Preface to the Three Volume Set

Geometry measures space (*geo* = earth, *metry* = measurement). Einstein's theory of *relativity* measures space-time and might be called *geochronometry* (*geo* = space, *chrono*= time, *metry* = measurement). The arc of mathematical history that has led us from the geometry of the plane of Euclid and the Greeks after 2500 years to the physics of space-time of Einstein is an attractive mathematical story. Geometrical reasoning has proved instrumental in our understanding of the real and complex numbers, algebra and number theory, the development of calculus with its elaboration in analysis and differential equations, our notions of length, area, and volume, motion, symmetry, topology, and curvature.

These three volumes form a very personal excursion through those parts of the mathematics of 1- and 2-dimensional geometry that I have found magical. In all cases, this point of view is the one most meaningful to me. Every section is designed around results that, as a student, I found interesting in themselves and not just as preparation for something to come later. Where is the magic? Why are these things true?

> *Where is the tension? Every good theorem should have tension between hypothesis and conclusion.* — Dennis Sullivan

Where is the Sullivan *tension* in the statement and proofs of the theorems? What are the key ideas? Why is the given proof natural? Are the theorems almost false? Is there a nice picture? I am not interested in quoting results without proof. I am not afraid of a little algebra, or calculus, or linear algebra. I do not care about complete rigor. I want to understand. If every formula in a book cuts the readership in half, my audience is a small, elite audience. This book is for the student who likes the magic and wants to understand.

> *A scientist is someone who is always a child, asking 'Why? why? why?'.* — Isidor Isaac Rabi, Nobel Prize in Physics 1944

> *Wir müssen wissen, wir werden wissen.* [We must know, we will know.] — David Hilbert

The three volumes indicate three natural parts into which the material on 2-dimensional spaces may be divided:

Volume 1: The geometry of the plane, with various historical attempts to understand lengths and areas: areas by similarity, by cut and paste, by counting, by slicing. Applications to the understanding of the real numbers, algebra, number theory, and the development of calculus. Limitations imposed on the measurement of size given by nonmeasurable sets and the wonderful Hausdorff-Banach-Tarski paradox.

Volume 2: The topology of the plane, with all of the standard theorems of 1- and 2-dimensional topology, the fundamental theorem of algebra, the Brouwer fixed-point theorem, space-filling curves, curves of positive area, the Jordan curve theorem, the topological characterization of the plane, the Schoenflies theorem, the R. L. Moore decomposition theorem, the open mapping theorem, the triangulation of 2-manifolds, the classification of 2-manifolds via orientation and Euler characteristic, dimension theory.

Volume 3: An introduction to non-Euclidean geometry and curvature. What is the analogy between the standard trigonometric functions and the hyperbolic trig functions? Why is non-Euclidean geometry called *hyperbolic*? What are the gross intuitive differences between Euclidean and hyperbolic geometry?

The approach to curvature is backwards to that of Gauss, with definitions that are obviously invariant under bending, with the intent that curvature should obviously measure the degree to which a surface cannot be flattened into the plane. Gauss's Theorema Egregium then comes at the end of the discussion.

Prerequisites: An undergraduate student with a reasonable memory of calculus and linear algebra, but with no fear of proofs, should be able to understand almost all of the first volume. A student with the rudiments of topology—open and closed sets, continuous functions, compact sets and uniform continuity—should be able to understand almost all of the second volume with the exeption of a little bit of algebraic topology used to prove results that are intuitively reasonable and can be assumed if necessary. The final volume should be well within the reach of someone who is comfortable with integration and change of variables. We will make an attempt in many places to review the tools needed.

Comments on exercises: Most exercises are interlaced with the text in those places where the development suggests them. They are an essential part of the text, and the reader should at least make note of their content. Exercise sections which appear at the end of most chapters refer back to these exercises, sometimes with hints, occasionally with solutions, and sometimes add additional exercises. Readers should try as many exercises as attract them, first without looking at hints or solutions.

Comments on difficulty: Typically, sections and chapters become more difficult toward the end. Don't be afraid to quit a chapter when it becomes too difficult. Digest as much as interests you and move on to the next chapter or section.

Comments on the bibliography: The book was written with very little direct reference to sources, and many of the proofs may therefore differ from the standard ones. But there are many wonderful books and wonderful teachers that we can learn from. I have therefore collected an annotated bibliography that you may want to explore. I particularly recommend [**1**, G. H. Hardy, *A Mathematician's Apology*], [**2**, G. Pólya, *How to Solve It*], and [**3**, T. W. Körner, *The Pleasure of Counting*], just for fun, light reading. For a bit of hero worship, I also recommend the biographical references [**21**, E. T. Bell, *Men of Mathematics*], [**22**, C. Henrion, *Women of Mathematics*], and [**23**, W. Dunham, *Journey Through Genius*]. And I have to thank my particular heroes: my brother Larry, who taught me about uncountable sets, space-filling curves, and mathematical induction; Georg Pólya, who invited me into his home and showed me his mathematical notebooks; my advisor C. E. Burgess, who introduced me to the wonders of Texas-style mathematics; R. H. Bing, whose Sling, Dogbone Space, Hooked Rug, Baseball Move, epslums and deltas, and Crumpled Cubes added color and wonder to the study of topology; and W. P. Thurston, who often made me feel like Gary Larson's character of little brain ("Stop, professor, my brain is full.") They were all kind and encouraging to me. And then there are those whom I only know from their writing: especially Euclid, Archimedes, Gauss, Hilbert, and Poincaré.

Finally, I must thank Bill Floyd and Walter Parry for more than three decades of mathematical fun. When we would get together, we would work hard every morning, then talk mathematics for the rest of the day as we hiked the cities, countrysides, mountains, and woods of Utah, Virginia, Michigan, Minnesota, England, France, and any other place we could manage to get together. And special thanks to Bill for cleaning up and improving almost all of those figures in these books which he had not himself originally drawn.

Preface to Volume 3

This is the last of a three volume set describing a very personal arc of thought that begins with earth measurement (that is, geo-metry), passes through the topology of 2-dimensional surfaces, and ends with space-time measurement (that is, geo-chrono-metry, where Einstein identifies gravity with the curvature of space-time). The volumes are (1) The Geometry of the 2- Dimensional Spaces; (2) The Topology of 2-Dimensional Spaces; and (3) An Introduction to Non Euclidean Geometry and Curvature.

Volume 3 gives a general introduction to hyperbolic non-Euclidean geometry in all dimensions, with an introduction to all of the standard models and their relationships to one another. It explains why the models are called hyperbolic. It discusses the ways in which this geometry differs from Euclidean geometry. It calculates the shortest paths in this geometry (the geodesics). It explains some of the reasons why this geometry is studied. Following the introduction to non-Euclidean geometry, Volume 3 approaches curvature in dimension 2 in a way that does not begin with the Gauss map. Instead, it begins with a natural metric-invariant definition of curvature that measures in an obvious way the difficulty of flattening a surface into the plane without distorting lengths and areas. It ends with the Gauss map and a proof of Gauss's Theorema Egregium. This approach, which is backward to the classical approach, is intuitive; but it entails difficulties in proving that certain limits exist.