

Preface to the Three Volume Set

Geometry measures space (*geo* = earth, *metry* = measurement). Einstein's theory of *relativity* measures space-time and might be called *geochronometry* (*geo* = space, *chrono* = time, *metry* = measurement). The arc of mathematical history that has led us from the geometry of the plane of Euclid and the Greeks after 2500 years to the physics of space-time of Einstein is an attractive mathematical story. Geometrical reasoning has proved instrumental in our understanding of the real and complex numbers, algebra and number theory, the development of calculus with its elaboration in analysis and differential equations, our notions of length, area, and volume, motion, symmetry, topology, and curvature.

These three volumes form a very personal excursion through those parts of the mathematics of 1- and 2-dimensional geometry that I have found magical. In all cases, this point of view is the one most meaningful to me. Every section is designed around results that, as a student, I found interesting in themselves and not just as preparation for something to come later. Where is the magic? Why are these things true?

Where is the tension? Every good theorem should have tension between hypothesis and conclusion. — Dennis Sullivan

Where is the Sullivan *tension* in the statement and proofs of the theorems? What are the key ideas? Why is the given proof natural? Are the theorems almost false? Is there a nice picture? I am not interested in quoting results without proof. I am not afraid of a little algebra, or calculus, or linear algebra. I do not care about complete rigor. I want to understand. If every formula in a book cuts the readership in half, my audience is a small, elite audience. This book is for the student who likes the magic and wants to understand.

A scientist is someone who is always a child, asking 'Why? why? why?'. — Isidor Isaac Rabi, Nobel Prize in Physics 1944

Wir müssen wissen, wir werden wissen. [We must know, we will know.] — David Hilbert

The three volumes indicate three natural parts into which the material on 2-dimensional spaces may be divided:

Volume 1: The geometry of the plane, with various historical attempts to understand lengths and areas: areas by similarity, by cut and paste, by counting, by slicing. Applications to the understanding of the real numbers, algebra, number theory, and the development of calculus. Limitations imposed on the measurement of size given by nonmeasurable sets and the wonderful Hausdorff-Banach-Tarski paradox.

Volume 2: The topology of the plane, with all of the standard theorems of 1 and 2-dimensional topology, the Fundamental Theorem of Algebra, the Brouwer Fixed-Point Theorem, space-filling curves, curves of positive area, the Jordan Curve Theorem, the topological characterization of the plane, the Schoenflies Theorem, the R. L. Moore Decomposition Theorem, the Open Mapping Theorem, the triangulation of 2-manifolds, the classification of 2-manifolds via orientation and Euler characteristic, dimension theory.

Volume 3: An introduction to non-Euclidean geometry and curvature. What is the analogy between the standard trigonometric functions and the hyperbolic trig functions? Why is non-Euclidean geometry called *hyperbolic*? What are the gross intuitive differences between Euclidean and hyperbolic geometry?

The approach to curvature is backwards to that of Gauss, with definitions that are obviously invariant under bending, with the intent that curvature should obviously measure the degree to which a surface cannot be flattened into the plane. Gauss's Theorema Egregium then comes at the end of the discussion.

Prerequisites: An undergraduate student with a reasonable memory of calculus and linear algebra, but with no fear of proofs, should be able to understand almost all of the first volume. A student with the rudiments of topology—open and closed sets, continuous functions, compact sets and uniform continuity—should be able to understand almost all of the second volume with the exception of a little bit of algebraic topology used to prove results that are intuitively reasonable and can be assumed if necessary. The final volume should be well within the reach of someone who is comfortable with integration and change of variables. We will make an attempt in many places to review the tools needed.

Comments on exercises: Most exercises are interlaced with the text in those places where the development suggests them. They are an essential part of the text, and the reader should at least make note of their content. Exercise sections which appear at the end of most chapters refer back to these exercises, sometimes with hints, occasionally with solutions, and sometimes add additional exercises. Readers should try as many exercises as attract them, first without looking at hints or solutions.

Comments on difficulty: Typically, sections and chapters become more difficult toward the end. Don't be afraid to quit a chapter when it becomes too difficult. Digest as much as interests you and move on to the next chapter or section.

Comments on the bibliography: The book was written with very little direct reference to sources, and many of the proofs may therefore differ from the standard ones. But there are many wonderful books and wonderful teachers that we can learn from. I have therefore collected an annotated bibliography that you may want to explore. I particularly recommend [1, G. H. Hardy, *A Mathematician's Apology*], [2, G. Pólya, *How to Solve It*], and [3, T. W. Körner, *The Pleasure of Counting*], just for fun, light reading. For a bit of hero worship, I also recommend the biographical references [21, E. T. Bell, *Men of Mathematics*], [22, C. Henrion, *Women of Mathematics*], and [23, W. Dunham, *Journey Through Genius*]. And I have to thank my particular heroes: my brother Larry, who taught me about uncountable sets, space-filling curves, and mathematical induction; Georg Pólya, who invited me into his home and showed me his mathematical notebooks; my advisor C. E. Burgess, who introduced me to the wonders of Texas-style mathematics; R. H. Bing, whose Sling, Dogbone Space, Hooked Rug, Baseball Move, epslums and

deltas, and Crumpled Cubes added color and wonder to the study of topology; and W. P. Thurston, who often made me feel like Gary Larson's character of little brain ("Stop, professor, my brain is full.") They were all kind and encouraging to me. And then there are those whom I only know from their writing: especially Euclid, Archimedes, Gauss, Hilbert, and Poincaré.

Finally, I must thank Bill Floyd and Walter Parry for more than three decades of mathematical fun. When we would get together, we would work hard every morning, then talk mathematics for the rest of the day as we hiked the cities, countrysides, mountains, and woods of Utah, Virginia, Michigan, Minnesota, England, France, and any other place we could manage to get together. And special thanks to Bill for cleaning up and improving almost all of those figures in these books which he had not himself originally drawn.

Preface to Volume 2

The first of three volumes in this set was devoted to the measurement of lengths and areas, and to some of the consequences that study had in number theory, algebra, and analysis. Euclid was able to solve quadratic equations by geometric construction. But when mathematicians tried to extend those results to equations of higher degree and to differential equations, a number of fascinating difficulties arose, all involving limits and continuity, best modelled by topology.

In this second volume we assume that the reader has had a first course in topology and is comfortable with open and closed sets, connected sets, compact sets, limits, and continuity. Two good references are W. S. Massey [25] and J. R. Munkres [24].

The following discoveries led to the topics of this second volume.

(1) The solution of cubic and quartic equations required serious consideration of complex numbers, thought at first to be mysterious. But the mystery disappeared when it was seen that complex numbers simply model the Euclidean plane. Abel and Galois proved that equations of degrees 5 and higher could not be solved in the relatively simple manner by formula as had sufficed in equations of degrees 1 through 4. But Gauss, without giving explicit solutions, managed to prove the Fundamental Theorem of Algebra that ensured that complex numbers sufficed for their solution. Gauss gave proofs involving the geometry and topology of the plane.

(2) Newton showed that the study of motion could be greatly simplified if, instead of examining standard equations, one examined differential equations. Proving the existence of solutions to rather general differential equations led to problems in topology. One of the standard proof techniques involves Brouwer's Fixed Point Theorem. This volume proves that theorem in dimension 2 and outlines the proof in general dimensions.

(3) Descartes demonstrated that mechanical devices other than straight edge and compass can construct curves of very high degree. Once curves of very general form are accepted as interesting, further delicate questions of length and area arise: finite curves of infinite length, finite curves of positive area, space filling curves, disks whose interiors have smaller areas than their closures, 0-dimensional sets through which no light rays can penetrate, continuous functions that are nowhere differentiable, sets of fractional dimension. This volume gives examples of many of these phenomena.

(4) The study of solutions to equations became more unified when all variables were considered to be complex variables. Riemann modelled complex curves by surfaces, which are 2-dimensional manifolds and are called *Riemann surfaces*. The analysis of 2-dimensional manifolds led naturally to notions, such as triangulation, genus, and Euler characteristic. These notions are explained in this volume.

All of these considerations required the study of limits and continuity, and the abstract notion that models limits and continuity in their most general settings is the notion of *topology*. Henri Poincaré wrote:

As for me, all of the diverse paths which I have successively followed have led me to topology. I have needed the gifts of this science to pursue my studies of the curves defined by differential equations and for the generalization to differential equations of higher order, and, in particular, to those of the three body problem. I have needed topology for the study of nonuniform functions of two variables. I have needed it for the study of the periods of multiple integrals and for the application of that study to the expansion of perturbed functions. Finally, I have glimpsed in topology a means to attack an important problem in the theory of groups, the search for discrete or finite groups contained in a given continuous group.