
Preface

This book discusses the combinatorial theory of chip-firing on finite graphs, a subject that has its main sources in algebraic and arithmetic geometry on the one hand and statistical physics and probabilistic models for dispersion on the other. We have structured the text to reflect these two different motivations, with Part 1 devoted to the divisor theory of graphs (a discrete version of the algebraic geometry of Riemann surfaces) and Part 2 devoted to the abelian sandpile model (a toy model of a slowly-driven dissipative physical system). The fact that these seemingly different stories are in some sense two sides of the same coin is one of the beautiful features of the subject.

To provide maximal coherence, each of the first two parts focuses on a central result: Part 1 presents a quick, elegant, and self-contained route to M. Baker and S. Norine's Riemann-Roch theorem for graphs, while Part 2 does the same for L. Levine's threshold density theorem concerning the fixed-energy sandpile Markov chain. In the exposition of these theorems there are many tempting tangents, and we have collected our favorite tangential topics in Part 3. For instructors, the ability to include these topics should provide flexibility in designing a course, and the reader should feel free to pursue them at any time. As an example, a reader wanting an introduction to simplicial homology or matroids could turn to those chapters immediately. Of course, the choice of topics included in Part 3 certainly reflects the biases and expertise of the authors, and we have not attempted to be encyclopedic in our coverage. In particular, we have omitted a thorough discussion of self-organized criticality and statistical physics, the relationship to arithmetic geometry, pattern formation in the abelian sandpile model, and connections to tropical curves.

The audience we had in mind while writing this book was advanced undergraduate mathematics majors. Indeed, both authors teach at undergraduate liberal arts colleges in the US and have used this material multiple times for courses. In addition, the second author has used this subject matter in courses at the African Institute for Mathematical Sciences (AIMS) in South Africa, Ghana, and Cameroon.

The only prerequisites for reading the text are first courses in linear and abstract algebra, although Chapter 8 assumes rudimentary knowledge of discrete probability theory, as can be easily obtained online or through a standard text such as [82]. In fact, one of the charms of this subject is that much of it can be meaningfully and entertainingly presented even to middle school students! On the other hand, this text is also suitable for graduate students and researchers wanting an introduction to sandpiles or the divisor theory of graphs. We encourage all readers to supplement their reading with computer experimentation. A good option is the free open-source mathematics software system SageMath ([33], [77]), which has extensive built-in support for divisors and sandpiles.

In addition to presenting the combinatorial theory of chip-firing, we have the ulterior motive of introducing some generally useful mathematics that sits just at the edge of the standard undergraduate curriculum. For instance, most undergraduate mathematics majors encounter the structure theorem for finitely generated abelian groups in their abstract algebra course, but it is rare for them to work with the Smith normal form, which arises naturally in the computation of sandpile groups. In a similar way, the following topics all have connections to chip-firing, and the reader can find an introduction to them in the text: the matrix-tree theorem, Markov chains, simplicial homology, M -matrices, cycle and cut spaces for graphs, matroid theory, and the Tutte polynomial. Further, for students intending to study algebraic geometry, learning the divisor theory of graphs is a fantastic stepping stone toward the theory of algebraic curves. In fact, some of the proofs of the corresponding results are the same in both the combinatorial and the algebro-geometric setting (e.g., Clifford's theorem).

We now provide a more detailed description of the text. A reader wanting to start with sandpile theory instead of divisors should be able to begin with Part 2, looking back to Part 1 when needed for vocabulary.

Part 1:

- Chapter 1 introduces the dollar game in which vertices of a finite graph trade dollars across the edges in an effort to eliminate debt. This simple game provides a concrete and tactile setting for the subject of divisors on graphs, and the chapter ends with a list of motivating questions.
- Chapter 2 introduces the Laplacian operator, which is really our central object of study. We reinterpret the dollar game in terms of the Laplacian, and introduce Smith normal form as a computational tool.
- Chapter 3 discusses algorithms for winning the dollar game or certifying unwinnability. The first section presents a greedy algorithm for winnability, while later sections describe the more sophisticated Dhar's algorithm along with the attendant concepts of q -reduced divisors and superstable configurations.
- Chapter 4 continues the study of Dhar's algorithm, using it to establish a crucial bijection between acyclic orientations of the graph and maximal unwinnable divisors.
- Chapter 5 draws on all of the previous chapters to establish the Riemann-Roch theorem for graphs. Section 5.3 lays out the striking analogy of the divisor theory of graphs with the corresponding theory for Riemann surfaces.

Part 2:

- In Chapter 6 we begin anew by imagining grains of sand stacked on the vertices of a graph, with new grains arriving slowly at random locations. When too many grains accumulate at a vertex, the vertex becomes unstable and topples, sending grains along edges to neighboring vertices. An avalanche may ensue as new vertices become unstable due to the toppling of other vertices. One vertex is designated as the sink, having the capacity to absorb an unlimited amount of sand—this ensures that every avalanche eventually comes to an end. As in Part 1, the Laplacian operator plays the central role in this story, but now our attention turns to the recurrent sandpiles: those configurations of sand on non-sink vertices that occur over and over again as we drop sand on the graph.
- Chapter 7 presents the burning algorithm for determining whether a sandpile is recurrent. As a consequence, we establish a duality between recurrent sandpiles and superstable configurations, thereby revealing one of many connections between the sandpile theory of Part 2 and the divisor theory of Part 1.
- Chapter 8 provides a brief introduction to Markov chains and then presents the threshold density theorem for the fixed-energy sandpile. In this model there is no sink vertex, so an avalanche might continue forever. Starting with a highly stable state, we imagine dropping additional grains of sand on the graph just as before, allowing time for the avalanches to occur in between successive grains. How much sand will be on the graph when it first passes the critical threshold where the avalanche never ends? The threshold density theorem provides a precise answer to this question in the limit where the starting state becomes “infinitely stable”.

Part 3:

- Chapter 9 contains two proofs of the matrix-tree theorem, which computes the number of spanning trees of a graph as the determinant of the reduced Laplacian matrix. This tree-number is also the number of recurrent sandpiles on the graph. The second section contains several corollaries, while the final two sections discuss tree-bijections and the remarkable rotor-router algorithm, providing an action of the sandpile group on the set of spanning trees.
- Chapter 10 returns to the setting of Part 1 and studies harmonic morphisms between graphs. These are discrete analogues of holomorphic mappings between Riemann surfaces, and we prove a graph-theoretic version of the classical Riemann-Hurwitz formula at the end of the first section, explaining the original result for surfaces in Section 10.2. In the final section, we interpret harmonic morphisms as generalized solutions to the dollar game in which individuals pool their money by forming households.
- Chapter 11 presents two related topics concerning the divisor theory of complete graphs. In the first section, the superstable configurations on a complete graph are shown to be essentially the same as parking functions—certain integer sequences arising frequently in combinatorics, introduced here via a simple story about parking cars. In the second section, we present the Cori-Le Borgne algorithm for computing ranks of divisors on complete graphs, exploiting the connection with parking functions in an appealing way.

- Chapter 12 addresses several additional topics related to sandpiles: the dependence of the sandpile group $\mathcal{S}(G, s)$ on the choice of sink vertex s , the minimal number of generators for $\mathcal{S}(G, s)$, and a generalization of the sandpile dynamics in which the reduced Laplacian is replaced by a non-singular M -matrix. The chapter concludes with a brief discussion of the concept of self-organized criticality.
- Chapter 13 introduces the algebraic theory of cycles and cuts, and discusses the connection to the sandpile group. As an application, we prove that the sandpile group of a plane graph is isomorphic to the sandpile group of its planar dual.
- Chapter 14 provides a brief introduction to matroids and their Tutte polynomials. As applications, we show that the sandpile group of a graph G depends only on the cycle matroid of G , prove Merino's theorem identifying the number of superstable of each degree with the coefficients of the Tutte polynomial, state Stanley's conjecture concerning h -vectors of matroid complexes and, finally, present Merino's proof of Stanley's conjecture in the case of cographic matroids.
- Chapter 15 provides a brief introduction to higher-dimensional versions of many of our topics. We begin by introducing the relevant setting of simplicial complexes and their associated homology theory. We then define higher-dimensional critical groups, generalizing the Jacobian of a graph. We define simplicial spanning trees and present a generalized matrix-tree theorem due to A. Duval, C. Klivans, and J. Martin. We conclude with some brief comments about higher-dimensional versions of the dollar game and the sandpile model.

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