(x)$ defined on the interval $[0, P)$ can be uniquely extended to a periodic function $g(x)$ of period P whose domain is all of $\mathbb{R}$. This is done by setting $g(x) = f(x - nP)$ for $x \in [nP, (n+1)P)$ for all integers n. We will refer to this procedure as "extending from $[0, P)$ to $\mathbb{R}$ by periodicity".

Effect of Shifting and Stretching on Periodicity. If $y = f(x)$ is a periodic function with period P, then the vertical and horizontal shifts $y = f(x) + c$ and $y = f(x - c)$ for $c \in \mathbb{R}$ are also periodic of period P, as is the vertical stretch $y = cf(x)$. However the horizontal stretch $y = f(x/c)$ will have period cP. So the effect of stretching horizontally by a factor of c is to divide the frequency of $f(x)$ by c. The proofs of these assertions will be left as an exercise.

Shifting and Stretching Sine and Cosine. The two trigonometric functions $y = \sin x$ and $y = \cos x$ play a central role in the remaining discussion, and they are related as follows: The graph of $y = \cos x$ is obtained by shifting the graph of $y = \sin x$ to the left by $c = \frac{\pi}{2}$. This is because the sine and cosine functions have the relationship

$$\cos x = \sin\left(x + \frac{\pi}{2}\right),$$

which is a special case of the "summation formula"

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\,. \tag{10.1}$$

Note that the former equation is obtained from the latter by setting $\alpha = x$ and $\beta = \frac{\pi}{2}$, since $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$.

More generally, if we treat (10.1) as a functional equation by replacing α by the independent variable x and letting β be some fixed number (we might wish to think of β as being an angle measured in radians), we have

$$\sin(x + \beta) = \cos\beta\sin x + \sin\beta\cos x\,. \tag{10.2}$$

The numbers $\cos\beta$ and $\sin\beta$ are the coordinates of the point Q on the unit circle (i.e., the circle of radius one) centered at the origin, such that the arc length counter-clockwise along the circle from $(1, 0)$ to Q is β.

Let $k, d \in \mathbb{R}$ with $d \geq 0$. Replacing x by kx in (10.2) and multiplying both sides of the above equation by d yields the equation of the function $g(x)$ obtained by starting with $f(x) = \sin x$, shifting to the left by β, compressing horizontally by a factor of k (i.e., stretching by $1/k$), and stretching vertically by a factor of d. The resulting general transformation of $\sin x$ is

$$\boxed{g(x) = d\sin(kx + \beta) = d\left(\cos\beta\sin kx + \sin\beta\cos kx\right).} \tag{10.3}$$

Now let us consider an arbitrary function of the form

$$h(x) = A\sin kx + B\cos kx\,, \tag{10.4}$$

where $A, B \in \mathbb{R}$ are any numbers. The point (A, B) has distance $\sqrt{A^2 + B^2}$ from the origin. If A and B are not both zero, then letting

$$a = \frac{A}{\sqrt{A^2 + B^2}}\,, \quad b = \frac{B}{\sqrt{A^2 + B^2}}\,,$$

the point (a, b) has distance 1 from the origin, hence lies on the unit circle centered at the origin. Thus there is an angle β for which $a = \cos\beta, b = \sin\beta$, and letting $d = \sqrt{A^2 + B^2}$ we have

$$\begin{aligned} h(x) &= d\,(a \sin kx + b \cos kx) \\ &= d\,(\cos\beta \sin kx + \sin\beta \cos kx) \\ &= d \sin(kx + \beta)\,. \end{aligned}$$

Therefore $h(x)$ is a transformation of $\sin x$ having the form (10.3), where $d = \sqrt{A^2 + B^2}$. The angle β is called the *phase shift*, and the number $d \geq 0$ is the *amplitude*.

Example. Consider the function $h(x) = 3 \sin x + 2 \cos x$. We have $A = 3$, $B = 2$, $d = \sqrt{3^2 + 2^2} = \sqrt{13}$, $a = \frac{3}{\sqrt{13}}$, and $b = \frac{2}{\sqrt{13}}$. The angle β is an acute angle (since the point $(3, 2)$ lies in the first quadrant, so β can be found on a calculator by taking $\arcsin \frac{2}{\sqrt{13}} \approx 0.588$. Thus we have

$$\begin{aligned} h(x) &= \sqrt{13}\,(\frac{3}{\sqrt{13}} \sin x + \frac{2}{\sqrt{13}} \cos x) \\ &= \sqrt{13}\,(\cos\beta \sin x + \sin\beta \cos x) \\ &= \sqrt{13} \sin(x + \beta)\,, \end{aligned}$$

where $\beta \approx 0.588$. The amplitude is $\sqrt{13}$ and the phase shift is $\beta \approx 0.588$.

Vibrations. We will use the term *vibration* to mean an oscillation having a pattern which repeats every interval of P units of time. The frequency of the vibration, i.e., the number of repetitions of its pattern per unit of time, is $1/P$. For our purposes, time will be measured in seconds, thereby giving frequency in hertz (vibrations per second). If we realize a vibration as the up and down motion of a point, the vibration is given by a function $y = f(t)$ where y is the position of the particle at time t. The function will be periodic, the period being the number P above. For the remainder of this section t will be used as the independent variable of periodic functions, to suggest time.

Vibrating motion can arise from the strings of a violin, a column of air inside a trumpet, or the human vocal cords. The vibration is transmitted through the air by contraction and expansion (this is called a sound wave) and received by the human ear when the ear drum is set in motion, vibrating in the same pattern as the vibrating object. The brain interprets the vibration as a musical tone. If the vibration has period P, measured in seconds, then the pitch, or frequency, of the tone will be $F = 1/P$ Hz.

Musical Tones and Periodic Functions. Given any periodic function $y = f(t)$ of period P, we can contemplate an oscillating object whose position at time t is $f(t)$ and ask what is the sound of such a vibration. We would expect the pitch of the tone to be $1/P$ Hz, but we wish to investigate what other aspects of $y = f(t)$ determine the character, or timbre, of the sound we are hearing.

If a function $y = f(t)$ did in fact represent the position of an object, we would expect the function $f(t)$ to be continuous. This is based on the supposition that the object's position does not "jump" instantly. Although this is indeed a reflection of reality, our discussion will nevertheless associate a vibration with any periodic function $y = f(t)$ of period $P \in \mathbb{R}^+$ satisfying the following more general properties:

1. f has only finitely many discontinuities on $[0, P)$.

2. f is *bounded*, i.e., there are numbers $b, B \in \mathbb{R}$ such that for all $t \in \mathbb{R}$, $b < f(t) < B$.

We interpret the discontinuities as moments at which the vibrating object's position changes very quickly, so that the transition from one location to another seems instantaneous. This exemplifies the fact that mathematics presents models of physical phenomena, not exact representations.

Suppose $y = f(t)$ is a periodic function, with period P, satisfying the above two conditions. As described above, $f(t)$ is associated to a tone of pitch (frequency) $F = 1/P$. According to our observations about the effect of shifting on periodicity, the pitch is not changed if we alter $f(t)$ by a horizontal shift. Since such a shift can be thought of as a delay, we would not expect it to affect the timbre of the tone, and in fact it does not. The vertical shift describes a motion with altered amplitude, but with the same pitch and the same basic "personality". Observation confirms that such a stretch adjusts the loudness, with very little effect, if any, on the timbre of the tone. The horizontal compression $y = f(ct)$ changes the period to P/c, hence the pitch to $1/(P/c) = c/P = cF$. So the effect of compressing horizontally by a factor of c is to multiply the frequency of $f(t)$ by c.

Effect of Horizontal Stretching on Pitch. The final observation above tells us how to apply a horizontal compression to $f(t)$ to achieve any desired pitch (frequency) r. Suppose the period P is given in seconds. We want $r = cF = \frac{c}{P}$, which gives $c = rP$. Thus the function

$$y = f(rPt)$$

represents a tone having frequency r cycles per second, i.e., r Hz.

Example. Suppose $y = \sin t$ gives motion in seconds. Here $P = 2\pi$, so the frequency is $1/2\pi$ Hz (which is way below the threshold of human audibility). Let us adjust the pitch to give A_4, tuned to $r = 440$ Hz. Accordingly we write $y = \sin(rPt)$, i.e.,

$$y = \sin(880\pi t)\,.$$

The tone given by a sine function as above is sometimes called a "pure tone". It is a non-descript hum, very similar to the tone produced by a tuning fork.

Fourier Theory. We will describe how all periodic functions having reasonably good behavior can be written in terms of the functions $\sin t$ and $\cos t$. This is a fundamental result of harmonic analysis, more specifically Fourier theory, which is based on work of the French mathematician and physicist Joseph Fourier (1768-1830). We first make the following observations.

The first is that if $f(t)$ and $g(t)$ are two functions which are periodic of period P, then so is $(f+g)(t)$, which is defined as $f(t)+g(t)$. This is elementary: $(f+g)(t+P) = f(t+P)+g(t+P) = f(t)+g(t) = (f+g)(t)$. More generally, one sees that if $f_1(t), \ldots, f_n(t)$ are periodic of period P, then so is $\sum_{k=1}^{n} f_k(t)$.

Secondly, suppose $f(t)$ is periodic of period P, and $k \in \mathbb{Z}^+$. As we have seen, the function $f(kt)$ has as its graph the graph of $f(t)$ compressed horizontally by a compression factor of k, and it has period P/k. However, it also has period P, since $f(k(t+P)) = f(kt+kP) = f(kt)$. Obviously the function $af(kt)$, for any $a \in \mathbb{R}$, is also periodic of period P. Therefore a sum $\sum_{k=1}^{n} a_k f(kt)$, where $a_1, \ldots, a_n \in \mathbb{R}$, is again periodic of period P. In particular, a sum $\sum_{k=1}^{n} a_k \sin(kt)$ has period 2π.

The following theorem, basic to harmonic analysis, entails two concepts from calculus which go well beyond the scope of this course: the derivative and the infinite summation.

THEOREM. *Suppose $f(t)$ is periodic of period 2π which is bounded and has a bounded continuous derivative at all but finitely many points in $[0, 2\pi)$. Then there is a real number C and sequences of real numbers $A_1, A_2, A_3, \ldots$ and $B_1, B_2, B_3, \ldots$ such that, for all t at which $f(t)$ is continuous we have $f(t)$ represented by the convergent sum*

$$f(t) = C + \sum_{k=1}^{\infty} \left[A_k \sin(kt) + B_k \cos(kt)\right]\,. \tag{10.5}$$

Note that there is a condition on $f(t)$ beyond the conditions 1 and 2 stated earlier in this chapter. It involves the concept of derivative, which one learns in calculus. The condition roughly says that, away from finitely many points, the graph of $f(t)$ is smooth and that it doesn't slope up or down too much.

The real numbers whose existence is asserted in the above theorem are called *Fourier coefficients*. The infinite summation (10.5), called the *Fourier series* for f, is based on the notions of limit and convergence, also from calculus. With the proper definitions and development, it becomes possible for an infinite sum to have a limit, i.e., to "add up" (converge) to a number. An example is the sum $\sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, which has 2 as its limit. This is the same sum as formula (2.2) from Chapter 2, encountered in our discussion of dotted notes.

The moral of the story told in the above theorem is that well-behaved periodic functions can be approximated by a series of multiples of the sine and cosine functions. There is more to the story, which, again, can be understood by anyone familiar with calculus: The coefficients in formula (10.5) are uniquely determined by the integrals below.

$$\begin{aligned} C &= \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt\,, \\ A_k &= \frac{1}{\pi} \int_0^{2\pi} \sin(kt) f(t)\, dt\,, \\ B_k &= \frac{1}{\pi} \int_0^{2\pi} \cos(kt) f(t)\, dt. \end{aligned} \tag{10.6}$$

If $g(t)$ is a function of arbitrary period P, then $g(\frac{P}{2\pi}t)$ has period 2π, hence we have

$$g\left(\frac{P}{2\pi}t\right) = C + \sum_{k=1}^{\infty} [A_k \sin(kt) + B_k \cos(kt)]$$

by the theorem. Recovering $g(t)$ by replacing t by $\frac{2\pi t}{P}$ in the above, we get the Fourier series for an arbitrary function of period P, satisfying the other hypotheses of the theorem:

$$\boxed{g(t) = C + \sum_{k=1}^{\infty} \left[A_k \sin\frac{2\pi kt}{P} + B_k \cos\frac{2\pi kt}{P}\right]\,.} \tag{10.7}$$

Harmonics and Overtones. Associating the function $g(t)$ having period P as above with a musical tone of pitch $F = 1/P$, let us note that

$$g(t) = C + \sum_{k=1}^{\infty} \left[A_k \sin(2\pi Fkt) + B_k \cos(2\pi Fkt)\right] .$$

Each summand $A_k \sin(2\pi Fkt) + B_k \cos(2\pi Fkt)$ in (10.7) has the form (10.4), and therefore represents a transformation of $\sin(2\pi Fkt)$ which can be written in the form (10.3) as

$$d_k \left[\cos\beta_k \sin(2\pi Fkt) + \sin\beta_k \cos(2\pi Fkt)\right] = d_k \sin(2\pi Fkt + \beta_k) ,$$

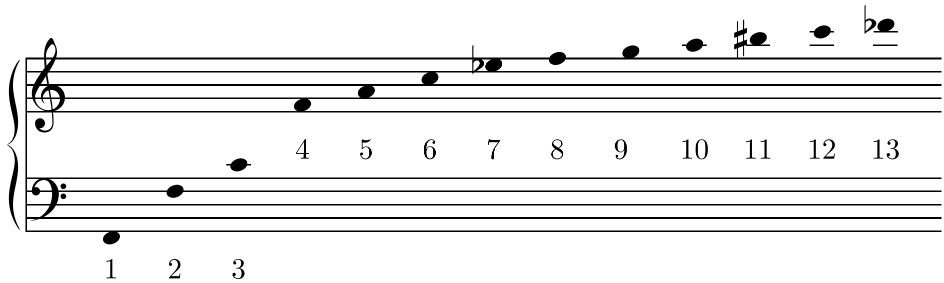
where

$$d_k = \sqrt{A_k^2 + B_k^2}, \qquad \cos\beta_k = \frac{A_k}{d_k}, \qquad \sin\beta_k = \frac{B_k}{d_k}$$

(provided A_k and B_k are not both zero). Hence we have

$$\boxed{g(t) = C + \sum_{k=1}^{\infty} d_k \sin(2\pi Fkt + \beta_k) .} \tag{10.8}$$

The k-th summand $d_k \sin(2\pi Fkt + \beta_k)$ is obtained from $\sin t$ via shifting by β_k (the k-th phase shift), compressing horizontally by a factor of k and stretching vertically by a factor of d_k (the k-th amplitude). This function has the same basic sound (pitch and timbre) as $\sin(2\pi kFt)$, with a volume adjustment resulting from the amplitude d_k. It is called the k-th *harmonic* of the function $g(t)$. For $k \geq 1$ it is also called the $(k-1)$-st *overtone* of $g(t)$. When isolated, this harmonic gives the pitch kF, so the sequence of pitches associated to the harmonics gives the sequence of integer ratios with the fundamental frequency F. These are the intervals discussed in Chapter 9; recall that if we take F_2 as the fundamental (first harmonic), the first 13 harmonics are approximated on the keyboard as follows:



For a given fundamental frequency F, the infinite sequence of pitches

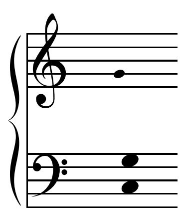
$$F,\ 2F,\ 3F,\ 4F,\ 5F,\ldots$$

is called its *overtone series.*

For a given pitch, it is the relative sizes of the (non-negative) amplitudes $d_1, d_2, d_3, \ldots$ that determine the timbre, or "personality", of a sustained tone, allowing us to distinguish between different musical voices and instruments. We can think of d_k as the "weight" or "degree of presence" of the k-th harmonic in the sound represented by $g(t)$. The timbre of the tone seems to depend on this sequence alone, independent of the sequence of phase shifts $\beta_1, \beta_2, \beta_3, \ldots$, which certainly affect the shape of the graph of $g(t)$, but not the sound.

Overtones (harmonics) are generally not perceived by the ear as pitches; rather the totality of those overtones that fall into audible range are heard as an integrated single tone, with harmonics determining the timbre as explained above. However there are times when overtones can actually be heard as pitches. Overtone singing is a type of singing in which the singer manipulates the resonating cavities in the mouth by moving the tongue and jaw so as to isolate specific overtones one at a time. The isolated overtone then becomes clearly audible. While holding a constant fundamental pitch the singer can thereby "play a tune" with the overtones.

Another situation where overtones can become audible occurs when a certain pitch appears as a *reinforced overtone,* i.e., is an overtone of two or more notes in a well-tuned chord. For example, suppose a chord has root C_3 and fifth G_3. Note, then, that G_4 is the third harmonic of the root[1] and is the second harmonic of the fifth.



G_4, the small note, appears in the overtone series of both C_3 and G_3.

Since G_3 is reinforced, it is sometimes heard as a pitch. Its audibility is even more likely if it lies within a formant (a term to be explained later in the chapter) for the vowel being sung or the instruments playing the

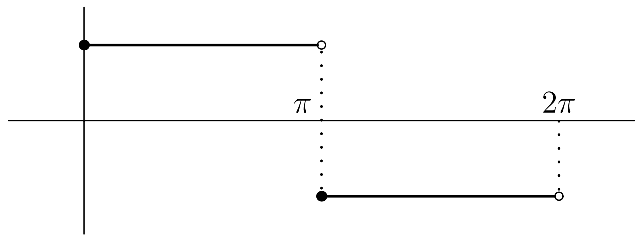
[1] Actually it is off by about two cents, as per the discussion in Chapter 9.

chord. Singers of *a cappella* music often say the chord "rings" when this phenomenon is experienced.

Example: The Square Wave. To illustrate the use of the theorem, and equations (10.6), (10.7), and (10.8), to calculate harmonics of a tone, we consider a so-called "square wave", the periodic function defined on the interval $[0, 2\pi)$ by

$$s(t) = \begin{cases} 1, & \text{for } 0 \le t < \pi, \\ -1, & \text{for } \pi \le t < 2\pi, \end{cases}$$

and extended by periodicity to a function whose domain is $\mathbb{R}$. The graph of one period, over $[0, 2\pi)$, appears below.



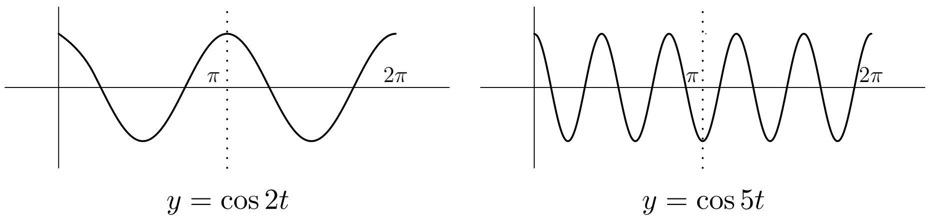
$y = s(t)$

This waveform, encountered in electronics and signal processing and available in most synthesizers, produces a distinctive timbre that vaguely resembles the sound of a clarinet. The function satisfies the hypothesis of the theorem, so we wish to use (10.6) to calculate the coefficients C, A_k, and B_k that appear in its Fourier series.

At this point we will fall back on the common interpretation of the integral which asserts that for a well-behaved function $y = f(x)$ the integral $\int_a^b f(x)\,dx$ gives the area enclosed between the graph of $f(x)$ and the x-axis between the vertical lines $x = a$ and $x = b$, with the caveat that area below the axis assumes negative value. Though not mathematically rigorous, this informal realization of the integral will serve here as a working definition, as it will allow the reader who is not familiar with calculus to follow the discussion.

According to (10.6) we have $C = \frac{1}{2\pi}\int_0^{2\pi} s(t)\,dt$. It now becomes apparent from the graph of $s(t)$ that $\int_0^{2\pi} s(t)\,dt = 0$, since the rectangle enclosed above the t-axis between 0 and π has the same area as the one below the axis between π and 2π. Hence $C = 0$.

Now let us consider the coefficients $B_k = \frac{1}{\pi}\int_0^{2\pi} \cos(kt)s(t)\,dt$ (again from (10.6)). First, let's observe that, for $k \in \mathbb{Z}$, the graph of $y = \cos(kt)$ is symmetric around $t=\pi$, in other words $\cos(k(\pi-t)) = \cos(k(\pi+t))$. This can be surmised from the graph, exhibited below for $k=2$ and $k=5$,



and it follows easily from the summation formula

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\,. \tag{10.9}$$

(The derivation using this formula will appear as an exercise at the end of this chapter.) Consequently we see that

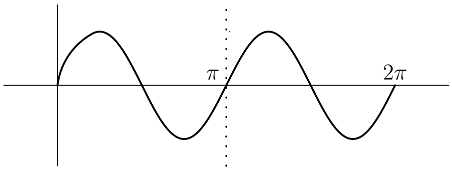
$$\int_0^{\pi} \cos(kt)\,dt = \int_{\pi}^{2\pi} \cos(kt)\,dt\,. \tag{10.10}$$

Appealing to basic properties of the integral which are apparent from our working definition, and recalling the definition of $s(t)$, we have

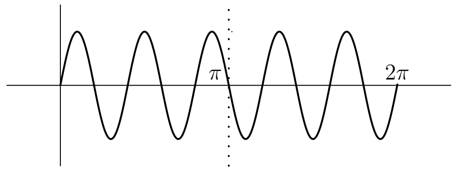
$$\begin{aligned}\int_0^{2\pi} \cos(kt)s(t)\,dt &= \int_0^{\pi} \cos(kt)s(t)\,dt + \int_{\pi}^{2\pi} \cos(kt)s(t)\,dt \\ &= \int_0^{\pi} \cos(kt)\cdot 1\,dt + \int_{\pi}^{2\pi} \cos(kt)\cdot(-1)\,dt \\ &= \int_0^{\pi} \cos(kt)\,dt - \int_{\pi}^{2\pi} \cos(kt)\,dt \\ &= 0\,,\end{aligned}$$

with the last equality owing to (10.10). This shows $B_k = 0$.

In order to evaluate $A_k = \frac{1}{\pi}\int_0^{2\pi} \sin(kt)s(t)\,dt$, we make some observations about the behavior of $\sin(kt)$ on the interval $[0,2\pi]$. Specifically, we want to see how its graph on $[0,\pi]$ compares to its graph on $[\pi,2\pi]$. Now $\sin(kt)$ has period $2\pi/k$, each period being a horizontal compression the graph of $y=\sin t$ over the interval $[0,2\pi]$, thus comprising an "upper lobe" and a "lower lobe", each enclosing the same amount of area between the graph and the t-axis. From this it follows that the integral of $\sin(kt)$ over any one period, or over any number of complete periods, is zero. Moreover, the point $t = \pi$ lies either at the point between two adjacent periods or at the midpoint of a period, depending on whether k is even or odd, respectively.



$y = \sin 2t$



$y = \sin 5t$

In the case k is even (see $k = 2$ above), the graph looks exactly the same on both intervals, thus the integral over both intervals is the same (and in fact equal to zero, since each period has the same amount of area below the t-axis as above). Therefore $\int_0^{2\pi} \sin(kt)s(t)\,dt = \int_0^{\pi} \sin(kt)s(t)\,dt + \int_\pi^{2\pi} \sin(kt)s(t)\,dt = \int_0^{\pi} \sin(kt)\cdot 1\,dt + \int_\pi^{2\pi} \sin(kt)\cdot(-1)\,dt = \int_0^{\pi} \sin(kt)\,dt - \int_\pi^{2\pi} \sin(kt)\,dt = 0 - 0 = 0$. Hence we have $A_k = 0$ when k is even.

When k is odd (see $k = 5$ above), write $k = 2n + 1$ and note that the interval $[0, \pi]$ contains n complete periods, plus the upper lobe of another; the interval $[\pi, 2\pi]$ has the lower lobe of the $(n + 1)$-st period followed by n complete periods. Since the integral over complete periods is zero we see then that $\int_0^{\pi} \sin(kt)\,dt = R$ and $\int_\pi^{2\pi} \sin(kt)\,dt = -R$, where R is the area under one upper lobe. Therefore

$$\begin{aligned}\int_0^{2\pi} \sin(kt)s(t)\,dt &= \int_0^{\pi} \sin(kt)s(t)\,dt + \int_\pi^{2\pi} \sin(kt)s(t)\,dt \\ &= \int_0^{\pi} \sin(kt)\cdot 1\,dt + \int_\pi^{2\pi} \sin(kt)\cdot(-1)\,dt \\ &= \int_0^{\pi} \sin(kt)\,dt - \int_\pi^{2\pi} \sin(kt)\,dt \\ &= R - (-R) = 2R\end{aligned} \tag{10.11}$$

and we are reduced to evaluating R. We appeal to another intuitive maxim: When a region is stretched horizontally by a factor of a the area of the stretched region equals the area of the original region multiplied by a. Accordingly,

$$R = \frac{1}{k}\int_0^{\pi} \sin t\,dt \tag{10.12}$$

and here, alas, we appeal to the Fundamental Theorem of Calculus for one brief calculation and ask the general reader's forbearance:

$$\int_0^{\pi} \sin t\,dt = -\cos t\,\Big|_0^{\pi} = -\cos\pi + \cos 0 = -(-1) + 1 = 2\,. \tag{10.13}$$

This says the area under one upper lobe of the standard sine (or cosine) curve is 2. Using (10.11), (10.12), and (10.13) we get

$$A_k = \frac{1}{\pi}\int_0^{2\pi} \sin(kt)s(t)\,dt = \frac{1}{\pi}2R = \frac{4}{k\pi}$$

for k odd.

To summarize, we have shown:

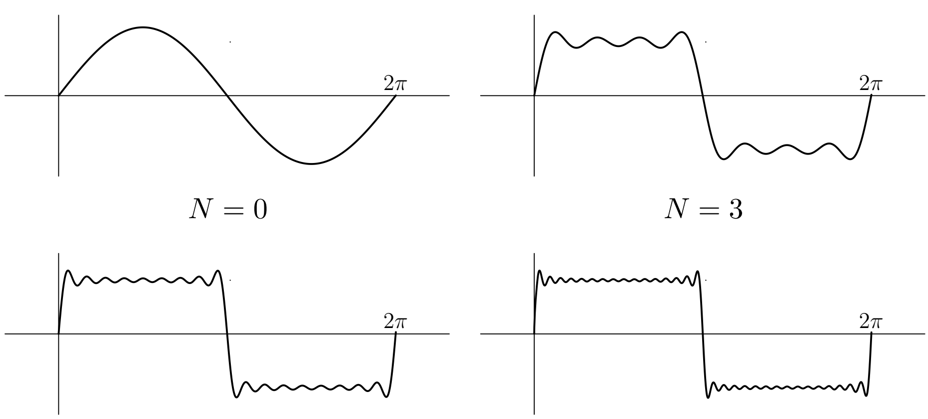
$$C = 0, \qquad B_k = 0 \text{ for all } k, \qquad A_k = \begin{cases} 0, & \text{for } k \text{ even,} \\ \frac{4}{k\pi}, & \text{for } k \text{ odd.} \end{cases}$$

Writing the odd positive integers as $k = 2n+1$ for $n = 0, 1, 2, \ldots$ and pulling the common factor $4/\pi$ to the left, the summation (10.7) for the function $s(t)$ reads:

$$s(t) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1}\sin((2n+1)t)\,. \tag{10.14}$$

Note that the absence of cosines in the series says that the phase shifts β_k are all zero and the amplitudes d_k are 0 for k even, $4/k\pi$ for k odd.

It is interesting (and fun!) to "watch" the summation (10.14) converge by plotting the graph of the truncated series $\frac{4}{\pi}\sum_{n=0}^{N}\frac{1}{2n+1}\sin((2n+1)t)$ for larger and larger N. Note how the graphs below increasingly resemble the graph of $s(t)$.



$N = 0$ $N = 3$

$N = 8$ $N = 15$

We should remark that all but finitely many overtones lie outside the range of human audibility. Hence some truncation of the Fourier series suffices to represent the audible sound.

For reasons rooted in physics of sound, the clarinet also has only odd harmonics, which explains the faint resemblance of its sound to that of the square wave.

Formants. Suppose a waveform is given by equation (10.8), and suppose we vary only the pitch F, keeping the numbers d_k fixed. (We won't worry about the numbers β_k since they don't contribute much to the character of the sound.) Then the amplitude of each harmonic remains unchanged. This would be the case if we sounded the square wave at different frequencies. The weights of the harmonics are not affected.

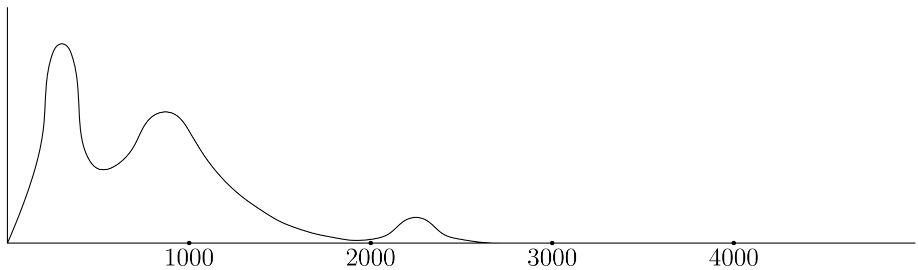
However this is not what happens when a musical instrument or a singer changes pitch. Rather, the harmonics that fall within certain frequency ranges will consistently have larger weights than those which do not. These frequency ranges, called *formants,* depend only on the musical instrument being played or the human vowel sound being sung; they remain unchanged as the pitch F varies. Thus each weight d_k will change from note to note, depending on whether the k-th harmonic lies within one of these formants.

This explains why speeding up or slowing down a recording distorts the sound beyond simply changing the pitch. When a recorded tone is played at a different rate from which it was recorded, the sound wave is simply stretched or compressed over time, i.e., the frequency F is changed, with all other parameters in (10.8) remaining unaltered. Thus the formants are not preserved, but rather shifted along with F.[2] Speeding up recorded music produces the familiar "chipmunk effect". Music which is slowed down sounds dark and muddy. In either case the character of the music is changed in a rather comical way.

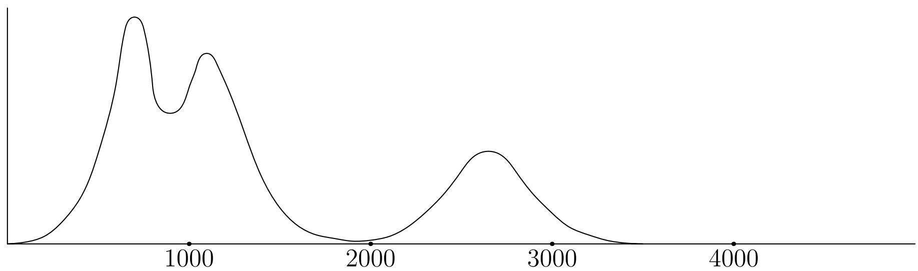
Musical sounds tend to have two or three formants. These formants are created by the resonating chambers inside the instrument or mouth of the singer. A chamber favors a certain frequency range, determined by its size and shape; frequencies within that range are amplified.

As examples, we will consider the three vowel sounds as typically spoken by Americans. The vowel *oo* (as in "food") has three formants centered respectively near 310 Hz, 870 Hz, and 2250 Hz. The vowel *ah* (as in "father") has formants around 710 Hz, 1100 Hz, and 2640 Hz. The vowel *ee* (as in "feed") has formants at 280 Hz, 2250 Hz, and 2900 Hz. The graphs which plot loudness (vertical axis) against pitch (in Hz) for these vowels look something like the following.

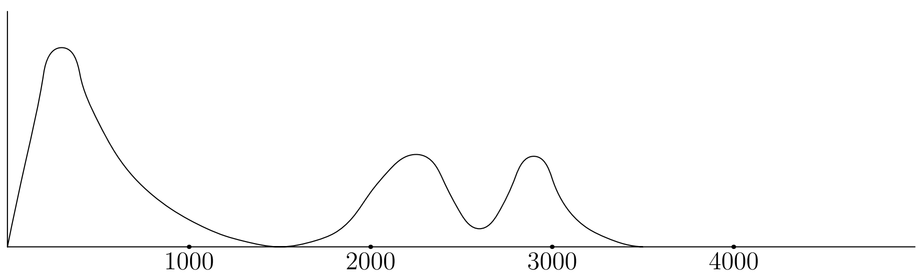
[2] Modern studio editing now allows recorded tones to be transposed (i.e., pitch altered) in a way that keeps the formants intact, thus preserving the character of the sound. This process is highly sophisticated and represents a great triumph in signal analysis technology.



formants for the *oo* vowel



formants for the *ah* vowel



formants for the *ee* vowel

We often use the word "bright" to describe sounds with one or more prominent high formants, and "dark" for sounds whose formants all lie low. Note that the *ee* vowel has higher second and third upper formants than the other two, which accounts for its relatively bright sound. Note also that a formant will have no effect on the timbre if the fundamental pitch being sung lies above that formant; hence if a soprano sings A_5 (880 Hz) on an *oo* vowel, the lowest formant, centered around 310 Hz, has no harmonics to amplify.

Musical instruments also possess characterizing formants. For example, the clarinet has formants in the ranges 1500-1700 Hz, and the trumpet has a formant in the range 1200-1400 Hz and another centered narrowly around 2500 Hz.

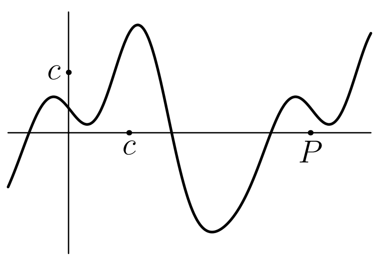
Finally we should acknowledge that the term "loudness" used above is subjective and difficult to quantify, as it varies from person to person. It is not directly proportional to mere amplitude. Physics attempts to measure it as a function of "sound pressure", measured in *decibels,* as well as frequency.

Exercises

1. Prove that if $y = f(t)$ has period P, then so does $y = f(t) + c$, $y = f(t - c)$, and $y = cf(t)$, for any $c \in \mathbb{R}$. Prove that $f(t/c)$ $(c \neq 0)$ has period cP.

2. Show that if a periodic function $f(t)$ has period P, then it also has period nP for any positive integer n. Note that the determination of the Fourier coefficients C, A_k, and B_k depend on the choice of period for $f(t)$ (see equation (10.7)). How do these coefficients for f as a function of period kP compare to those obtained from viewing f as a function of period P?

3. Suppose the function $y = f(t)$ is the periodic function of period P corresponding to a musical tone, and suppose the graph of $y = f(t)$ is:



   For each of the functions below, sketch its graph and explain how its associated tone compares with that of $f(t)$.

   (a) $y = \frac{1}{2}f(t)$ (b) $y = f(2t)$

   (c) $y = f(t) + c$ (d) $y = f(t + c)$

4. Find the value α for which the pitch associated to the periodic function $y = \sin(\alpha t)$, where t is time in seconds, is:

   (a) middle C (b) $A^\flat_2$ (c) $D^\sharp_6$

5. Find the period, frequency, amplitude, and phase shift for these functions, and express each in the form $A\sin(\alpha t) + B\cos(\alpha t)$.

   (a) $f(t) = 5\sin(30\pi t + \frac{\pi}{4})$

   (b) $g(t) = \sqrt{2}\sin(800t + \pi)$

   (c) $h(t) = -\frac{5}{3}\sin(2000t + \arcsin(0.7))$

6. Find the period, frequency, amplitude, and phase shift for these functions, and express each in the form $d\sin(\alpha t + \beta)$:

   (a) $f(t) = 4\sin(300t) + 5\cos(300t)$

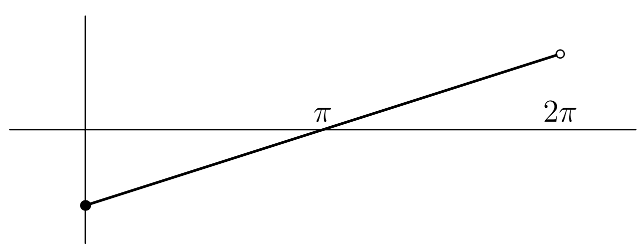
   (b) $g(t) = 2\sin(450\pi t) - 2\cos(450\pi t)$

   (c) $h(t) = -\sin(1500\pi t) + 3\cos(1500\pi t)$

7. Suppose musical tone with pitch B_4 has harmonics 1, 3, 5 only, with amplitudes 1, $\frac{1}{9}$, $\frac{1}{25}$, respectively, and phase shifts 0, π, $-\frac{\pi}{2}$, respectively. Suppose also that the vertical shift C is 0. Write its Fourier series in the form $\sum[A_k \sin(kt) + B_k \cos(kt)]$.

8. Verify the formula $\cos(k(\pi - x)) = \cos(k(\pi + x))$ using the formula (10.9). Recall this formula was used in showing the Fourier coefficients B_k for the square wave function are all zero.

9. This is a challenging exercise. Let $q(t)$ be defined by $q(t) = \frac{1}{\pi}t - 1$ on the interval $[0, 2\pi)$, extended to a periodic function on $\mathbb{R}$ by periodicity. This a *sawtooth wave.* Its graph on $[0, 2\pi)$ is:



$y = q(t)$

The sound of this waveform is a harsh buzz. Show that

$$q(t) = -\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin(kt)\,.$$

(Hint: Mimic the computation for the square wave. You will need the formula

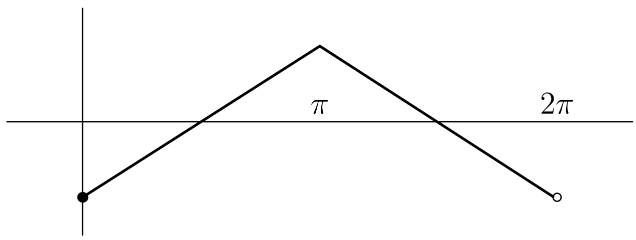
$$\int_0^{2\pi} t\sin(kt)\,dt = -\frac{2\pi}{k}\,,$$

which calculus students should verify using integration by parts.)

10. In the spirit of the last exercise, find the Fourier series for the *triangle wave* given on $[0, 2\pi)$ by

$$r(t) = \begin{cases} \frac{2}{\pi}t - 1, & \text{for } 0 \le t < \pi, \\ -\frac{2}{\pi}t + 3, & \text{for } \pi \le t < 2\pi, \end{cases}$$

which looks like



$y = r(t)$

You should find that the triangle wave, like the square wave, has only odd harmonics. However the weights of the higher harmonics diminish faster than those of the square wave, hence its sound is less harsh.

11. We saw that the square wave had only odd harmonics. What can you say about the periodicity of a waveform that has only even harmonics? This relates to Exercise 2.

12. The human *ee* vowel has a formant centered at 2900 Hz. What pitch should one sing in order for the fifth harmonic to be maximally amplified by this formant?

13. Two instruments play the pitches A_2 and E_3, making the interval of a keyboard fifth. Suppose they are playing the same kind of instrument,

and that the instrument has a formant centered at 3000 Hz. Suppose the formant amplifies pitches within 400 Hz of its center. Identify the harmonics produced by each instrument which will be amplified by the formant, and give their frequencies. How many pairs of these frequencies are almost aligned? Could this "near alignment" be perfected by slightly adjusting the interval? Might this induce the performers to make such an adjustment, if the instrument permitted?