

Acknowledgements

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Preface

We live in a highly connected world, with multiple self-interested agents interacting, leading to myriad opportunities for conflict and cooperation. Understanding these is the goal of game theory. It finds application in fields such as economics, business, political science, biology, psychology, sociology, computer science, and engineering. Conversely, ideas from the social sciences (e.g., fairness), from biology (evolutionary stability), from statistics (adaptive learning), and from computer science (complexity of finding equilibria) have greatly enriched game theory. In this book, we present an introduction to this field. We will see applications from a variety of disciplines and delve into some of the fascinating mathematics that underlies game theory.

An overview of the book

Part I: Analyzing games: Strategies and equilibria. We begin in **Chapter 1** with **combinatorial games**, in which two players take turns making moves until a winning position for one of them is reached.

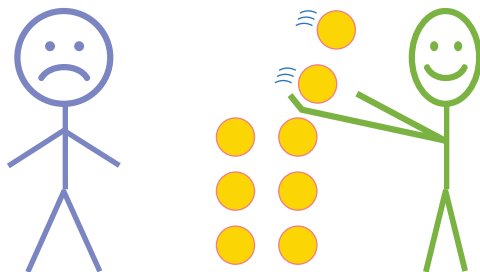


FIGURE 1. Two people playing Nim.

A classic example of a combinatorial game is **Nim**. In this game, there are several piles of chips, and players take turns removing one or more chips from a single pile. The player who takes the last chip wins. We will describe a winning strategy for Nim and show that a large class of combinatorial games can be reduced to it.

Other well-known combinatorial games are Chess, Go, and Hex. The youngest of these is **Hex**, which was invented by Piet Hein in 1942 and independently by John Nash in 1947. Hex is played on a rhombus-shaped board tiled with small hexagons (see Figure 2). Two players, Blue and Yellow, alternate coloring in hexagons in their assigned color, blue or yellow, one hexagon per turn. Blue wins if she produces

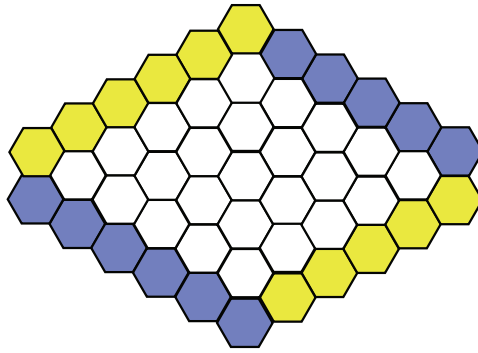


FIGURE 2. The board for the game of Hex.

a blue chain crossing between her two sides of the board and Yellow wins if he produces a yellow chain connecting the other two sides.

We will show that the player who moves first has a winning strategy; finding this strategy remains an unsolved problem, except when the board is small.

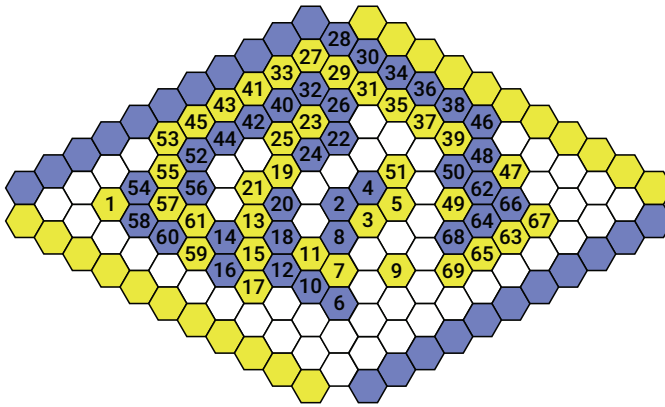


FIGURE 3. The board position near the end of the match between Queenbee and Hexy at the 5th Computer Olympiad. Each hexagon is labeled by the time at which it was placed on the board. Blue moves next, but Yellow has a winning strategy. Can you see why?

In an interesting variant of the game, the players, instead of alternating turns, toss a coin to determine who moves next. In this case, we can describe optimal strategies for the players. Such **random-turn combinatorial games** are the subject of **Chapter 9**.

In **Chapters 2–5**, we consider games in which the players simultaneously select from a set of possible actions. Their selections are then revealed, resulting in a payoff to each player. For two players, these payoffs are represented using the matrices $A = (a_{ij})$ and $B = (b_{ij})$. When player I selects action i and player II selects action j , the payoffs to these players are a_{ij} and b_{ij} , respectively. Two-person games where one player's gain is the other player's loss, that is, $a_{ij} + b_{ij} = 0$ for all i, j , are called **zero-sum games**. Such games are the topic of **Chapter 2**.

We show that every zero-sum game has a **value** V such that player I can ensure her expected payoff is at least V (no matter how II plays) and player II can ensure he pays I at most V (in expectation) no matter how I plays.

For example, in **Penalty Kicks**, a zero-sum game inspired by soccer, one player, the kicker, chooses to kick the ball either to the left or to the right of the other player, the goalie. At the same instant as the kick, the goalie guesses whether to dive left or right.

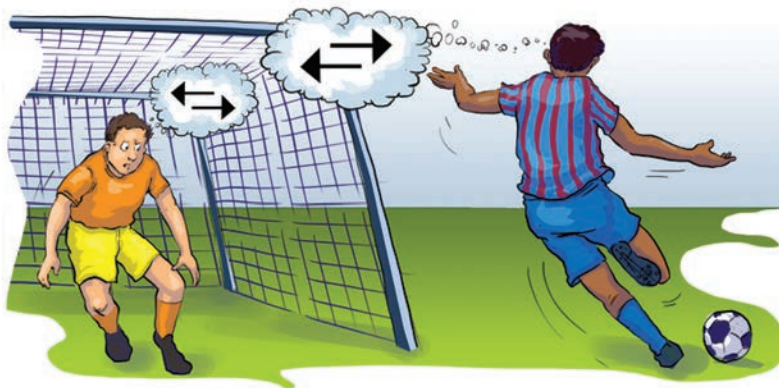


FIGURE 4. The game of Penalty Kicks.

The goalie has a chance of saving the goal if he dives in the same direction as the kick. The kicker, who we assume is right-footed, has a greater likelihood of success if she kicks right. The probabilities that the penalty kick scores are displayed in the table below:

		goalie	
		L	R
kicker	L	0.5	1
	R	1	0.8

For this set of scoring probabilities, the optimal strategy for the kicker is to kick left with probability $2/7$ and kick right with probability $5/7$ — then regardless of what the goalie does, the probability of scoring is $6/7$. Similarly, the optimal strategy for the goalie is to dive left with probability $2/7$ and dive right with probability $5/7$.

Chapter 3 goes on to analyze a number of interesting zero-sum games on graphs. For example, we consider a game between a **Troll and a Traveler**. Each of them chooses a route (a sequence of roads) from Syracuse to Troy, and then they simultaneously disclose their routes. Each road has an associated toll. For each road chosen by both players, the traveler pays the toll to the troll. We find optimal strategies by developing a connection with electrical networks.

In **Chapter 4** we turn to **general-sum games**. In these games, players no longer have optimal strategies. Instead, we focus on situations where each player's strategy is a *best response* to the strategies of the opponents: a **Nash equilibrium** is an assignment of (possibly randomized) strategies to the players, with the property that no player can gain by unilaterally changing his strategy. It turns out that

every general-sum game has at least one Nash equilibrium. The proof of this fact requires an important geometric tool, the **Brouwer fixed-point theorem**, which is covered in **Chapter 5**.

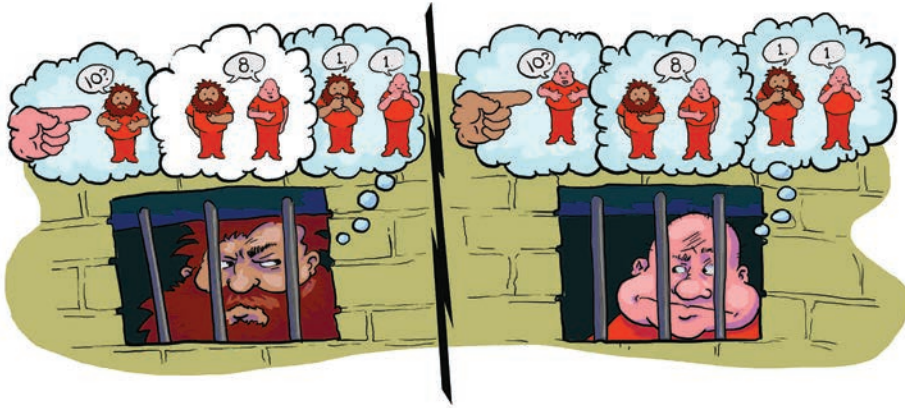


FIGURE 5. Prisoner's Dilemma: the prisoners considering the possible consequences of confessing or remaining silent.

The most famous general-sum game is the **Prisoner's Dilemma**. If one prisoner confesses and the other remains silent, then the first goes free and the second receives a ten-year sentence. They will be sentenced to eight years each if they both confess and to one year each if they both remain silent. The only equilibrium in this game is for both to confess, but the game becomes more interesting when it is repeated, as we discuss in **Chapter 6**. More generally, in Chapter 6 we consider games where players alternate moves as in Chapter 1, but the payoffs are general as in Chapter 4. These are called **extensive-form** games. Often these games involve **imperfect information**, where players do not know all actions that have been taken by their opponents. For instance, in the 1962 Cuban Missile Crisis, the U.S. did not know whether the U.S.S.R. had installed nuclear missiles in Cuba and had to decide whether to bomb the missile sites in Cuba without knowing whether or not they were fitted with nuclear warheads. (The U.S. used a naval blockade instead.) We also consider games of **incomplete information** where the players do not even know exactly what game they are playing. For instance, in poker, the potential payoffs to a player depend on the cards dealt to his opponents.

One criticism of optimal strategies and equilibria in game theory is that finding them requires hyper-rational players that can analyze complicated strategies. However, it was observed that populations of termites, spiders, and lizards can arrive at a Nash equilibrium just via natural selection. The equilibria that arise in such populations have an additional property called **evolutionary stability**, which is discussed in **Chapter 7**.

In the same chapter, we also introduce **correlated equilibria**. When two drivers approach an intersection, there is no good Nash equilibrium. For example, the convention of yielding to a driver on your right is problematic in a four-way intersection. A traffic light serves as a correlating device that ensures each driver is incentivized to follow the indications of the light. Correlated equilibria generalize this idea.

In **Chapter 8**, we compare outcomes in Nash equilibrium to outcomes that could be achieved by a central planner optimizing a global objective function. For example, in Prisoner's Dilemma, the total loss (combined jail time) in the unique Nash equilibrium is 16 years; the minimum total loss is 2 years (if both stay silent). Thus, the ratio, known as the **price of anarchy** of the game, is 8. Another example compares the average driving time in a road network when the drivers are selfish (i.e., in a Nash equilibrium) to the average driving time in an optimal routing.

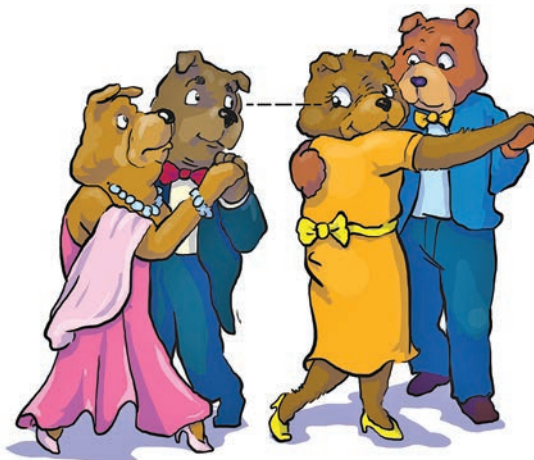


FIGURE 6. An unstable pair.

Part II: Designing games and mechanisms. So far, we have considered predefined games, and our goal was to understand the outcomes that we can expect from rational players. In the second part of the book, we also consider **mechanism design** where we start with desired properties of the outcome (e.g., high profit or fairness) and attempt to *design* a game (or market or scheme) that incentivizes players to reach an outcome that meets our goals. Applications of mechanism design include voting systems, auctions, school choice, environmental regulation, and organ donation.

For example, suppose that there are n men and n women, where each man has a preference ordering of the women and vice versa. A matching between them is **stable** if there is no *unstable pair*, i.e., a man and woman who prefer each other to their partners in the matching. In **Chapter 10**, we introduce the Gale-Shapley algorithm for finding a stable matching. A generalization of stable matching is used by the National Resident Matching Program, which matches about 20,000 new doctors to residency programs at hospitals every year.

Chapter 11 considers the design of mechanisms for **fair division**. Consider the problem of dividing a cake with several different toppings among several people. Each topping is distributed over some portion of the cake, and each person prefers some toppings to others. If there are just two people, there is a well-known mechanism for dividing the cake: One cuts it in two, and the other chooses which

piece to take. Under this system, each person is at least as happy with what he receives as he would be with the other person's share. What if there are three or more people? We also consider a 2000-year-old problem: how to divide an estate between several creditors whose claims exceed the value of the estate.

The topic of **Chapter 12** is **cooperative game theory**, in which players form coalitions in order to maximize their utility. As an example, suppose that three people have gloves to sell. Two are each selling a single, left-handed glove, while the third is selling a right-handed one. A wealthy tourist enters the store in dire need of a pair of gloves. She refuses to deal with the glove-bearers individually, so at least two of them must form a coalition to sell a left-handed and a right-handed glove to her. The third player has an advantage because his commodity is in scarcer supply. Thus, he should receive a higher fraction of the price the tourist pays. However, if he holds out for too high a fraction of the payment, the other players may agree between themselves that he must pay both of them in order to obtain a left glove. A related topic discussed in the chapter is **bargaining**, where the classical solution is again due to Nash.



FIGURE 7. Voting in Florida during the 2000 U.S. presidential election.

In **Chapter 13** we turn to **social choice**: designing mechanisms that aggregate the preferences of a collection of individuals. The most basic example is the design of **voting schemes**. We prove Arrow's Impossibility Theorem, which implies that all voting systems are strategically vulnerable. However, some systems are better than others. For example, the widely used system of runoff elections is not even monotone; i.e., transferring votes from one candidate to another might lead the second candidate to lose an election he would otherwise win. In contrast, Borda count and approval voting are monotone and more resistant to manipulation.

Chapter 14 studies **auctions** for a single item. We compare different auction formats such as **first-price** (selling the item to the highest bidder at a price equal to his bid) and **second-price** (selling the item to the highest bidder at a price



FIGURE 8. An auction for a painting.

equal to the second highest bid). In first-price auctions, bidders must bid below their value if they are to make any profit; in contrast, in a second-price auction, it is optimal for bidders to simply bid their value. Nevertheless, the **Revenue Equivalence Theorem** shows that, in equilibrium, if the bidders' values are independent and identically distributed, then the expected auctioneer revenue in the first-price and second-price auctions is the same. We also show how to design optimal (i.e., revenue-maximizing) auctions under the assumption that the auctioneer has good prior information about the bidders' values for the item he is selling.

Chapters 15 and 16 discuss truthful mechanisms that go beyond the second-price auction, in particular, the **Vickrey-Clarke-Groves (VCG)** mechanism for maximizing **social surplus**, the total utility of all participants in the mechanism. A key application is to **sponsored search auctions**, the auctions that search engines like Google and Bing run every time you perform a search. In these auctions, the bidders are companies that wish to place their advertisements in one of the slots you see when you get the results of your search. In Chapter 16, we also discuss **scoring rules**. For instance, how can we incentivize a meteorologist to give the most accurate prediction he can?

Chapter 17 considers **matching markets**. A certain housing market has n homeowners and n potential buyers. Buyer i has a value v_{ij} for house j . The goal is to find an allocation of houses to buyers and corresponding prices that are stable; i.e., there is no pair of buyer and homeowner that can strike a better deal. A related problem is allocating rooms to renters in a shared rental house. See Figure 9.

Finally, **Chapter 18** concerns **adaptive decision making**. Suppose that each day several experts suggest actions for you to take; each possible action has a reward (or penalty) that varies between days and is revealed only after you choose.



FIGURE 9. Three roommates need to decide who will get each room and how much of the rent each person will pay.

Surprisingly, there is an algorithm that ensures your average reward over many days (almost) matches that of the best expert. If two players in a repeated zero-sum game employ such an algorithm, the empirical distribution of play for each of them will converge to an optimal strategy.

For the reader and instructor

Prerequisites. Readers should have taken basic courses in probability and linear algebra. Starred sections and subsections are more difficult; some require familiarity with mathematical analysis that can be acquired, e.g., in [Rud76].

Courses. This book can be used for different kinds of courses. For instance, an undergraduate game theory course could include Chapter 1 (combinatorial games), Chapter 2 and most of Chapter 3 on zero-sum games, Chapters 4 and 7 on general-sum games and different types of equilibria, Chapter 10 (stable matching), parts of Chapters 11 (fair division), 13 (social choice), and possibly 12 (especially the Shapley value). Indeed, this book started from lecture notes to such a course that was given at Berkeley for several years by the second author.

A course for computer science students might skip some of the above chapters (e.g., combinatorial games) and instead emphasize Chapter 9 on price of anarchy, Chapters 14–16 on auctions and VCG, and possibly parts of Chapters 17 (matching markets) and 18 (adaptive decision making). The topic of stable matching (Chapter 10) is a gem that requires no background and could fit in any course. The logical dependencies between the chapters are shown in Figure 10.

There are solution outlines to some problems in Appendix D. Such solutions are labeled with an “S” in the text. More difficult problems are labeled with a *. Additional exercises and material can be found at:

<http://homes.cs.washington.edu/~karlin/GameTheoryAlive>

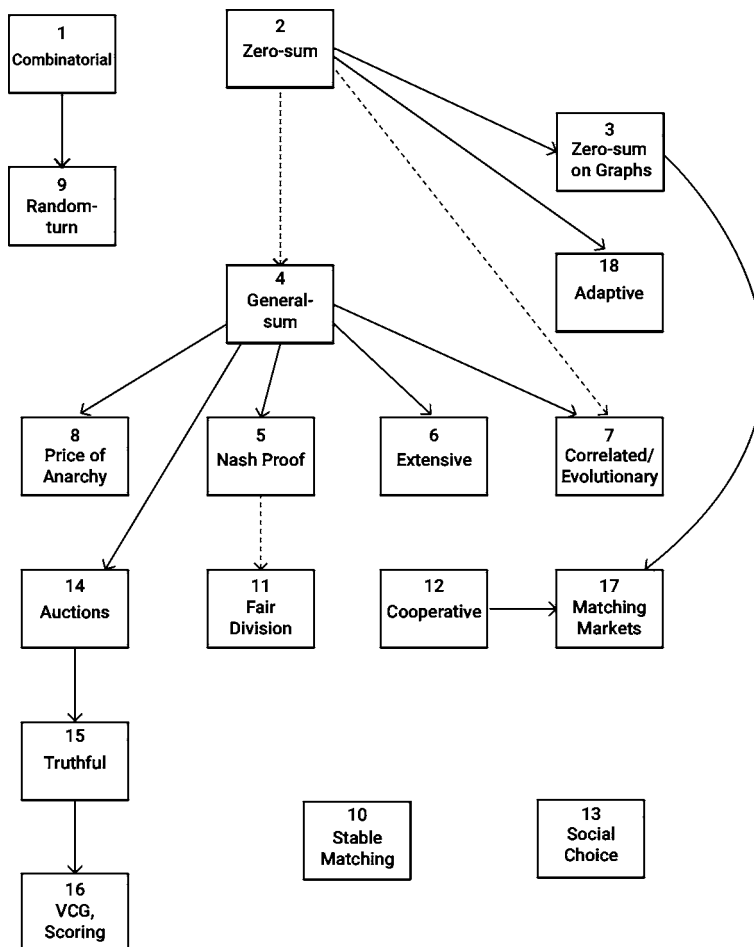


FIGURE 10. Chapter dependencies.

Notes

There are many excellent books on game theory. In particular, in writing this book, we consulted Ferguson [Fer08], Gintis [Gin00], González-Díaz et al. [GDGJFJ10], Luce and Raiffa [LR57], Maschler, Solan, and Zamir [MSZ13], Osborne and Rubinstein [OR94], Owen [Owe95], the survey book on algorithmic game theory [NRTV07], and the handbooks of game theory, Volumes 1–4 (see, e.g., [AH92]).

The entries in the payoff matrices for zero-sum games represent the **utility** of the players, and throughout the book we assume that the goal of each agent is maximizing his expected utility. Justifying this assumption is the domain of utility theory, which is discussed in most game theory books.

The Penalty Kicks matrix we gave was idealized for simplicity. Actual data on 1,417 penalty kicks from professional games in Europe was collected and analyzed by Palacios-Huerta [PH03]. The resulting matrix is

		goalie	
		L	R
kicker	L	0.58	0.95
	R	0.93	0.70

Here ‘R’ represents the dominant (natural) side for the kicker. Given these probabilities, the optimal strategy for the kicker is $(0.38, 0.62)$ and the optimal strategy for the goalie is $(0.42, 0.58)$. The observed frequencies were $(0.40, 0.60)$ for the kicker and $(0.423, 0.577)$ for the goalie.

The early history of the theory of strategic games from Waldegrave to Borel is discussed in [DD92].