

Chapter 1

The Basics

Against the disease of writing one must take special precautions, since it is a dangerous and contagious disease.

Peter Abelard
Letter 8, Abelard to Heloise

Judge an artist not by the quality of what is framed and hanging on the walls, but by the quality of what's in the wastebasket.

Anon., quoted by Leslie Lamport

*It matters not how strait the gate,
How charged with punishments the scroll,
I am the master of my fate;
I am the captain of my soul.*

W. E. Henley

Your manuscript is both good and original; but the part that is good is not original, and the part that is original is not good.

Samuel Johnson

In America only the successful writer is important, in France all writers are important, in England no writer is important, and in Australia you have to explain what a writer is.

Geoffrey Cotterel

It may be true that people who are merely mathematicians have certain specific shortcomings; however, that is not the fault of mathematics, but is true of every exclusive occupation.

Carl Friedrich Gauss
letter to H. C. Schumacher [1845]

In fifty years nobody will have tenure but everyone will have a Ph.D.

M. V. Wickerhauser

1.1 What It Is All About

1.1.1 How to Write and Why to Write

In order to write effectively and well, you must have something to say. You must also be *confident* that you have something to say, and that that something is worth saying. Finally, you have to figure out how to say it.

This sounds trite, but it is the single most important fact about writing. In order to write effectively and well, you must also have an audience. And you must *know consciously* who that audience is. Much of the bad writing that exists is performed by the author of a math research paper who thinks that all of his/her readers are Henri Poincaré, or by the author of a textbook who does not seem to realize that his/her readers will be students.

Good writing requires a certain confidence. You must be confident that you have something to say, and that that something is worth saying. But you also must have the confidence to know that “My audience is X and I will write for X .”

Indeed, many writers of a mathematical paper seem to be writing primarily to convince themselves that their theorem is correct, rather than as an effort to communicate. Such authors seem embarrassed to explain anything, and hide instead behind the details.

Many textbook authors seem embarrassed to speak to the student in language that the student will apprehend. Such authors instead find themselves making excuses and asides to the instructor (who either will not read the book, or will flip through it impatiently and entirely miss the author’s efforts).

Imagine penning a poem to your one true love, all the while thinking “What would my English teacher think?” or “What would my pastor think?” or “What would my mother think?” Have the courage of your convictions. Define your audience. Doing so, you should say to yourself, “My audience is X and I will write for X .” Speak to that person or to those people whom you are genuinely trying to reach. Know what it is that you want to say and then say it, all the while anticipating your specific audience’s specific needs.

I note here that I am an American author and I am writing with an American audience in mind. So the opinions that I express, and the rules that I enunciate, are American rules. British rules are often different. And no rule of grammar or syntax is etched in stone. These are ever-changing. It is what makes life interesting.

1.1.2 The Research Paper

For at least some mathematicians, the most important writing is the writing of a research paper. You have proved a nice theorem, perhaps a great theorem. You certainly have something to say. You also know exactly who your audience is: other research mathematicians who are interested in your field of study. Thus two of the biggest problems for a writer are already solved. The issue that remains is how to put it on paper. Remember that, as much as you might

admire your own results, if you pen a love letter to yourself, then it will have both the good features and the bad features of such a screed: it will exhibit both passion and fervor, but it will tend to exclude the rest of the world. What do these remarks mean in practice? In particular, they mean that as you write you must think of your reader—not yourself. Narcissistic writing is precisely that—narcissism is suitable in some contexts, but not in a research mathematics context. As a mathematics writer, you must place yourself at the service of your readers. You must consider their convenience and understanding, not just your own.

In the Sputnik era, in the late 1950s and early 1960s, when mathematics departments and journals were growing explosively and everyone was in a rush to publish, it was common to begin a paper by writing “Notation is as in my last paper.” Today, by contrast, there are truly gifted mathematicians who write papers that look like a letter home to Mom: they just start to write, occasionally starting a new paragraph when the text spills over onto a new page, never formally stating a theorem or even a definition, never coming to any particular point. The contents may be divine, but busy readers will likely lack all patience to discover and understand them.

These last are not the sorts of papers that you would want to read, so why torment your readers by writing papers like this? Much of the remainder of this book will discuss ways to write your work so that people *will* want to read it, and will enjoy it when they do so.

1.2 Who Is My Audience?

1.2.1 Identifying Your Audience

If you are writing a diary, then it may be safe to say that your readership is just yourself. Truthfully, even this may not be accurate. For you may have it in the back of your mind that (like Anne Frank’s or Samuel Pepys’s diaries) this piece of writing is something for the ages. If you are writing a letter home to Mom, then your audience is Mom and, on a good day, perhaps Pop. If you are writing a calculus exam, then your audience consists of your students, and perhaps some of your colleagues (or your Chair, if the Chair is in the habit of reviewing your teaching). If you are writing a tract on handle-body theory, then your audience is probably a well-defined group of fellow mathematicians (most likely topologists). Know your audience!

Keep in mind a specific person—somebody actually in your acquaintance—to whom you might be writing. If you are writing to yourself or to Mom, this is easy. If you are instead writing to your peers in handle-body theory, then think of someone in particular—someone to whom you could be explaining your ideas. This technique is more than a facile artifice; it helps you to picture what questions might be asked, or what confusions or objections might arise, or which details you might need to trot out and explain. It enables you to formulate the explanation of an idea, or the clarification of a difficult point.

1.3 Writing and Thought

1.3.1 Clear Thinking

The ability to think clearly and the ability to write clearly are inextricably linked. If you cannot articulate a thought, formulate an argument, marshal data, assimilate ideas, or organize a thesis, then you will not be an effective writer. By the same token, you can use your writing as a method of developing and honing your thoughts—your hypotheses, your theorems, your proofs. See [Hig] for an insightful discussion of this concept.

We all know that one way to work out our insights and organize our ideas is to engage in an animated discussion with someone whom we respect. But you can instead, à la Descartes, have that interchange with yourself. And a useful way to do so is by writing. When I want to develop my ideas on some topic—teaching reform, or the funding of mathematics, or the directions that future research in several complex variables ought to take, or my new ideas about domains with noncompact automorphism group—I often find it useful to write a little essay on the subject. For writing forces me to express my ideas clearly and in the proper order, to fill in logical gaps, to differentiate hypotheses from blind assumptions from conclusions, and to make my point forcefully and clearly. Sometimes I show the resulting essay to friends and colleagues, and sometimes, with many edits and revisions, I try to publish it. But, just as often, I file it away on my hard disk and forget about it until I have future need to refer to it.

1.3.2 Research Mathematics

The writing of research-level mathematics is a more formal process than that described in the last paragraph, but it can begin in the same way. You can begin your exposition with a little essay that organizes your ideas, sets your course, and defines your goals. This should probably be a draft that only you will see. You can judge this first draft, and indeed subsequent drafts, as any critical reader would. You may, and probably will, find that that “obvious lemma” is not so obvious after all. You may have to put in some extra work to make things come out right. You may, along the way, discover extraneous content, ideas that may have momentary interest but are not really germane to your main thesis.

When you write up your latest ideas for dissemination and publication, then you must finally face the music. That critical lemma must now be treated; the case that you did not really want to consider must be dispatched. The ideas must be put in logical order and the chain of reasoning forged and fixed. It can be a real pleasure to craft your latest burst of creativity into a compelling flood of logic and calculation. In any event, this skill is one that you are obliged to master if you wish to see your work in print, and read by other people, and understood and appreciated.

1.3.3 Various Drafts

You might think in terms of writing a sequence of drafts of whatever project you are now working on:

- The first draft just puts the basic ideas down on paper. Be as accurate as you can, but do not get bogged down at this early stage.
- The second draft considers organization, accuracy, flow of ideas, logic, and potential audience reaction.
- The third draft is the proofreading stage. You should focus on syntax, spelling, diction, sound, and grammar.
- By the fourth draft you should have a fairly polished piece in your hands. Read it aloud, listening for meaning and coherence.
- By the fifth draft you can assume that you have said what you want to say. Now check for spelling (a spell-checker can be useful here), typos, grammatical slips, and the like. Make sure that all the pages are formatted properly. Make sure that all the equations and displayed mathematics look as they should.

As a writer, you in some sense resemble the budding tennis pro who is practicing his/her backhand or the future concert pianist perfecting his/her scales. Little by little, the technical exercise of writing becomes less dreary and more pleasurable. And certainly more constructive and effective.

1.3.4 Writing as an Enabling Activity

Once you apprehend the principles just enunciated, writing ceases to be a dreary chore and instead turns into a constructive activity. It becomes a new challenge that you can aim to perfect—like your tennis backhand or your piano playing. If you are the sort of person who sits in front of the computer screen befuddled, frustrated, or even angry, and thinks “I know just what my thoughts are, but I cannot figure out how to say them,” then something is wrong. Perhaps you are impeding your mathematical creativity by assuming that your first draft is really your final draft (sadly, many if not most of us are guilty of this sin). Yet the first draft is almost inevitably thoughts-in-a-jumble, not as well or thoroughly reasoned out as you believe they are. Clarify your insights and put them in order—certainly the way that you order your ideas reflects critically on your thinking skills. Adopting the advice given here will not only make you a better writer but also a stronger thinker and a better mathematician.

Writing should *enable* you to express your thoughts, not hinder you. I hope that reading this book will help you to write, indeed will enable you to write, both effectively and well.

1.4 Say What You Mean, Mean What You Say

1.4.1 Obscure Language

You have likely often heard, or perhaps uttered, a sentence like

As a valued customer of XYZ Co., your call is very important to us.

✘

Or perhaps

To assist you better, please select one of the following from our menu.

✘

What is wrong with these sentences? The first suggests that “your call” is a valued customer. Clearly that concept is not the intention here. A more accurately formulated sentence would be

You are a valued customer of XYZ Co., and your call is very important to us.

Or perhaps

Because you are a valued customer of XYZ Co., your call is very important to us.

In the second misspoken example, the phrase “To assist you better” lacks a subject; it is clearly intended to modify an invisible “we”; therefore a stronger construction is

So that we may better assist you, please select an item from our menu

or perhaps

We can assist you more efficiently if you will make a selection from the following menu.

What is my point here? Am I just pompously nit-picking? Assuredly not. Mathematics cannot tolerate imprecision. And while the nature of mathematical *notation* tends to rule out imprecision, the *words* that connect our formulae can *lead to* imprecision. If you formulate your thoughts inaccurately, then your point may well be lost. Here are a few more examples of sentences that do not convey what their authors intended:

Having spoken at hundreds of universities, the brontosaurus was a large green lizard. ✘

(Amazingly, this sentence is a slight variant of one that was uttered by a distinguished scholar who is world famous for his careful use of prose.)

As in our food, we strive to be creative with keeping the highest quality in mind, we have in our wine selections also. ❖

(This sentence was taken from the menu of a rather good St. Louis restaurant.)

To serve you better, please form a line. ❖

(How many times have you heard this at your local retailer's, or at the bank?)

The message here is a simple one: Make sure that your subject matches your verb. Make sure that your referents actually refer to the intended person or thing. Make sure that your participles do not dangle. Make sure that your clauses cohere. *Read each sentence aloud.* Does each one make sense? Would you *say this in a conversation*? Would you understand it if someone else said it?

1.4.2 Linear Versus Nonlinear

In life, we receive many different streams of ideas simultaneously, and we parallel-process them in that greatest of all CPUs—the human brain. We absorb and process information and knowledge in a nonlinear fashion. But mathematical writing must strive for linear order. Word k proceeds directly after word $(k - 1)$. The distinction between written language as a medium of communication and the information flow that we commonly experience helter-skelter is one of the barriers between you and good writing. I myself struggle with this matter constantly. Indeed, as you read this book (which purports to tell you how to write), you will see passages in which I say “now I will digress for a moment” or “here is an aside.” (In other places I put sentences in parentheses or brackets; or I use a footnote.) These are junctures at which I could not fit the material being discussed into strictly logical order.

But I can guarantee that you will have to learn to wrestle with similar problems in your own writing. One version of writer's block is a congenital inability to address this linear vs. nonlinear problem. In this situation, nothing succeeds like success. I recommend that, next time you encounter such a difficulty, address it head on. Find devices to help you work around the block. Work the writer's block into submission by forcing the words to say what you want to say. After you have defeated this problem a few times (not without a struggle!), then you will be confident that you can handle it in the future.

1.4.3 Global Issues

I have discoursed on accurate use of language in the technical sense. Now let me remark on more global issues. As you write, you must think not only about whether your writing is correct and appropriate, but also about where your writing will go and what it will do when it gets there.

And you must think about organization. This is one of many ways that you exhibit respect for your audience. Cogent organization makes your writing

compelling, helps the reader to be satisfied with the reading experience, and effects successful communication.

I have already admonished you to know when to start writing. Namely, you begin writing when you have something to say and you know clearly to whom you wish to say it. You also must know when to stop writing. Stop when you have said what you have to say. Say it clearly, say it completely, say it forcefully, say it without leap or lacuna, but then shut up. To prattle on and on is not to convince further.

1.4.4 Language as a Weapon

When you are a person of some accomplishment, and some clout, then your writing carries considerable responsibility. Your words may have great effect. You must weigh the words, and weigh their anticipated impact, carefully.

It is possible to cause considerable damage with what you write. You can have a considerable, and not very constructive, effect on people's lives. You can change the course of events.

The lesson here is to take writing seriously. Think about how people will read your words, and how they will react to your words. If you are writing something delicate, whose ultimate impact is difficult to predict, then take the time to give the matter careful thought. It is only the sensitive and courteous thing to do.

1.5 Diction

Diction is concerned with word choice, and how that choice can affect your meaning and your message. In this rather long section we shall treat a number of different aspects of diction in mathematical writing.

1.5.1 Careful Use of Words

Use words carefully. A well-trained mathematician is not likely to use the word "continuous" to mean "measurable" nor "pseudoconvex" to mean "one-connected." However, we sometimes lapse into sloppiness when using ordinary prose. Treat your dictionary as a close friend: consult it frequently. Do not use "enervate" to mean "invigorate" nor "fatuous" to mean "overweight" nor "provenance" to denote a geopolitical entity. When I am being underhanded, it is not because I am short of help.

1.5.2 Attitude

And never doubt that language is a weapon. "Sticks and stones may break my bones but words will never hurt me" is perhaps the most foolish sentence ever uttered. You can inflict more damage, sometimes permanently, with words than you can with any weapon. You can manipulate more minds, and more people, with words than with any other device.

When a police officer addresses you by

Sir, may I see your driver's license? Did you notice that red light back there?

then he/she is sending out one sort of signal. (Namely, you are clearly a law-abiding citizen and he/she is just doing his/her job by pulling you over and perhaps giving you a ticket.) When instead a cop in the station house says

OK, Billy. Why don't you spill your guts? You know that those other bums aren't going to do a thing to protect you. All they care about is saving their own skins. Jacko already confessed to the heist and told us that you held the gun, Billy. Now we need to hear it from you. Make it easy on yourself, Billy: play ball with us and we'll play ball with you.

then he is sending out quite a different sort of signal. (Namely, by using the first name—and not “William,” but “Billy”—he is undercutting the addressee's dignity; he is treating the person like a wayward child. Further, the policeman is cutting off the individual from his peers, making him feel as though he is on his own. He is suggesting—albeit vaguely—that he may be willing to cut a deal.)

1.5.3 Corruption of Language

I am going to turn now to a brief homily. (I promise that there will be no additional homilies in the book; you may even ignore this one if you wish.) John Locke said that the most effective way to bring down a society is to corrupt its language. You need only look around you to perceive the truth of this statement. When language is corrupted, then people do not communicate effectively. When they do not communicate effectively, then they cannot cooperate. When they cannot cooperate, then the fabric of civilization begins to unravel.

Some of us use the word “bad” to mean “good.” We use the phrase “let us keep our neighborhoods safe and clean” to mean “let us segregate our schools and arm every home”; we use the phrase “I am for gun control and freedom of choice” to mean “I'm a liberal and you're not.” We say “account executive” when we mean “sales clerk” and “sanitation engineer” when we mean “garbage man.” We use the words “interesting” to mean “foolish,” “imaginative” to mean “irresponsible,” and “naive” to mean “idiotic.” These observations are not just idle cocktail party banter. They are in fact indicative of barriers between certain social groups and of the use of language to manipulate and even to coerce.

It is just the same in mathematics. When we use the word “proof” to mean “guesses based on computer printouts” (see [Hor]), when we use “theoretical mathematics” to mean “speculative mathematics” (see [JQ]), when we use the phrase “Charles mathematicians” to belittle the practitioners of traditional and hard-won modes of reasoning that have been developed over many centuries (see [Ati, pp. 193–196]), when we use the phrase “new mathematics” to mean “facile

intuition” (see [PS], [Ati, pp. 193–196]), then we are trivializing our subject. These are gross examples, but the same type of corruption occurs in the small when we write our work sloppily or not at all. It is the responsibility of today’s scholars to develop, nurture, and record our subject for future generations. Good writing is of course not an end in itself; writing is instead the means for achieving the important goal of communicating and preserving mathematics.

1.5.4 What Is in a Title?

Consider these simple examples. Suppose that the Hemingway novel *For Whom the Bell Tolls* were instead entitled *Who the Dingdong Rings For*; or that the Thornton Wilder play *Our Town* were called *My Turf*. Even though the sense of the titles has not been changed appreciably, we see that the alternative titles eschew all the poetry and imagery we sense in the originals. *For Whom the Bell Tolls* evokes powerful emotions; the proffered alternative falls flat. The title *Our Town* suggests one value system, while *My Turf* brings to mind another. One fancies that, if *The Scarlet Letter* had had a less poetic title (how about *Bad Girls Finish Last*), then perhaps Hester Prynne would have garnered only an “A–,” or maybe even an “Incomplete.”

Mathematicians rarely have to wrestle with these poetic questions. But we need to choose names for mathematical objects; we need to formulate definitions. We need to come up with new terminology. We need to describe and to explain. My Ph.D. advisor thought very carefully about his choices of notation and choices of terminology. He figured that his ideas would have considerable influence and lasting value, and he wanted them to work.

As an instance of these ideas, the word “continuous” is a perfect name for a certain type of function; the alternative terminology “nonhypererratic” would be much less useful. The phrase “the point x lies in a relative neighborhood of P ” conveys a world of meaning in an elegant and memorable fashion. Not by accident has this terminology become universal. You should strive for this type of precision and elegance in your own writing.

William Shakespeare said that “. . . a rose by any other name would smell as sweet.” This statement is true, and an apt observation by Juliet, in the context of the dilemma that faced Romeo and Juliet. But the name of a person, place, or thing can profoundly affect its future. There will never be a great romantic leading man of stage and screen who is named Eggs Benedict and there will never be a Fields Medalist or other eminent mathematician named Turkey Tetrizzini. The name of an object may not change its properties (consult Saul Kripke’s *New Theory of Reference* for more on this thought), but it can definitely affect the way that the object is perceived by the world at large. Bear this notion in mind as you create terminology, formulate definitions, and give titles to your papers and other works.

Have you ever noticed that, when you are reading a menu or listening to an advertisement, it never fails that the food being described contains “fresh creamery butter” and “pure golden honey”? The marketing people never say “this grub contains butter and honey,” for there is nothing appealing about

the latter statement. But the first two evoke images of delicious food. As mathematicians, we are not in the position of hawking victuals. But we must still effectively convey our message, and the spirit of that message. We want to inform, and also to inspire.

1.5.5 Accuracy in Writing

As I have already advised, do not agonize over each word as you write a first draft. Just get the ideas down on the page. But *do* agonize a bit when you are editing and proofreading. A passage that reads

This is a very important operator, that has very specific properties,
culminating in a very significant theorem. ✖

is fine as a first try, but does not work well in the long run. It overuses the adverb “very.” It does not flow particularly smoothly. It makes the writer sound dull witted. Consider instead the following:

This operator will be significant for our studies. Its spectral properties, together with the fact that it is smoothing of order 1, will lead to our first fundamental theorem.

The second passage differs from the first in that it has *content*. It communicates with some real specificity. It flows nicely, and makes the writer sound as though he/she has something worthwhile to offer.

An amusing piece of advice, taken from [KnLR, p. 102], is never to use “very” unless you would be comfortable using “damn” in its place!

1.5.6 Alliteration

A good, though not ironclad, rule of thumb is not to use the same word, nor even the same sound, in two consecutive sentences. Of course you may reuse the word “the,” and the nouns that you are discussing will certainly be repeated; but, if possible, do not repeat descriptive words. In addition, do not place words that sound similar in close proximity.

Be especially wary of alliteration. Vice President Spiro Agnew, with the help of speech writer William Safire, earned for himself a certain reputation by using phrases like “pampered prodigies,” “pusillanimous pussyfooters,” “vicars of vacillation,” and “nattering nabobs of negativism.” This rhetoric encouraged, in certain circles, mockery of our vice president—certainly not the reaction that you as a serious mathematician might desire. Lyndon Johnson led us into an escalated Vietnam War by deriding “nervous nellies.” To be sure, the alliterative device is often suitable for poetry or other creative writing, and even perhaps for political polemics; but it is almost never appropriate for mathematics. When alliteration is absolutely necessary, for correct mathematical terminology, then you should de-emphasize it.

For example

This semisimple, sesquilinear operator serves to show sometimes that subgroups of S are sequenced. ✦

does not sound like mathematics. The typical reader probably will pause, reread the sentence several times, and wonder whether the writer is putting him/her on. Better is

Observe that this operator is both semisimple and sesquilinear. These properties can lead to the conclusion that if G is a subgroup of S then G is sequenced.

Notice how the division of one sentence into two is used to break up the alliteration, and in the process enhances comprehensibility.

The last two points—not to repeat words or sounds, and to avoid intrusive alliteration—illustrate the principle of “sound and sense.” If you read your work aloud as you edit and revise, then you will pick out offending passages quickly and easily. With practice, you also will learn how to repair them. The result will be clearer, more effective writing.

1.5.7 “Hence” and Related Words

It is tempting, indeed it is a trap that we all fall into, to overuse a single word that means “hence” or “therefore.” An experienced mathematical writer will have a clutch of words (such as “thus,” “so,” “it follows that,” “as a result,” “consequently,” and so on) to use instead. A paragraph in which every sentence begins with “hence,” or with “therefore,” can be uncomfortable to read. Have alternatives at your fingertips.

There are a number of words that are overused by everyone. These include “very,” “nice,” “get,” “and so forth. Use one of your proofreadings to look for words like this and eliminate them.

1.5.8 Overused and Unnecessary Notation

In general, you should avoid introducing unnecessary notation. Mary Ellen Rudin’s famous statement

Let X be a set. Call it Y .

is funny because it is so ludicrous. But this example is not far from the way we write when we are seduced by notation. Consider, for example

Let X be a compact, metric subspace of the space Y . If f is a continuous, \mathbb{R} -valued function on X , then it assumes both a maximum and a minimum value. ✦

This passage suffers from giving names to the metric space, its superspace, the function, and the target space, and then never using any of them. Slightly better is

Let X be a compact metric space. If f is a continuous, real-valued function on X , then f assumes both a maximum and a minimum value.

Better still is

A continuous, real-valued function on a compact metric space assumes both a maximum value and a minimum value.

The last version of the statement uses no notation, yet conveys the message both succinctly and clearly.

Paul Halmos [Ste] asserts that mathematics should be written so that it reads like a conversation between two mathematicians who are on a walk in the woods. The implementation of this advice may require some effort. If what you have in mind is a huge commutative diagram, or the determinant of a big matrix whose entries are all functions, then you will likely be unsuccessful in conveying your thoughts orally. You must think in terms of how you, or another reasonable person, would *understand* such a complicated object. Of course such understanding is achieved in bits and pieces, and it is achieved conceptually. Such is how you will communicate your ideas during a walk in the woods.

One corollary of the “walk in the woods” approach to writing is that you should write for a reader who is not necessarily sitting in a library, with all the necessary references at his/her fingertips. To be sure, almost any reader will have to look up a few things. But if the reader must race to the stacks, or boot up the computer and do a `Google` search, at every other sentence, then you are making the job too hard. Your paper is far too difficult to follow. Supply the necessary detail and the proper heuristic so that, even if the reader is not sure of a notion, he/she will be able temporarily to suspend his/her disbelief and move on.

As a parting thought, I shall say this: We want our use of notation to be precise and accurate, but we do not want it to be burdensome. While an overwhelming number of superscripts and subscripts may give an extraordinarily accurate description of the space you are discussing, it is no help if it is too tedious and impenetrable. Use Occam’s razor, and only indicate the parameters that are really needed.

1.5.9 Effective Use of Notation

An aspect of writing peculiar to mathematics is the use of notation. Without good notation, many mathematical ideas would be difficult to express. Indeed, the development of mathematics in the Middle Ages and the early Renaissance was hobbled by a lack of notation. With good notation, our writing has the potential to be forceful and direct.

But you need to be alert to pitfalls. A common misuse of notation is to put it at the beginning of a sentence or a clause. For example,

Let f be a function. f is said to be *semicontinuous* if . . . ✦

and

For all points x , $x \in S$. \boxtimes

Even in these two simple examples you can begin to apprehend the problem: the eye balks at a sentence or clause begun with a symbol. The practice is unsightly, and you find yourself rereading the passage a couple of times in order to discern the correct sense. Much better is:

A function f is said to be *semicontinuous* if ...

and

We see that $x \in S$ for all points x .

Observe that both of these revisions are easily comprehended the first time through. This is one of the goals of good writing.

In the same spirit, it is very common for mathematical writers to confuse f and $f(x)$. What is correct is that f is the *name* of a function while $f(x)$ is the *value* of that function at the point x . So, for instance, you should not say, “The function $f(x)$ has these properties ...” Correct is “The function f has these properties ...”

Mathematical notation is often so elegant and compelling that we are tempted to overuse, or misuse, it. You must be diligent to catch even small or seemingly innocuous errors in notation, for these can alter your meaning and confuse the reader. For example, the notation in the sentence “If $x > 0$, then $x^2 > 0$ ” is no hindrance, is easy to read, and tends to make the sentence short and sweet (nonetheless, there are those who would tender cogent arguments for “If a number is positive, then so is its square.”). By contrast, the phrase

Every real, nonsquare $x < 0$... \boxtimes

is objectionable. The reason is that it is not clear, on a first reading, what is meant. Are you saying that “Every real, nonsquare x is negative” or are you saying “Every real, nonsquare x less than zero has the additional property ...” By strictest rules, the notation $<$ is a *binary connective*. The notation is designed for expressing the thought $A < B$. If such is not the exact phrase that fits into your sentence, then you had best not use this notation.

When you are planning a paper, or a book, you should try to plan your notation in advance. You want to be consistent throughout the work in question. To be sure, we have all seen works that, in Section 9, say “For convenience we now change notation.” All of a sudden, the author stops using the letter H to denote a subgroup and instead begins to use H to denote a biholomorphic mapping. Amazingly, this abrupt device actually works much of the time—at least with professional mathematicians. But you should avoid it. If you can, use the same notation for a domain in Section 10 (or Chapter 10) of your work

that you used in Section 1 (or Chapter 1). Try to avoid local contradictions—like suddenly shifting your free variable from x to y . Try not to use the same character for two different purposes.¹

This last stipulation is admittedly not always easy to follow. Many of us commonly use i for the index of a sequence or series: $\{a_i\}$ or $\sum_{i=1}^{\infty} a_i$. No problem so far, but suppose that you are a complex analyst, and also use i to denote a square root of -1 . And now suppose that this last i occurs in some of your sequences and series. You can see the difficulties that would arise. It is best to use j or k as the index of your sequence or series. A little planning can help with this problem, though in the end it may involve a great deal of tedious work to weed out all notational ambiguities.

Many a budding mathematician is seduced by mathematical notation. There was a stage in my education when I thought that all of mathematics should be written without words. I wrote long, convoluted streams of $\forall, \exists, \ni, \Rightarrow, \equiv$, and so forth. This style would have served me well had I been invited to coauthor a new edition of *Principia Mathematica* (see [WR]). In modern mathematics, however, you should endeavor to use English—and to minimize the use of cumbersome notation. Why burden the reader with

$$\forall x, x \geq 0 \Rightarrow \exists y, y^2 = x \quad \spadesuit$$

when you can instead say

Every nonnegative real number has a square root.

1.5.10 Words as Objects

Sometimes you need to write a sentence that treats a word as an object. Here is an example:

We call Γ the *fundamental solution* for the partial differential operator L . We use the definite article “the” because, suitably normalized, there is only one fundamental solution.

I have oversimplified the mathematics here to make a typographical point. First, when you define a term (for the first time), you should italicize the word or phrase being defined. Second, when you refer to a word (in this case “the”) as the object of discussion, then put that word in quotation marks. For a variety of psychological reasons, writers often do not follow this rule. It is helpful to recall W. V. O. Quine’s admonition:

“Boston” has six letters. However Boston has 6 million people and no letters.

¹When André Weil was writing his book *Basic Number Theory* [Wei], he strove mightily to follow this advice. He used up all the Roman letters, all the Greek letters, all the fraktur letters, all the script letters, all the Hebrew letters, and all the other commonly used characters that are seen in mathematics. He ended up resorting to Japanese characters.

Word order can have a serious, if subtle, effect on the meaning (or at least the nuance) of a sentence. Consider the following examples:

Yellow is the color of my true love's hair.

My true love's hair has the color yellow.

The yellow hair of my true love is lovely.

These each say something different. All three sentences have the same *meaning*, but decidedly different emphases. To wit, the first of these sentences stresses the color yellow; the second emphasizes the *blonde's* hair; and the third stresses the yellow hair itself. These differences may seem to be relatively minor, but in mathematics they can make a big difference and change the reader's perception of what you are trying to say. (As an exercise, insert the word "only" into all possible positions in the sentence

I helped Carl prove quadratic reciprocity last week.

and watch the meaning change.)

In mathematics, word order can seriously alter the meaning of a sentence, with the result that the sentence is not immediately understood—if at all. When you proofread your own work, you tend to supply meaning not actually present in the writing; the result is that you can easily miss obscurity imposed by word order. Reading your work aloud can help cut through the problem.

1.5.11 Singular Versus Plural

Whenever possible, use singular constructions rather than plural. Consider the sentence

Domains with noncompact automorphism groups have orbit accumulation points in their boundaries. ✖

First, such a construction does not communicate well: should it be "groups" or "group"? More importantly, do all the domains share the same automorphism group, or does each have its own? Does each domain have several orbit accumulation points, or just one? Clearer is the sentence

A domain with noncompact automorphism group has an orbit accumulation point in its boundary.

1.5.12 Big Words and Pretension

Avoid the use of big words when small ones will do. Do not say "peregrinate" for "walk," nor "omphaloskepsis" for "thought," nor "floccinaucinihilipilificate" for "trivialize" unless the longer word conveys some important nuance that the shorter word does not. The urge to so bloviate should be resisted. To indulge in hippopotomonstrosesquipedalian tergiversation is not to show your erudition;

rather, it is to be superficial. Also remember that many of your readers will be foreign born, not native English speakers. Make some effort to write simple, straightforward English that they will easily apprehend. Save your high-flown rodomontade for ceremonial occasions.

Likewise—and I have said this elsewhere in the book—stick to simple sentence structures. Even the subjunctive mood can lead to confusion when it is used in mathematical writing. An example of the subjunctive is

He acted as though he were in a daze.

Notice the choice of verb here. Usually mathematics is formulated in simple, declarative sentences. We do not often lapse into the subjunctive. But when we do so we should exercise special care that the meaning is clear. Generally speaking, let the mathematics speak for itself; do not try to dress it up with fancy language.

1.5.13 Foreign Words and Phrases

You can have some fun peppering your prose with *bon vivant* and *Gemütlichkeit* and *ad hominem* and *samizdat*, but the careless use of foreign words and phrases does not add anything to most writing. And it will confuse many readers. Use foreign phrases sparingly. If you do use them, typeset them in italics. (An exception should be made for foreign words like “pâté” and “etc.” (short for *et cetera*), which have become standard parts of the English language and should be set in Roman.) The books [Hig], [SG], [Swa] give more detailed treatments of this topic.

Good mathematics is difficult. Do not let your writing be a device for making it more so. Use simple, declarative sentences—short ones. Use short paragraphs, each with a simple point. To understand my meaning, put yourself in the position of the reader. You are slugging your way through a tough paper. You come to the proof of the main theorem. After killing yourself for a couple of hours, you finally come to the crux of the argument. And it is a single, dense paragraph spanning two pages. Such a daunting prospect is truly depressing. You do not want to abuse your readers in this fashion. Break up the ideas into palatable bites. Do *not* inject stumbling blocks like foreign or unfamiliar words.

1.5.14 Flippancy and Faddish Prose

And now a note on flippancy. A friend of mine once wrote a truly elegant—and important—book that included the phrase “the reader should review enough functional analysis so that he does not barf [*sic*] at the sight of a Banach or Frechet space.” At the reviewer’s insistence, the phrase was toned down before publication. Another friend published a book with the phrase “we leave the details of this proof for the mentally infirmed.” I would advise against this sort of sarcasm. This suggestion is not simply a nod to propriety. You want to be proud of your work. Remember that your thesis advisor and the authorities in the field are likely to look at it. Such puerile prose is not what you want them

to see. Most likely, ten years hence, you will wish fervently that you had not included such phrases. Anyone who continues to grow intellectually will look on his/her work of ten years ago with some disdain. But there is no percentage in adding embarrassment to the mix.

Suit your tone, and your choice of words, to the subject at hand. It might be suitable to use the phrase “He had all the efficiency and dexterity of a ruptured snail” to describe a clumsy waiter; this is probably not appropriate language for describing your thesis advisor.

Finally, stay away from faddish prose. If you say “fraternally affiliated, ethically challenged young male” to mean “gang member” or “peregrinating, fashion-challenged, pulchritudinally advanced hostess” to mean “prostitute,” then you may be politically correct today but you will be strictly out to lunch tomorrow. Today, many a writer or speaker wants to work the word “dis” (gang talk for “disrespect”) or “flame” (yuppie talk for “disrespect”) into his/her prose. This practice is a mistake, because in ten years the words will have no meaning.

Mathematical writing is serious writing. You do not want to be flippant, you do not want to use faddish language, you do not want to crack jokes, you do not want to be careless. You must show respect for your audience and respect for yourself.

By the same token, avoid old-fashioned modes of expression. In 1827 it was appropriate for a physician to diagnose a patient with “falling crud and palpitation of the pluck”; in 1930 it was fashionable for a woman to complain of “the fantods.” Today these phrases are meaningless. It might exhibit devotion to Fermat to use “adæquibantur” instead of “=” (as did he), but such a practice would lead to boundless confusion today.

Some American writers think that it is tony to pepper their writing with British English. They use “humour” for “humor,” “lorry” for “truck,” and “spanner” for “wrench.” Such language is out of place, and can only lead to obfuscation. It would be just as foolish for an American cookbook to give recipes for spotted dick, bubble-and-squeak, and stodge. Nobody would know what the author was talking about. Use your dictionary to check that you are using the appropriate American words and spellings.

For the same reasons I advise against using contractions, abbreviations, or slang—at least in formal writing. Even acronyms (abbreviations created by using the first letter of each word—such as AMS for American Mathematical Society) are dangerous (see Section 1.12); use them with caution. We write because we want our thoughts to last, and to be comprehensible both now and in years hence. Do not let language stand in the way of that goal.

1.5.15 Pronouns

When I was a child, I once asked a mathematician why mathematics was usually written in the first person plural: “We now prove this”; “Our next task is thus”; “We conclude our story as follows.” His rejoinder was “This is so that the reader will think that there are a lot of you.”

More seriously, when you are writing up mathematics, then you must make a choice about your expository voice. You can say “I will now prove Lemma 5” or “We will now prove Lemma 5” or “One may now turn one’s attention to Lemma 5.” Which is correct?

As with many choices in writing, this one involves a degree of subjectivity. Every usage varies according to context. The first option is rarely chosen. Most people consider it pompous and slightly disrespectful. The only instance where I find the first person singular to be a comfortable choice is the following: sometimes at the end of a paper one says “At this time I do not know how to prove Conjecture A.” The first person singular is appropriate for this particular statement because in fact the writer is imparting to the reader some specific information about his/her own lack of knowledge. Writing in this fashion, I envision myself speaking to you somewhat informally, as a teacher does to his/her students. It would be misleading, and a trifle affected, to say “At this time *one* does not know . . .” Likewise for “At this time *we* do not know . . .” The writer could perhaps say, “At this time it is not known whether . . .” This last choice could be misleading, however, as it suggests that the writer’s ignorance is shared by the world at large.

In this book I often use “you,” but such usage would be out of place in a formal mathematics paper. As usual, one has to consider the context and the audience.

The first person plural, or “we,” is generally your best choice. Unlike the first person singular, or “I,” which sounds elitist and irritating, “we” stresses the participatory nature of the enterprise, and encourages the readers to push on. Moreover, since “we” is what people are accustomed to hearing, it is less prone than one of the other choices to jar their ears, or to distract them. The use of third person singular, or “one,” often leaves both writer and reader struggling with awkward sentence structures. If you endeavor to write in that mode, then you will probably find yourself soon breathing a sigh of relief as you abandon it.

With a little craftsmanship, you can avoid entirely the use of the first person in your writing. Rather than say “We now turn to the proof of Lemma 4,” instead say “Next is the proof of Lemma 4” or perhaps “The next task is the proof of Lemma 4.” Rather than say “We see that the proof is complete,” say “The proof is now complete” or “This completes the proof.” The book [Dup, Ch. 2] has a sensible and compelling discussion of the question of “We” vs. “I” vs. “one.”

Your subject, your purpose, and your audience will usually point you towards which of the words “I,” “we,” or “one”—or perhaps none of these—you wish to use. I am offering “we” as the default. But the sense of what you are writing may dictate another choice.

1.5.16 The Role of English

And now a coda on the role of English in mathematical writing. More and more, English is becoming the language of choice in mathematics. Therefore those of us who are native speakers set the standard for those who are not. We should

exercise a bit of care. I have a good friend, also an excellent mathematician, who is widely admired; his fans like to emulate him. He is fond of saying (informally)

What you need here is to cook up a function f such that . . .

Mathematicians of foreign extraction, who have been hearing him make this statement for years, have now developed the habit of saying

Take a function f . Now cook it for a while . . . ✂

It is a bit like having your children emulate (poorly) all your bad habits. A word to the wise should suffice.

1.5.17 Acronyms

Do not use acronyms, abbreviations, or jargon unless you are dead certain that your audience knows these shortcuts. Speaking of an ICBM, the NAFTA treaty, ARVN, and MIRV is fine for those well read in the current events of the past twenty-five years—and who have an excellent memory to boot. But most of us need to be reminded of the meanings of these acronyms. The best custom is to define the acronym parenthetically the first time it is used in a piece of writing. For example,

The Strategic Arms Limitation Talks (SALT) were progressing poorly, so we broke for lunch. A few hours later, we resumed our efforts with SALT.

I have served on many AMS (American Mathematical Society) committees, and am somewhat horrified at the extent to which I have become inured to certain acronyms. How many of these do you know: CPUB, COPROF, JSTOR, LRPC, ECBT, COPE? I am conversant with them all, and none has done me a bit of good. In practice, you may not even safely assume that your reader knows what the AMS is—what if he/she is Turkish?

We must keep in mind that there are certain acronyms that most all mathematicians will know (AMS, NSF, NSA, for example) but nonmathematicians probably will not. As always, keep your audience in mind when you write.

1.5.18 Jargon

I was once at a meeting to discuss the writing of a new grant proposal—to apply for renewal of funds from a generous source which, we hoped, would be inclined to give again. One of the PIs (“PI” denotes “Principal Investigator”) said, in all seriousness, “I think that we are going to need more blue sky in this proposal if we want to generate more bottom line.” Of course his meaning was “We must endeavor to paint an enlarged picture of long-term goals and anticipated achievements if we want to increase the size of this grant.” The first mode of expression might be appropriate among venture capitalists, who are inured to such language. It is probably inappropriate among academics.

1.6 Proofreading, Reading for Sound, Reading for Sense

1.6.1 Proofreading

Proofreading is an essential part of the writing process. And it is not a trivial one. (You do not simply write the words and then quickly scan them to be sure that there are no gross errors.) Paul Halmos [Hig] said that he never published a word before he had read it six times. Not all of us are that careful, but the spirit of his practice is correct:

- One proofreading should be for *mathematical accuracy*. Are the theorems correctly stated? Do the proofs cohere? Are the definitions on point?
- One proofreading should be for *organization* and for *logic*.
- One proofreading should be for *sense*, and for the meaning, flow, and integrity of the ideas.
- One proofreading should be to check *spelling* and simple *syntax* errors (software can help with the former, and even with the latter—see Section 6.4).
- One proofreading should be for *sound*.
- One proofreading should be for overall coherence. Does the piece make sense? Does it convince?

It is a good idea, after proofreading your work several times, to put it away for 48 hours or more. Go for a run. Go see a play. Take your spouse out to dinner. Read a good book. Just do something to get your mind going on another track. The point is that you can get so absorbed in your work that you do not see it objectively anymore. You cannot effectively detect errors. You are more like a rubber stamp than a critic. Taking some time away can resharpen your focus and make you a more effective proofreader and critic.

The great English stage actor Laurence Olivier used to rehearse Shakespeare by striding across the countryside and delivering his lines to herds of bewildered cattle. Understandably, you may be disinclined to emulate this practice when developing your next paper on p -adic L functions—especially if you live in Brooklyn. However, note this: all the best writers whom I know read their work aloud to themselves. Reading your words aloud *forces* you to slow down, to hear each word and each sentence precisely, to better understand what you have written, and to deliver it as a coherent whole. If you have never tried this technique, then your first experience with it will be a revelation. You will find that you quickly develop a new sensitivity for sound and sense in your writing. You will develop an “ear.” You will learn instinctively what works and what does not.

1.6.2 Writing with Good Sense

Consider the statement

The conjecture of Gauss (1830) is false. ✕

Contrast this rather bald assertion with

The lemmas of Euler (1766) and the example of Abel (1827) led Gauss to conjecture (1830) that all semistable curves are modular. The conjecture was widely believed, and more than fifty papers were written by Jacobi, Dirichlet, and Galois in support of it. To everyone's surprise and dismay, a counterexample was produced by Frobenius in 1902. This counterexample opened many doors.

This second passage puts the entire matter in context, tells the reader who worked on the conjecture and why, and also how the matter was finally resolved. Although written mathematics is traditionally terse, at least consider in your own writing the advantage of telling the reader what is going on.

You will become accustomed to what mathematical writing should look like by doing a lot of mathematical reading. But you really have to *think* about what you are seeing in these books and papers. What is the form of the sentences? What is the form of the paragraphs? What words are most commonly used? What phraseology is most commonly used? How long is a typical sentence? How long is a typical paragraph?

A math book does not read like a mystery novel, nor like a restaurant menu, nor like a religious tract. What makes it different? It is *not* just the notation. It is the *turns of phrase*, it is the *form* of the writing, it is the *organization*. It is the *logic*. Mathematical writing is forceful and focused. It is *not* tentative. It says what needs to be said directly and plainly with simple, incisive sentences. It proceeds step-by-step. It uses argumentation skillfully and accurately. It is scholarly and compelling.

1.7 Compound Sentences, Passive Voice

1.7.1 Overly Complex Sentences

It would be splendid if we could all write with the artistry of Flaubert, the elegance of Shakespeare, and the wisdom of Goethe. In mathematical writing, however, such an abundance of talent is neither necessary nor called for. In developing an intuitionistic ethics ([Moo]), for example, one presents the ideas as part of a ritualistic dance: there is a certain intellectual pageantry that comes with the territory. In mathematics, by contrast, what is needed is a clear and orderly presentation of the ideas.

Mathematics is already, by its nature, logically complex and subtle. The sentences that link the mathematics are usually most effective when they are simple, declarative sentences. Compound sentences (two or more independent

clauses joined by a coordinating conjunction such as “and” or independent clauses joined to a dependent clause by a subordinating conjunction such as “although”) should be broken up into simple sentences. Avoid run-on sentences at all cost. Here are some examples:

Rather than saying

As x tends ever closer to x_0 then $f(x)$ approaches $f(x_0)$, from which we see that f is continuous and in fact one can use this argument to see that f is uniformly continuous. ✘

instead say

As $x \rightarrow x_0$ we see that $f(x) \rightarrow f(x_0)$. Thus f is continuous. The same reasoning shows that f is uniformly continuous.

Of course mathematical notation allows us to write $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ instead of either of these first phrases; this abbreviated presentation will, in many contexts, be more desirable.

Rather than saying

If g is positive, f is continuous, the domain of f is open, and we further invoke Lemma 2.3.6, then the set of points at which $f \cdot g$ is differentiable is a set of the second category, provided that the space of definition of f is metrizable and separable. ✘

instead say

Let X be a separable metric space. Let f be a continuous function that is defined on an open subset of X . Suppose that g is any positive function. Using Lemma 2.3.6, we see that the set of points at which $f \cdot g$ is differentiable is of second category.

An alternative formulation, even clearer, is this:

Let X be a separable metric space. Let f be a continuous function that is defined on an open subset U of X . Suppose that g is any positive function on U . Define $S \subset U$ to be the set of points x such that $f \cdot g$ is differentiable at x . Then, by Lemma 2.3.6, S is of second category in U .

Note the use of the words “suppose” and “define” to break up the monotony. Observe how the formal definition of the set S clarifies the slightly awkward construction in the penultimate version of our statement.

1.7.2 Passive Voice

Most authorities believe that writing in the passive voice is less effective than writing in the active voice. To write in the active voice is to identify the agent of the action, and to emphasize that agent acting on the subject (see [Dup] for a powerful discussion of active voice vs. passive voice). In the passive voice, the subject is acted upon. In plain English,

The dog ate the cat.

is active voice. By contrast,

The cat was eaten by the dog.

is passive voice.

For a mathematical example, consider

The manifold M is acted upon by the Lie group G as follows: ✖

is less direct, and requires more words, than

The Lie group G acts on the manifold M as follows:

Likewise, the statement

It follows that the set Z will have no element of the set Y lying in it. ✖

can be more clearly expressed as

Therefore no element of Y lies in Z .

Even better is

The sets Y and Z are disjoint.

or

Therefore $Y \cap Z = \emptyset$.

Notice that the last version of the statement used 1 word, while the first version used 17. Also, a mathematician much more readily apprehends $Y \cap Z = \emptyset$ than he/she does a string of verbiage. Finally, coming up with the succinct fourth formulation required not only restating the proposition, but also thinking about its meaning. The result was plainly worth the effort.

In spite of these examples, and my warnings against passive voice, I must admit that passive voice gives us certain latitude that we do not want to forfeit. If, in the first example, you have reason to stress the role of the manifold M over the Lie group G , then you may wish to use passive voice. In the second example, it is unclear how the use of passive voice could add a useful nuance to your thoughts.

1.8 Technical Aspects of Writing a Paper

1.8.1 Details of Your Draft

Even when your paper is in draft form, your name should be on it. A date is helpful as well. Number the pages. Write on only one side of the paper. Give the paper a working title.²

Is all this just too compulsive? No.

First, you must always put your name on your work to identify it as your own. If it contains a good idea, then you do not want someone else to walk off with it. Because you tend to generate so many different drafts and versions of the things that you write, you should date your work. I have even known mathematicians who put a time of day on each draft. (Of course a computer puts a date and time stamp on each *computer file* automatically; here I am discussing hard copy or paper drafts.)

Your academic affiliation should appear—even on the draft. If you are usually at Harvard, then write that down. If instead you are spending the year in Princeton, write that down. The draft could, at some point, be circulated. People need to know where to find you. With this notion in mind, include your current email address.

1.8.2 Numbering Systems

Take a few moments to think about the numbering of theorems, definitions, and so forth. This task is important both in writing a paper and in writing a book. Some authors number their theorems from 1 to n , their definitions from 1 to k , their lemmas from 1 to p , their corollaries from 1 to r —each item having its own numbering system. Do not laugh: this describes the default system in L^AT_EX. As a reader, I find this method maddening; for the upshot is that I can never find anything. For instance, if I am on the page that contains Lemma 1.6, then that gives me no clue about where to find Theorem 1.5. If, instead, all displayed items are numbered in sequence—Theorem 1.2 followed by Corollary 1.3 followed by Definition 1.4, etc.—then I always know where I am.

Having decided on the logic of your numbering system, you also need to decide how much information you want each number to contain. What does this mean? My favorite numbering system (in writing a book) is to let “ $\langle\langle Item \rangle\rangle$ 3.6.4” denote the fourth enunciated item (theorem or corollary or lemma or definition) in the sixth section of Chapter 3. If there is a labeled, displayed equation in the statement of the $\langle\langle Item \rangle\rangle$, then I label it (3.6.4.1). The good feature of this system is that the reader always knows precisely where he/she is, and can find anything easily. The bad feature is that the numbering system is a bit cumbersome. Other authors prefer to number displayed items within each section. Thus, in Section 6 of Chapter 3 the displayed items are numbered simply 1, 2, 3, When reference is later made to a theorem, the reference

²Some mathematicians do their composition directly on the computer. They never pick up a pen or pencil. But the basic principles and admonitions presented here still apply.

is phrased as “by Theorem 4 in Section 6 of Chapter 3” or “by Theorem 4 of Section 3.6.” As you can see, this ostensibly simpler numbering system is cumbersome in its own fashion.

The main point is that you want to choose a numbering system that suits your purposes, and then to use it consistently. You want to make your book or paper as easy as possible for your reader to study and navigate. Achieving this end requires that you attend to many small details. Your numbering system is one of the most important of these.

A final point is this: do not number every single thing in your manuscript. This dictum applies whether you are writing a paper or a book. I have seen mathematical writing in which every single paragraph is numbered. Such a device certainly makes navigation easy. But it is cumbersome beyond belief. Likewise, do not number all formulas. You will only be referring to some of them, and the reader knows that. To number all formulas will create confusion in the reader’s mind; he/she will no longer be able to discern what is truly important and what is less so.

1.8.3 Use of Paper and Ink

As I have already mentioned, when writing your draft (by hand), write on one side of the paper only. If you do not, and if you are writing something fairly technical and complicated (like mathematics), then you can become hopelessly confused when trying to find your place. In addition, you will find that you must frequently set two pages side by side—for the sake of comparing formulas, for instance. This move is easy with a manuscript written on one side, and nearly impossible with one written on two sides.

If you are scrupulous about not wasting paper, and insist on using both sides, then my advice is this: write drafts of your mathematical papers on one side of fresh paper. When that work is typed up and out the door, boldly *X*-out the writing on the front side of each page of your old drafts. Turn the paper over, and use it as scratch paper, or for your laundry list, or perhaps for the first draft of your next paper.

I suggest writing in ink. Pencil can smear, and erasing can tear the page, and it is difficult to read a palimpsest. Also pencil-written material does not photocopy well. Blue pens do not photocopy well either. I always write with a black pen on either white or yellow paper. I write either with a fountain pen or a rolling writer or a fiber-tip pen so that the pen strokes are *dense* and *sharp* and *dark*. I write with a pen that does not skip or blot. If it begins to do either, I immediately discard it and grab a new one.

Of course you cannot erase words that are written with a pen; but you can cross them out, and such a practice is much cleaner. It is easier to read a page written in bold black ink, and which includes some crossed out passages, than to decipher a page of chicken scratch layered over erased smears written with a pencil or written with a pen that is not working properly.

Be sure that your desk is well stocked with paper, pens, Wite-Out[®], Post-it

notes[®], a stapler, staples, a staple remover, cellophane tape, paper clips, manila folders, manila envelopes, scissors, a dictionary, and anything else you may need for writing. Have them all at your fingertips. You do not want to interrupt the precious writing process by running around and looking for something trivial.

Do not write much on each page. I advise writing *large*, and double- or triple-spaced. The reason? First, you want to be able to insert passages, make editorial remarks, make corrections, and so forth. Second, a page full of cramped writing on every line is hard to read. Third, you can more easily rearrange material if there is just a little on each page. For example, if one page contains the statement of the main theorem and nothing else, another contains key definitions and nothing else, and so forth, then you can easily change the location of the main theorem in the body of the paper. If the main theorem is buried in a page with a great deal of other material, then moving it would involve either copying, or photocopying, or cutting with scissors.³ I read recently of a famous novelist who writes his books on 3" × 5" index cards. Really! This way he has about one sentence on each card, and it is easy to move them around.

Do not hesitate to use colored pens. For instance, you could be writing text in black ink, making remarks and notes to yourself (like "find this reference" or "fill in this gap") in red ink, and marking unusual characters in green ink. This may sound compulsive, but it makes the editing process much easier.

1.8.4 Bibliographic References

A good bibliography is an important component of scholarly work (more on bibliographies can be found in Sections 2.6, 5.5). Suppose that you are writing a paper with a modest number of references (about 25, say), and you are assigning an acronym to each one. For instance, [GH] could refer to the famous book by Griffiths and Harris. When you refer to this work while you are writing, use the acronym. Keep a sheet of notes to remind yourself what each acronym denotes. Do not worry about looking up the detailed bibliographic reference while you are engaged in writing; instead, compartmentalize the procedure. When you are finished writing the paper, you will have a complete, *informal* list of all your references. You can go to MathSciNet (Section 7.2) online and find most of your references in an instant. You can also go to your library's catalog online to find locally obtainable references. L^AT_EX can be a great help in eliminating much of the tedium of assembling and formatting bibliographies. See the discussion in Sections 2.6 and 5.5.

1.8.5 The Writing Process

Finally, let me make a few general remarks about the writing process. As you are writing a paper, there will be several junctures at which you feel that you need to look something up: either you cannot remember a theorem, or you have

³Of course, if you are writing on a computer, then all cutting and pasting and moving of passages is trivial. You can have several windows open at once, and can move from one part of the document to another with ease.

lost a formula, or you need to imitate someone else's proof. My advice is *not* to interrupt yourself while you are writing. Take your red pen and make a note to yourself about what is needed. But *keep writing*. When you are in the mood to write, you should take advantage of the moment and do just that. Interrupting yourself to run to the library, or for any other reason, is a mistake.

Write on a desk free of clutter. It is romantic, to be sure, to watch a film in which the writer labors furiously on a desk awash with papers, books, hamburger bags, ice cream containers, old coffee cups, last week's underwear, and who knows what else. Leave that stuff to the movies. Instead imagine tearing into page 33 of your manuscript and accidentally spilling a week-old cup of coffee and a piece of pepperoni pizza all over your project. Think of the time lost in mopping up the mess, separating the pages, trying to read what you wrote, recopying your pages, and so forth. Enough said.

If you are going to drink coffee or a soda or eat a sandwich while you work, I suggest having the food on a small separate side table. This little inconvenience will force you to be careful, and if you do have an accident then it will not make a mess of your work.

Write in a place where you can concentrate without interruption. Whether you have music going, or a white noise machine playing, or a strobe light flashing is your decision. But if you are going to concentrate on your mathematics, it may take up to an hour to get the wheels turning, to fill your head with all the ideas you need, and to start formulating the necessary assertions. After you have invested the necessary time to tool up, you want to use it effectively. Therefore you do not want to be interrupted. Close the door and unplug the telephone if you must. Victor Hugo used to remove all his clothes and have his servant lock him in a room with nothing but paper and a pen. Moreover, the servant guarded the door so that the great man would not be interrupted by so much as a knock. This method is not very practical, and is perhaps not well suited to modern living, but it is definitely in the right spirit.

1.9 More Details of Mathematical Writing

1.9.1 Effective Sentence Structure

For the most part, the writing of mathematics is like the writing of English prose. Indeed, it *is* a part of the writing of English. (*Caveat*: I hope that my remarks have some universality, and apply even if you are writing mathematics in Tagalog or Coptic or Lingit.) If you read your work aloud (I advocate this practice in Section 1.5), then you should be reading complete sentences that flow from one to the next, just as they do in good prose.

It is all too easy to write a passage like

Look at this here equation:

$$x^n + y^n = z^n. \quad \times$$

If you read your work aloud to yourself—being sensitive both to large and small issues—then you will catch a blunder like this one immediately. Much smoother is the passage

The equation

$$x^n + y^n = z^n$$

tells us that Fermat’s Last Theorem is still alive.

Another example of good sentence structure is

Since

$$A < B,$$

we know that

Notice that the sentence (fragment) reads well aloud: “Since A is less than B we know that” The phrase is compact and straightforward and gets its message across directly.

1.9.2 Overuse of Commas

Do not overuse commas. I become distressed when I see a sentence like

We went to the store, to buy some potatoes. ❖

This sentence requires no punctuation but the period at the end. Slightly more subtle, but still irksome, is

Now that we have our hypotheses in place, we state our theorem, with the point in mind, that we wish to understand the continuity, of functions in the class \mathcal{S} . ❖

We certainly use a comma to indicate a pause. But the comma indicates a *logical pause*, not a lack of air or lack of good sense. Read the last displayed sentence out loud, with suitable pauses where the commas occur. It sounds like someone huffing and puffing; the pauses have no reason to them. This sentence is not a representative example of the way that we speak, hence it is not indicative of the way that we should write. Much more attractive is

Our hypotheses are now in place, and we next state our theorem. The point is to understand the continuity properties of functions belonging to the class \mathcal{S} .

While there is no universal agreement on the placement of commas, there are logical guidelines that one should follow.

1.9.3 Miscellaneous Stipulations

Mathematicians like the word “given.” We tend to overuse and misuse it—especially in instances where the word can be discarded entirely. Consider the example “Given a metric space X , and a point $p \in X$, we see that” More direct is “If X is a metric space and $p \in X$, then” We are often tempted to transcribe spoken language and call that written language; such laziness should be defeated. Our misuse of “given” is an example of such sloth.

When you are putting the final polish on a manuscript, look it over for general appearance. In mathematical writing, several consecutive pages of dense prose are not appealing, nor are several consecutive pages of tedious calculation. For ease of reading, the two types of mathematical writing should be interwoven. It requires only a small extra effort to produce a paper or book with comfortable stopping places on every page. The reader needs to take frequent breathers, to survey what he/she has read, to pause and look back. Make it easy for him/her to do so.

While you are thinking about the counterpoint between prose and formulas, think also about the use of displayed math versus in-text math (in \TeX (see Section 6.5), the former is set off by double dollar signs $\$$ while the latter is set off with single dollar signs). Long formulas are usually better displayed, for they are difficult to read when put in text. Of course *important* formulas should be displayed no matter what their length—and provided with numbers or labels if they will be mentioned later. Do not display every single formula, for that will make your paper a cumbersome read. Also do not put every formula in text, as that will make your writing tedious. A little thought will help you to strike a balance, and to use the two formats to good effect.

1.10 Essential Rules of Grammar, Syntax, and Usage

1.10.1 Introduction

I have intentionally put this discussion of the rules of grammar, syntax, and usage at the end of Chapter 1. The reasons are several. I want, in a gentle way, to de-emphasize them. I do not, however, wish to trivialize them. I am not one of those who says “the battle against ‘hopefully’ is lost,” “the battle for ‘which’ vs. ‘that’ is lost,” “the battle for ‘lay’ vs. ‘lie’ is lost,” and so forth. I find such statements facile, and they miss the point that careful writing requires some precision. The argument “You know what I mean; whether I use ‘that’ or ‘which’ is incidental” abrogates the fact that accurate writing, and accurate expression of your thoughts, requires accurate use of language. It takes some linguistic skill to recognize that minuscule errors in usage can change or obfuscate your intended meaning.

The intent of this book is that you should learn to write logically and cogently; to say precisely what you mean, using the right words and the right

number of words; to eschew obfuscation. You want to develop an ear, so that clear writing becomes natural. To be sure, exact use of the language is not the primary goal of most mathematicians; but it must certainly be a secondary goal, absolutely crucial to the successful dissemination of your ideas.

Fortunately, most of the rules of English usage are succinct and logical. A particularly concise enunciation of the basic rules appears in [SW]. Since I cannot improve on that presentation, I certainly shall not repeat all of its insights. Here I shall mention just a few sticky points that come up frequently in the writing process. I hope that you will find this section and the next to be a useful “quick-and-dirty” reference. With that goal in mind, I have presented the topics in alphabetical order. See also [Chi], [Dup], [Fow], [Fra], [Hig], and [MW] for a more thorough treatment of issues of grammar, syntax, and usage.

Bear in mind, as you read these precepts, that no rule of English grammar is etched in stone. There will certainly be times that a sentence or phrase formed according to the strictest rules will sound just awful. In such an instance, you must override the rules and use your good sense and taste. More will be said about this technique as the book develops.

1.10.2 Rules of Grammar and Syntax

Now for some rules:

- **All, Any, Each, Every** In mathematics we commonly formulate statements such as “Show that any continuous function f on the interval $[0, 1]$ has a point M in its domain such that $f(M) \geq f(x)$ for $x \in [0, 1]$.” For cognoscenti it is clear that, when we say “any” here, we mean “all.” But for others—for students, or for non-native speakers—this slight abuse of language could cause confusion. For example, a student reading this sentence could (perfectly correctly) construe it to mean “Demonstrate that for *some* function $f \dots$ ” Thus, if this sentence were part of an exercise, the student might answer

The function $f(x) = -(x - 1/2)^2$ is continuous on $[0, 1]$ and the point $M = 1/2$ satisfies the conclusion. ✘

The lesson? Avoid using “any” when “all” or “each” or “every” is intended.

Conversely, even when you are writing for experts you can cause confusion by misusing quantifiers. Avoid using “all” when “every” or “each” is intended. Experts themselves can be confused by far-too-common sentences like

All continuous functions have a maximum. ✘

Notice that the sentence suggests that all continuous functions share the *same* maximum. Of course what was intended was

Every continuous function has a maximum.

or, more precisely,

Each continuous function has a maximum.

(Once again we see the advantage, from the point of view of clarity, of the singular over the plural.) As you proofread your work, you must learn to take the part of the reader (who is not *a priori* sure of what is being said) in order to weed out misused quantifiers.

- **Brevity** Endeavor to formulate your thoughts briefly and succinctly. For example, you *could* say

In point of fact, we devolved upon the decision to solicit opinions, form an enumeration, and produce a tally. ✖

Such a sentence sounds mellifluous, sanguine, and high toned. But why not instead say

We decided to take a vote.

The second sentence says in 6 words what the first said in 19; and it presents the message more clearly and forcefully. Strunk and White [SW] give a thorough and engaging treatment of the topic of brevity, and they speak particularly cogently of eliminating extra or extraneous words. Mathematics is difficult to read under the best of circumstances. Do not make the reader's job even more difficult by weighing down your prose with excess baggage.

I once saw a sign in the elevator of a Washington, D.C., hotel that said

Do not carry lighted tobacco products in the elevator. ✖

I can only suppose that some politician created this sign. Why not just say

No smoking.

Being concise and to-the-point is not simply a pose. It is essential to good writing and effective communication. You do not want to omit important details, but you also want your text to be of a "high information" nature. You should think of your reader as a quite impatient person who will be easily turned off by a wordy, vague, expendable sentence or a paragraph with no useful information. You should ask yourself whether each sentence is worth the space that it occupies. Does it really say anything? Are we better off without it?

- **British Spelling vs. American spelling** Many readers of this book will be American, and will be inured to American spelling. But, in an effort to be tony, we are sometimes tempted to write “armour” instead of “armor,” “aluminium” instead of “aluminum,” and “centre” instead of “center.” Please resist. There is no place for British spelling in an American document (and vice versa). It adds nothing, and can only cause confusion.
- **Comprise vs. Compose** People use the word “comprise” because they think it makes them sound tony. Unfortunately, because most everyone misuses the word, they instead sound uneducated. The correct use of the word “comprise” is

The standing committee comprises two women, three men, and a donkey.

The formula is “*A* comprises *B*.” What people often say, or write, instead is

The committee is comprised of two women, three men, and a donkey. ❌

What *should* have been used in this last instance is “composed,” *not* “comprised.” Never say “is comprised of.”

- **Contractions** Do *not* use contractions in formal writing. Thus the words “don’t,” “can’t,” “shouldn’t,” “I’m,” “you’re,” etc., are taboo. Of course you should never write “ain’t.” You also should avoid abbreviations. Particularly avoid using informal abbreviations like “cuz” for “because,” “tho” for “though,” and so forth. You will probably never be tempted to work “bar-b-q” into your next paper on para-differential operators, but you might be tempted to use “rite inverse.” Please resist.

More generally, do not use colloquial language. This is confusing for foreigners (and you will have *many* foreign readers) and annoying for native speakers. It is also a good idea to avoid words like “get.” Strictly speaking, “get” is not colloquial. But it is an overused word that has too many meanings. There are so many other words that communicate the idea more precisely.

Occasionally you will find it suitable to use contractions in various kinds of *informal* writing. It can be a way of drawing in your audience, or of warming yourself up to your subject. For example, in the book [Kr2], I intentionally used an occasional contraction in an effort to create a friendly air about the book. By contrast, the present book is a book about writing, and I wish to set a more formal example—so there are no contractions.

- **Denote** Use the word “denote” carefully. It has a special purpose in mathematics (and in logical positivist philosophy and modal logic); we should take care to preserve its usefulness. Suppose that a certain mathematical symbol A stands for, or represents, the item or set of ideas B (ideally, you should be able to excise any occurrence of A and replace it with B and preserve exactly the intended meaning). Under these circumstances, and *only under these circumstances*, do we say that “ A denotes B .” For example,

Let X denote the set of all semisimple homonoids with stable quonset hut.

If the above discussion seems obvious, then consider the following shade of difference:

(1) Let f be a continuous function.

and

(2) Let f denote a continuous function. ✕

The intended meaning of the first sentence here is “let f be *any* continuous function.” Thus the first statement is both customary and correct. The second is neither customary *nor* correct. For we use “denote” when we want to say that a certain specific item stands for some other specific item. This is not what we are trying to say here.

Lack of familiarity with English, or lack of familiarity with the precise meaning of “denote,” sometimes leads to dreadful abuses of the word. A common one is “Denote X the set of all left-handed polyglots.” I leave it to you to decide whether failing English or failing intellect might be the correct provenance of such a sentence; the lesson for you is not to use “denote” in such a fashion.

The word “connote,” rarely used in mathematical writing, can be (but should not be) confused with “denote.” The dictionary teaches us that “ A connotes B ” means that A *suggests* B , but not in a logically direct fashion. For example,

To a young man, “love” connotes flowers, beautiful music, and happiness.

is an appropriate use of the word “connote.”

- **Enervate** Often we are lazy, and we use a word according to how it sounds, rather than according to what it actually means. This text offers “enervate” as an instance of this phenomenon. What the word actually

means is “to lessen the vitality or strength of.” But, intuitively, we confuse “enervate” with “energize” and give it essentially the opposite meaning. The lesson is to be careful with words with which you are unfamiliar.

- ***He and she*** It used to be the custom that, if one referred to an abstract person in one’s writing, then one used the male pronouns “he” or “him” or “his.” These were routinely used in scholarly writing. Now this practice is considered to be politically incorrect. One must treat women the same as men.

One solution to this problem is to replace “he” with “he/she,” “him” with “him/her,” and “his” with “his/her.” But this practice is a bit clumsy. Another possibility is to replace “he” with “she,” replace “him” with “her,” and replace “his” with “her.” This does not really seem to solve the problem; instead it replaces one conundrum with another. A third possibility, commonly taught at colleges and universities, is to replace the gender-specific pronouns with “they,” “them,” and “their.” This unfortunately results in some rather awkward constructions. A fourth possibility is to preclude all offense by using the words dreamed up by Michael Spivak [SPI]. Spivak replaces “he” and “she” with “e,” replaces “him” and “her” with “em,” and replaces “his” and “her” with “eir.”

The really best approach, though it requires some extra time and effort, is to phrase your sentences so that they omit pronouns altogether. As an example, instead of saying

Ask her whether she wants a new computer.

one could say

Ask whether a new computer is desired.

- ***Hyphen vs. en dash*** It is common in mathematics, if two mathematicians have proved a result together, to call it something like “the Riemann-Lebesgue lemma.” Nowadays this is considered to be inappropriate. The use of the hyphen here may suggest that Riemann and Lebesgue have more than a professional relationship. More grammatically correct in today’s climate is to write “the Riemann–Lebesgue lemma.” What is the difference? In the second example I used the so-called *en dash* rather than the hyphen. The en dash is a dash which is about the width of the letter “n” in the current font, and it typically indicates spans or differentiation, where it may be considered to replace “and” or “to.” For instance, the en dash can be used to denote a range of numbers (as in “pages 324–386”). It is also used in phrases like “the U.S.–Canada border.” It carries less emotional baggage and is therefore a better choice to denote a mathematical collaboration.

You may think this discussion ludicrous, but I can tell you that, if you do not conform to the prescription described in the last paragraph, then your copy editor will change all your hyphens to en dashes.

It may be worth mentioning, and Knuth emphasizes this point in the $\text{T}_{\text{E}}\text{X}$ Book, that there are three types of dashes in typesetting. The hyphen, rendered in $\text{T}_{\text{E}}\text{X}$ with a single $-$, is a punctuation mark used to join words and to separate syllables of a single word. We might say “semi-continuous” or “egg-beater” or “two-thirds majority.” And, as we all know, hyphens are used to indicate a broken word at the end of a line of type. The en dash (typically the width of the letter “n” in the current font), typeset in $\text{T}_{\text{E}}\text{X}$ as $--$, is used to indicate a span or differentiation. The em dash (typically the width of the letter “m” in the current font), typeset in $\text{T}_{\text{E}}\text{X}$ as $---$, denotes a break in a sentence or a parenthetical remark. There is also the minus sign—often longer than an em dash—but that is part of mathematics and not of English.

- ***If . . . Then*** The most important logical syllogism for the mathematician is *modus ponendo ponens*, or “if . . . then.” If you begin a sentence with the word “If,” then do not forget to include the word “then.” Consider this example:

If $x > 4$, $y < 2$, the circle has radius at least 6, the sky is blue,
the circle can be squared. ✕

Which part of this sentence is the hypothesis and which the conclusion? After a few readings you may be able to figure it out. If it were sensible mathematics, then the mathematical meaning would probably give you some clues. But it is clearer to write

If $x > 4$, $y < 2$, the circle has radius at least 6, and the sky is
blue, then the circle can be squared.

Following the dictum that shorter sentences are frequently preferable to longer ones, you can express the preceding thought even more succinctly as

Suppose that $x > 4$, $y < 2$, the circle has radius at least 6, and
the sky is blue. Then the circle can be squared.

The word “Then” is pivotal to the logical structure here. It acts both as a connective and as a sign post. And never doubt that such linguistic clues are absolutely necessary. For although the reader can (often) figure out what is meant if the word “then” is omitted, he/she should not *have* to do so. Your job as the writer is to perform this task *for* the reader, to

make the reader's job easier. You want your audience to concentrate on the beauty of your mathematics, not on the ambiguities of your prose.

Mathematicians have a tendency to want to jam everything into one sentence. However, as the last example illustrates, greater clarity can often be achieved by breaking things up; this device also forces you to think more clearly and to organize your thoughts more effectively.

Mathematicians commonly write "If f is a continuous function, then prove X ." A moment's thought shows that this is not the intended meaning: the desire to prove X is not contingent on the continuity of f . What is intended is "Prove that, if f is a continuous function, then X ." In other words, the hypothesis about f is part of what needs to be proved.

The phrase "if and only if" is a useful mathematical device. It indicates logical equivalence of the two phrases that it connects. While the phrase is surely used in some other disciplines, it plays a special role in mathematical writing; we should take some care to treat it with deference. Some people choose to write it as "if, and only if,"—with two commas. Such a practice is perfectly kosher, if a little stilted. One unacceptable habit (because it sounds artificial and is difficult to read) is beginning a sentence with this phrase. For instance,

If and only if x is nonnegative, can we be sure that the real number x has a real square root. ✖

What a painful sentence to read, whether the reading is done aloud or *sotto voce*. Better is

A real number x has a real square root if and only if $x \geq 0$.

An alternative form, not with universal appeal (but better than beginning a sentence with "if and only if"), is

Nonnegative real numbers, and only those, have real square roots.

It is a fact that many definitions ought to be formulated using "if and only if." For instance,

Definition: The function f is *continuous* at the point P in its domain if and only if $\lim_{x \rightarrow P} f(x) = f(P)$.

But we often, either out of habit or out of laziness, write "if" instead of "if and only if." It is too bad, because this can confuse neophytes and non-English speakers. Strive to be careful about this matter.

Incidentally, the neologism “iff,” reputed to have been popularized by Paul Halmos, is a generally accepted abbreviation for “if and only if.” It provides a useful bridge between the formality of “if and only if” and the convenience of “if.” It is also common to use the symbol \iff for “if and only if.”

- **If vs. Whether** The words “if” and “whether” have different meanings, and are suitable for different contexts. Follow the example of master editor George Piranian:

Go to the window and see *whether* it is raining; *if* it is raining,
then let Fido inside.

- **Infer and Imply** The words “infer” and “imply” are often confused in everyday usage. It should not be difficult for a mathematician to keep these straight. A set of assumptions can *imply* a conclusion. But one *infers* the conclusion from the assumptions. It is that simple.
- **Its and It’s** Use “it’s” only to denote the contraction for “it is.” Otherwise use “its.” For example “Give the class its exam” and “A place for everything and everything in its place.” Compare with “It’s a great day for singing the blues.”

More generally, the apostrophe is never used to denote the possessive of a pronoun: the correct forms are “its,” “hers,” “his,” and “theirs.”

- **Latin Abbreviations** By these we mean

cf., e.g., i.e., n.b., q.v.

and the like. These are abbreviations for specific Latin expressions: *confer* (compare), *exempli gratia* (for example), *id est* (that is), *nota bene* (note well), *quod vide* (which see). They have particular meanings, and you should strive to use them accurately. In particular, “cf.” is often misused to mean “see.” It actually means “compare.” Sometimes “e.g.” and “i.e.” are interchanged in error; the first of these means “for example,” and the second means (literally) “the favor of an example” or (more familiarly) “for the sake of example.” It is difficult to use “n.b.” with grace. If you are unsure, then use the English equivalent of which you *are* sure.

In fact it is difficult to make a compelling case for “i.e.” in favor of “that is,” or for any of the other Latin substitutes in favor of their English equivalents. The punctuation and font selection questions connected with these Latin abbreviations are tricky (see [Hig] or [Fow] or [Chi] or [SaK]). For instance, many people do not know that you are always supposed to put a comma after “i.e.” or “e.g.” To repeat, use these items with care.

- **Lay and Lie** “Lay” is a transitive verb and “lie” is intransitive. This means that “lay” is an action that you perform on some object, while “lie” is not. For instance, “Lay down your weary head,” “Now I will lay down the law,” and “I shall lay responsibility for this transgression at your feet”; compare with “I am tired and I shall lie down” or “Let sleeping dogs lie.” Note, however, that the past tense of “lie” is “lay.” Therefore you may say “Yesterday I was so tired that I laid down my books and then I lay down.”
- **Less and Fewer** How many times have you been in the grocery store and gravitated toward the line labeled *Ten Items or Less*? Of course what is intended here is *Ten Items or Fewer*, and I have a special place in my heart for those few grocery stores that get it right. The word “fewer” is for comparing two numbers while “less” is for comparing quantity.⁴ In mathematics we certainly say “3 is less than 5,” and we do so because the discrete set \mathbb{Z}^+ is embedded in the continuum \mathbb{R}^+ . You would never say “I have fewer milk than you do,” just because milk is a continuum.

You should also be wary of “smaller,” which designates not number but size. Avoid saying “3 is smaller than 5,” because “smaller” is a word about *size*: perhaps the digit 3 is smaller than the digit 5. It *could* also be correct to say “‘5’ is smaller than ‘3’” if comparison of digit size is what you intended: 5 versus 3.⁵

- **Lists Separated with Commas (the Serial Comma)** When you are presenting a list, separated with commas, then you should put a comma after every item in the list except the last. For example, say “the good, the bad, and the ugly” rather than “the good, the bad and the ugly.” A moment’s thought reveals that the former conveys the intended meaning of these distinct individuals; the latter may not, for the reader could infer that “bad” and “ugly” are simply two complementary descriptions of the same individual.
- **Numbers** Some sources will tell you that (whole) numbers less than 101 should be written out in words; larger numbers should be expressed in numerals. (Other sources will put the cutoff at twenty or some other arbitrary juncture.) A detailed discussion appears in [SG]. Such considerations are, for a mathematician, next to nonsensical. The main thing, and this advice applies to spelling and to many other *choices*, is to select a standard and to be consistent.
- **Obviously, Clearly, Trivially** These words have become part of standard mathematical jargon. This is too bad. In the best of circumstances,

⁴Another way to think about the matter is that “fewer” is used to compare discrete sets while “less” is used to compare continua.

⁵Note the extra single quotation marks to tell the reader that we are talking about the *digits* rather than the numbers.

when one uses these phrases, he/she is endeavoring to push the reader around. In the worst of circumstances, he/she is throwing up a smoke screen for something that he/she himself/herself has not thought through. It would be embarrassing to count the number of major published mathematical errors that have been prefaced with “Obviously” or “Clearly” or “Trivially.” (No doubt the supreme deity’s way of reminding us that “Pride goeth before the fall.”) The indiscriminant use of these words is one of the ways that we have of kidding ourselves.⁶

As you proofread your manuscript relentlessly, and endeavor to weed out superfluous words, pay particular attention to the use, abuse, and overuse of these trite words. They add nothing to what you are saying, and are frequently a cover-up.

- **Overused Words** Many other words in the English language are also grossly overused. Among these are “very” and “most” and “nice” and “interesting.” It is certainly very pleasant and most insightful to express great appreciation for a very nice and supremely interesting theorem; but I encourage you not to do so—at least not with these banal words. If such language represents how you wish to express yourself, then perhaps you have nothing to say. Instead think carefully about the real substance of what you are endeavoring to convey, and then find the substantive vocabulary to express it.

Be aware that the language is littered with overused expressions that come into and out of fashion. The words “awesome,” “totally,” “dude,” and “righteous” are current examples. The phrase “today I’m not 100%,” foisted upon us by some semiliterate sports announcer, is currently the bane of our collective existence. Each field of mathematics has its own set of stock phrases and tiresome clichés. Endeavor not to propagate them.

A good general principle is to put every word in every sentence under the microscope: What does it add to the sentence? Will the sentence lose its meaning if the word is omitted? Can the thought be expressed with fewer words? Strunk and White [SW] have a splendid discussion of the concept of weighing each word.

- **Plural Forms of Foreign Nouns** We all grind our teeth when we hear our freshmen say “And this point is the *maxima* of the function.” To no avail we explain that “maxima” is plural, and “maximum” is singular. Yet we make a similar error when we do not differentiate “data” (plural) from “datum” (singular) and “criteria” (plural) from “criterion” (singular). To be sure, Latinisms take on a life of their own when transferred

⁶In a moment of exasperation, a friend of mine said of her soon-to-be-ex-husband, a mathematician, “You look at anything and you either say that it is ‘very interesting’ or ‘trivial’.”

to English. “Data,” for example, can be construed as a collective singular noun. Similar ambiguities attend “agenda.” This is a complicated business, because the word “data” is sometimes used as a collective noun and, in that context, is singular. Similar comments apply to “agenda” (plural to “agendum”) and a few other words. In general, you should opt for the historically correct Latin forms. Above all, be consistent. As usual, exercise special care when dealing with foreign words.

- ***The Possessive*** When you express the possessive of a singular noun, always use ’s. Thus you should say “Pythagoras’s society,” “the dog’s day,” “Stokes’s theorem,” “Bliss’s book,” “baby’s bliss,” and “van der Corput’s lemma.” The terminal “s” is omitted when you are denoting the possessive of a plural noun: “the boys’ trunk,” “the dogs’ food,” “the students’ confusion.”

“Collective nouns” are treated in a special manner. For instance, we write “the people’s choice” and “the children’s folly”: even though the nouns are plural, we denote the possessive *with* a terminal “s.”

Just because we frequently see such misuse in advertising and other informal writing, we sometimes get sucked into using extraneous apostrophes. Above all, remember that the apostrophe signals possession. Do *not* get bogged down with extraneous apostrophes. As an example, one often sees expressions like “This sentence contains a lot of TLA’s.” Here a TLA is a “three-letter acronym.” What is wrong with this sentence? The last “word” in the sentence is supposed to be a plural—*not* a possessive. So the apostrophe is out of place. It should be “TLAs,” not “TLA’s.” Likewise, do not write, “I surely miss the 1960’s.” It should be “1960s,” *not* “1960’s.”

- ***Precision and Custom*** At times, the goal of precision in writing flies in the face of custom. Antoni Zygmund once observed that the World Series of American baseball might more properly be called the “World Sequence.” I am inclined to agree (in no small part out of fealty to my mathematical grandfather), but I must be over ruled by custom: if you use the phrase “World Sequence,” then nobody will know what you are talking about. Bear this thought in mind when you are tempted to invent new terminology or new notation (see also the remarks in Section 2.4 on terminology and notation).
- ***Principal vs. Principle*** These words are easily confused. “Principal” means the main or the significant choice. “Principle” is a tenet or point that you want to make. Thus you speak of a “principal investigator” on a grant but the “localization principle” in partial differential equations.
- ***Subject and Verb, Agreement of*** Make sure that subject and verb agree in your sentences. A mismatch not only grates on the sensitive

ear, but can seriously distort meaning. Consider the example “The set of all morphisms are compact.” This syntax is incorrect. The *subject* (i.e., the person or thing performing the action) in this sentence is *set* (which is a singular noun). We should conjugate the verb “to be” so that it agrees with this subject. The point is that a singular subject requires a singular verb and a plural subject requires a plural verb. As a result, the grammatically correct statement is “The set of all morphisms *is* compact.” (Note, in passing, that the original form of the sentence might have misled the reader into thinking that the writer was—rather clumsily—discussing a collection of compact morphisms.)

Of course the test is easy: omit the prepositional phrase “of all morphisms” and analyze the root sentence. Clearly “The set is compact” is correct while “The set are compact” is not. You will find the device of focusing on the root statement, or breaking into pieces (see our analysis of Subject and Object below), to be a valuable tool in analyzing many grammatical questions. Another way to look at the matter is that prepositional phrases can cause confusion.

As a parting exercise, consider the phrases “the sequence $\{z_n\}$ converges to p ” while “the numbers z_n converge to p .” Think carefully about why both statements are correct.

- **Demonstrative Pronouns: *This* and *That*** We often hear, especially in conversation, phrases like “Because of this, we decided that.” If we exercise the full force of logic, then we must ask “‘Because of’ *what?*” and “‘we decided’ *what?*” And this niggling query raises an entire body of common errors that I would like to point out. This corpus is not composed so much of errors in English usage, but rather errors in logic and precision. Consider the following examples:

Shakespeare was an important writer. This tells us a lot about English literature. ❖

A triangle is a three-sided polygon. This means that . . . ❖

The day was bright and beautiful. Because of this, Mary smiled. ❖

In each of these examples, my objection is “‘this’ *what?*” (Notice that I did *not* say “In each of these, my objection is . . .” I was careful to say *precisely* what I meant.) The following passages convey the same spirit as the preceding three, but they actually *say* something:

Shakespeare was an important writer. The forms of his plays and poems as well as his use of language have had a strong influence on English literature.

A triangle is a three-sided polygon. The trio of sides satisfies the important *triangle inequality*.

The day was bright and beautiful. Observing the weather caused Mary to smile.

Here is a delightful example that was contributed by G. B. Folland:

Saddam Hussein was determined to resist attempts to force Iraqi troops out of Kuwait, although George Bush made it clear that he did not want to be seen as a wimp. This caused the Gulf War.

If you were to ask someone to which clause “This” refers, then the answer you received would probably depend on that person’s politics.

The message here is fundamental: as a default, do not use “this” or “that” or “these” or “those” without a clear point of reference. When the occurrence of “this” or “that” is fairly close to the referent, then the intended meaning is often clear from context. When instead the distance is greater (as in Folland’s example), then confusion can result.

Repetition is a good thing, so repeat your nouns rather than refer to them with a potentially vague pronoun. There *will* be cases where the casual use of “this” or “that” is both natural and appropriate, but such instances will be exceptions. As a general rule, repeat your nouns.

Copy editor Rosalie Stemer says that a hallmark of good writing is that it answers more questions than it raises. Applying this philosophy will lead naturally to many of the points raised in this book, including the present one.

- **Where** One of the most common types of run-on sentence in mathematics is a statement with a dangling concluding phrase such as “where A is defined to be . . .” An example is

Every convex polynomial function is of even degree, where we define a function to be convex if . . . ✘

We see this abuse so often that we are rather accustomed to it. This is also an easy crutch for the writer: he/she did not bother to plant the definition before this statement, so he/she just tacked the definition onto the end.

This practice is sloppy writing and there is no excuse for it: before you use a term, define it. You need not use a formal, displayed definition. But you must put matters in logical order. The example I have given is quite trivial; but in serious mathematical writing it is taxing on the reader to

have to pick up definitions on the fly. Especially if you are writing with a computer, it is very easy for you to scroll up and put the needed definition where it belongs.

- ***Who and Whom; Subject and Object***

Here we discuss relative pronouns.

Be conscious of the difference between “who” and “whom.” The word “whom” is an object; used properly, it denotes a person who is *acted upon*. Put in other words, “who” is subjective, but “whom” is objective. The word “who” acts as the subject of a clause; the word “whom” acts as a direct object or as the object of a preposition. An example of the common misuse of the word “whom” is

The pastor, whom expected a large donation, smiled warmly.



Here the issue is what is the correct subject to put in front of the verb “expected.” The word “whom” cannot act as a subject. The correct word is “who”: “The pastor, who expected a large donation, smiled warmly.” In the same vein, it is correct to say “To whom am I speaking?” and “Is he the man who was awarded the Nobel Prize?”

Also do not confuse “I”—used for the subject of a verb—and “me.” The latter is an object, the former not. For example, “The teacher was addressing Bobby and I” is plainly wrong, since (here “I” is used incorrectly as the object of) the verb “addressing” calls for a direct object—“me.” President Clinton’s famous misstatement “Give Al Gore and I a chance to bring America back” is a dreadful error; nobody would say “Give I a chance . . .” That sort of sentence analysis—breaking a sentence down to its component parts—is the method you should use to detect the error. The sentence

Him and me proved the isotopy isomorphism theorem in 1967.



is an abomination. Unfortunately, even smart people make mistakes like this. Anyone can see that “Him proved the isotopy isomorphism . . .” and “Me proved the isotopy isomorphism . . .” are incorrect. But, somehow, the ganglia are more prone to misfire when we put the two sentences together. Conclusion: test the correctness of a sentence with compound subject (or any compound element) by breaking it into its component sentences.

1.11 More Rules of Grammar, Syntax, and Usage

1.11.1 Introduction

Here I include additional rules of grammar and syntax that are dear to my heart. They come up frequently in general writing, less so in specifically mathematical writing. They should prove useful in your expository work, and sometimes in your research work as well.

1.11.2 More Rules of Grammar and Syntax

- ***Adjectives vs. Adverbs*** An adjective is designed to describe, or to modify, a noun. An adverb is designed to describe, or to modify, a verb. Correct is to say

This is a good book.

and

This is an expensive car.

and

The quick, brown fox jumped over the stupid, lazy dog.

because “good,” “expensive,” “quick,” “brown,” “stupid,” and “lazy” are adjectives. They modify the nouns “book,” “car,” “fox” (twice), and “dog” (twice), respectively. You may also say

She shouts loudly.

and

He sings beautifully.

and

She strove sporadically to master her homework thoroughly.

because “loudly,” “beautifully,” “sporadically,” and “thoroughly” are adverbs. They modify the verbs “shouts,” “sings,” “strove,” and “to master.” Learn to distinguish between adjectives and adverbs, and learn to use both correctly.

A nice example of the principles discussed here is

I want to speak good English because I want people to think that I speak English well.

Here “good” is an adjective modifying the noun “English” while “well” is an adverb modifying the verb “speak.”

After Paul Halmos had seen an early draft of the first edition of this book, he sent me the message “You write good.” One can guess effortlessly that he was joking mischievously about this silly, little book. It may be noted, however, that “good” *can* be a noun. Consider, for instance, the sentence

The good that men do is oft interred with their bones.

We close this piece of advice by noting the very common problem with “good” and “well.” “Good” is *always* an adjective. “Well” can be either a verb or an adjective. It makes sense to say “I feel well,” because the adverb “well” modifies the verb “feel.” It does *not* make sense to say “I feel good.” Just because good is an adjective. But you *can* say “I am good.”

- **Alternate vs. Alternative** The adjectives “alternate” and “alternative” have traditionally different meanings, though they are often, and erroneously, used interchangeably. The word “alternate” (most commonly used in the form “alternately”) refers to some pair of events that occur repeatedly in successive turns; the word “alternative” refers to a choice between two mutually exclusive possibilities. For example:

Pierre alternately dated Mimi and Fifi. He had considered monogamy, but had instead chosen the alternative lifestyle of a concupiscent lothario.

- **The Verb *To Be*** The verb “to be” is a linking verb; it implies a state of being, and can never take an object. Probably you have been hearing this assertion all your life. What does it mean?

When you formulate the sentence

I hit the ball.

then “I” is the subject (of the verb “hit”) and “ball” is the object (of the verb “hit”). But when you formulate the sentence

I am the walrus.

then “I” is the subject (of the verb “to be,” conjugated as “am”), but “walrus” is the *predicate nominative* (also sometimes called the *predicate noun* or *subjective complement*). The word “walrus” is *not* receiving any action; it is simply restating or describing the subject (which is “I”).

When you are using nouns, such as “ball” and “walrus,” you are unlikely to run into serious difficulties. With pronouns, however, you may. For

you must carefully distinguish between “I”—the nominative singular used for *subjects*—and “me”—the accusative singular used for objects and also objects of prepositions. Likewise, you need to differentiate “we” (the nominative plural) from “us” (the accusative plural).

Thus it is technically incorrect (though rather common) to answer the query (over the telephone) “Is this Napoleon Bonaparte?” with the answer “This is me.” This is because “is” requires a predicate nominative, *not* an accusative. Hence the correct rejoinder is “This is I” or “This is he.”

To make a long story short, your writings should not include the statement “The person who proved Fermat’s Last Theorem is me.” Grammatically correct is “The person who proved Fermat’s Last Theorem is I” or “It is I who proved Fermat’s Last Theorem” or “I am the one who proved Fermat’s Last Theorem.” You should not, however, pen any of these statements unless you are Andrew Wiles.

- **Compare and Contrast** The words “compare” and “contrast” have different meanings. One compares two or more items in order to bring out their similarities; one contrasts two or more items in order to emphasize their differences. For instance, we can compare groups and semigroups because they are both associative. We can contrast them because one contains all inverse elements and the other need not.
- **Different from and Different than** The phrase “different from” is generally preferable to the phrase “different than.” Notice that “from” is a preposition while “than” is a conjunction.

Examples are

His view of grammar is different from mine. (*)

and

His syntax is different from what I expected. (**)

Modern usage (see [Fra, p. 266]) suggests, however, that “different than” is permissible when it introduces a new clause. Thus, in the sentence (**), you could instead say

His syntax is different than I expected.

You will have to decide which usage you prefer, but do be consistent.

- **Due to** Mathematicians commonly use the phrase “due to,” and we often use it incorrectly. We sometimes say “due to the fact that” when instead

“because” will serve nicely. The phrase “due to” tempts us to wordiness best resisted.

- **Farther and further** It is common to interchange the words “farther” and “further,” but there is a loss of precision when you do so. The word “farther” denotes distance, while “further” suggests time or quantity. For example, one might say “I wish to study *further* the question of whether Lou Gehrig could throw the baseball *farther* than Ty Cobb.”
- **Good taste and good sense** Suit your prose to the occasion. The writer of a Harlequin romance novel might write

Clutched in the gnarled digits of the syphilitic Zoroastrian homunculus was a dazzling Fabergé egg. ✦

while Raymond Chandler would have written something more like

The dwarf held a gewgaw.

In mathematics, simpler is usually better. Flamboyant writing is out of place.

- **Hopefully and I hope** With due homage to Edwin R. Newman [New], I note that it is incorrect (at least in my view) to use “hopefully” (at the beginning of a sentence) when you mean to say “It is hoped that” or “I hope.” The word “hopefully” is an adverb. It is intended to modify a verb. For example, consider the sentence

She wanted so badly to marry him, and she looked at him hopefully while she waited for a proposal.

Note that the word “hopefully” modifies “looked.” It is incorrect to say

Hopefully the weather will be better today. ✦

because the weather cannot hope. What you mean to say, of course, is

I hope that the weather is better today.

By the same token, do not say “This situation looks hopeful.” People can be hopeful, objects or things never.

Monty Python tells us that “Mitzi was out in the garden, hopefully kissing frogs.” If you are comfortable with the common misuse of “hopefully,” then you will probably misunderstand this sentence.

Actually, things are a bit more complicated than we have just indicated. In modern usage, an adverb can be used to modify almost anything except a noun. There is a concept of a *sentence adverb* that can modify an entire phrase. For example,

How is she doing, in terms of her math classes?



are simply dreadful. Usually the phrase “in terms of” is gratuitous, and can be omitted entirely. Consider instead

What is his surname?

and

How is she doing in her math classes?

As English speakers, we often rely on quite meaningless idioms. As writers, we should try to avoid these pitfalls. Expressions such as “at this point in time” and “in terms of” are gratuitous and excessively wordy. Neither idiom adds any substance to your meaning. Instead of saying, “At this point in time we will consider . . .,” instead say “Now we will consider . . .” Instead of saying “How is your food, in terms of tastiness?” instead say “How does your food taste?”

- ***Need Only; Suffices to*** In written mathematics, we often find it convenient to say “We need only show that . . .” or “It suffices to show that . . .” These are lovely turns of phrase. Strive not to overuse them, or to misuse them. Too often we see instead “We only need to show that . . .” or “Suffice it to show that . . .” With these alterations, the message still comes across—but in a more halting and less compelling manner.
- ***Parallel Structure*** The principle of parallel structure is that proximate clauses which have similar or related content and purpose are (often) more effective if they have similar form. The use of parallel structure is an advanced writing skill: good writing can be made better, more forceful, and more memorable with the use of parallel structure. Consider the dicta

Candy is dandy, but liquor is quicker.

or

Virtue is good, but sin is more fun.

Whether you approve of the sentiment or not, the first thought is memorably expressed—using a quintessential example of parallel structure. The second is somewhat parallel, but less so. As an exercise, try expressing the thoughts with more desultory prose, and see for yourself what is lost in the process.

The first inspirational quotations (from Sir Francis Bacon) in Chapters 3 and 5 provide less frivolous examples of parallel structure.

- ***Dangling and misplaced modifiers*** Dangling and misplaced modifiers are a frequent cause for discomfort. For example,

Shining like the sun, the man gazed happily upon the heap of gold coins. ✘

The participial phrase “shining like the sun” modifies “man,” whereas it should modify “the heap of gold coins.” Better would be

The man gazed happily upon the heap of gold coins, which shone like the sun.

Harold Boas contributes the following useful maxim: “When dangling, don’t use participles.”

Just as common, and even sillier, than misplaced modifiers are the dangling sort. To wit:

By giving daily quizzes, the students’ grades dramatically improved.

Who is giving daily quizzes here? The professor, of course. But he/she does not even appear in the sentence! Of course the phrase “By giving daily quizzes” requires a subject. The thought is expressed much more clearly as “By giving daily quizzes, the professor helped the students to improve their grades.”

Be aware that infinitive phrases can also dangle. For example, “To get tenure, her mathematics must be superb.” Here it is *not* the mathematics that gets tenured; it is the professor. A much clearer phraseology is “If she wants to get tenure, then her mathematics must be superb.”

- ***Prepositions, Ending a Sentence with*** Many people object to ending a sentence with a preposition (words such as “to,” “at,” “of,” etc.). Rather than say “Where do we stop playing at?” the purists suggest “At what point do we stop playing?” Better still is “When do we stop playing?” Rather than say “What book are you speaking of?” opt instead for “Of which book do you speak?” or “Which book is that?”

Like objections to the split infinitive, the distaste for preposition-ending sentences probably derives from a desire to apply Latin rules to English. Yet it is precisely because of this generalized distaste that you should at least be wary of this usage.

Often, when you are ending a sentence with a preposition, what is in fact occurring is that the errant preposition is a spare word—not needed at all. The preceding examples, and the suggested alternatives, illustrate the point.

Above all, try to keep your syntax simple and easily readable. Sometimes a sentence ending with a preposition sounds much more natural than a sentence that goes into contortions trying to avoid this problem.

An old joke has a yokel trying to find his way across the Harvard campus. A Brahmin student corrects him sternly for posing the question “Excuse me. Where’s the library at?” After the Harvardian explains at length that one does not end a sentence with a preposition, the yokel tries again: “Excuse me. Where’s the library at—*jerk?*” The yokel here is making a good point: Grammatical rules, if stilted, are best ignored.

As an exercise, find a better way to express the following sentence (which ends with five prepositions, and which I learned from Paul Halmos by way of [KnLR]):

What did you want to bring that book I didn’t want to be read
to out of up for?

Harold Boas cautions: “Watch out for prepositions that sentences end with.”

- **Quotations** We do not often include quotations in mathematics papers. If you decide to include a quotation, then be aware of the following technicality. Logically, it makes sense to write a sentence of the following sort:

As Methuselah used to say, “When the going gets tough, the
tough get going”. ❖

What is logical here is that the quotation itself is a proper subset of the entire sentence; therefore it stands to reason that the terminal double quotation mark should occur *before* the period that terminates the sentence. Unfortunately, logic fails us here. Admittedly typesetters are still debating this point, but the current custom in the United States is to put the period *before* the closing double quotation mark. Open any novel and see for yourself. Thus the sentence *should* be written

As Methuselah used to say, “When the going gets tough, the
tough get going.”

Like periods, commas should also be placed inside the quotation marks: “When the going gets tough,” as Methuselah used to say, “the tough get going.”

By the rules of *American* usage, commas and periods should be placed inside quotation marks, and colons and semicolons outside quotation marks (see [SG, p. 222] and [Dup, p. 192]). Surprisingly, perhaps (but logically), colons and semicolons should be placed *outside* the quotation

marks. Placing exclamation points and question marks inside or outside of quotation marks depends on context. British usage is even more ambiguous. This is all a bit like the infield fly rule in baseball. But do be consistent, and be prepared to arm-wrestle with your publisher or with your copy editor if you have strong opinions in the matter.

If your quotation is n paragraphs in length, then there is an opening double quotation mark on every paragraph. There is no closing double quotation mark on paragraphs 1 through $(n - 1)$; but there certainly *is* a closing double quotation mark on paragraph n . Again, check any published novel to see that this is the case.

- **Redundancy** Logical redundancy, used with discretion, can be a powerful teaching device. By contrast, avoid (local) verbal redundancy. The phrases “old adage,” “funeral obsequies,” “refer back,” “advance planning,” “strangled to death,” “invited guest,” “body of the late,” and “past history” display an ignorant and superfluous use of adjectives. Avoid constructions of this sort.
- **Shall and Will** In common speech, the words “shall” and “will” are often used interchangeably, or according to what appeals to the speaker. In formal writing, traditionalists note a distinction: when expressing belief regarding a future action or state, “shall” is used for the first person (“I” or “we”) and “will” is used for the second person (“you”) or third person (“he,” “she,” “it,” or “they”). To express determination, the first person could use “will.” These rules, taken from [SW], are illustrated whimsically in that source by

Bather in Distress: “I shall drown and no one will save me.”

but

Suicide: “I will drown and no one shall save me.”

In practice, these distinctions are largely lost in modern American usage. “Will” has become the general all-purpose choice for most people. Still, there are instances in which both “shall” and “will” can be used very effectively—if not for differences of meaning then certainly for differences of sound and emphasis.

A more modern example of careful usage of “shall” and “will” comes from President Lyndon Johnson:

I shall not seek, nor will I accept, the nomination of my party.

- **That and Which** The relative pronoun “that” is used to denote *restriction*, while the relative pronoun “which” denotes *amplification*. For

example, “I am speaking of the vase that sits on the table” and “The book that is by Gibbons is in the study.” Compare with “The vase, which is red, sits on the table” and “The book, which is by Gibbons, is fascinating.”

In these instances, “that” introduces essential information: the vase that is on the table, *not* the one that is on the floor; the book by Gibbons, *not* the one by Pascal. By contrast, “which” introduces descriptive but nonessential information: the vase is on the table and the book is fascinating; and, by the way, the vase is red and the book is by Gibbons. It is worth noting that nonrestrictive clauses are set off by commas, while restrictive clauses are not (see the examples in the last paragraph).

In mathematics, the difference between “that” and “which” can sometimes be crucial. Consider this example:

A holomorphic function on a connected open set that vanishes on S must be identically zero.

Compare with

A holomorphic function on a connected open set which vanishes on S must be identically zero.

Which is correct? Think about the logic. What we are saying is that a holomorphic function f on a connected open set such that $f(z) = 0$ for $z \in S$ must be identically zero. (For the mathematics, note that, in one complex variable, a set S with an interior accumulation point will suffice for the truth of the statement.) Phrased in this way, the statement is restrictive: a holomorphic function with a certain additional property must be zero. Thus the correct choice is “that” rather than “which.”

Modern grammarians approve of the use of “which” for “that” in suitable contexts. Consult a grammar book, such as [SG], for the details.

I have already noted that it is sometimes useful to let your ear overrule the strict code of grammar. In particular, there are times when “which” sounds more weighty, or more formal, than “that.” Thus some writers will make the technically incorrect choice, just to achieve a certain effect.

As already noted, the rules of grammar and syntax are not absolute. English usage is constantly evolving. While some current aspects of usage are fads and nothing more, others become common and are finally adopted by the best writers and speakers. Those tend to stay with us. But there is a more subtle point. Sometimes a sentence formed according to the strict rules of usage *sounds awkward*. A classic example (usually attributed to Winston Churchill) is

That is the sort of behavior up with which I will not put.



Notice that the speaker is going into verbal contortions to avoid ending the sentence with a preposition. The result is a sentence so absurd that it defeats the main purpose of a sentence—to *communicate*. Better is to say

That is the sort of behavior that I will not put up with.

While technically incorrect—because the preposition is at the end of the sentence—this statement nevertheless will not grate on the ears of the listener, will convey the sentiment clearly, and will get the job done. Of course it would be even better to say

I will not tolerate that sort of behavior.

This sentence conveys exactly the same meaning as the first two. But it has the advantage that it is direct and forceful. In most contexts, the last sentence would be preferable to the first two. This is again a matter of thinking about what the message is intended to be. And here is a point that I will make several times in this book: often it is a good idea *not* to wrestle with a sentence that is not working; instead, reformulate it. Make it sound more natural. We did precisely this with the last example.