Chapter 1

Scales of Measurement

1.1. Introduction

Dealing with real-world data in a quantitative way means we must select one of the scientific systems of *units* for our measurements. We will begin by reviewing the *metric system*, different forms of which are used almost universally in science. We will also recall the techniques needed to convert measurements given in metric units to the perhaps more familiar English units used in the United States. To prepare for material to come in later chapters, we will recall some of the mathematics of *logarithms* and discuss measurements on logarithmic scales.[1] Each chapter ends with an extended chapter project. The first of these will apply the material developed to estimate the total amount of water contained in the Greenland ice sheet and the expected increase in average sea level if all the freshwater ice sheets melt and that water is introduced into the oceans.

1.2. Metric Units

The metric system commonly used in science today was originally standardized by the First French Republic in 1799 as part of the "new start" following the revolution that deposed the king Louis XVI and abolished the French monarchy. In its basic form, the basic units are the following.

- The *meter*, abbreviated as m, was originally an idealized measure of length equal to one ten millionth of the distance from the North Pole to the Equator along the line of longitude through Paris.
- Areas can be measured in square length units (e.g., square kilometers). Another unit, the *hectare*, abbreviation ha, is often used for plots of land and

[1] Depending on how comfortable a class is with this material, this might be reviewed briefly or omitted entirely.

other everyday uses. One hectare is the area of a square 100 meters on a side, or $10,000$ square meters.

- The *kilogram*, abbreviation *kg*, is an idealized measure of the mass of a cubical volume of pure water $1/10$ of a meter on a side, at the melting point of ice.
- The *liter*, abbreviation *l*, is the volume given by the cube $1/10$ of a meter on a side.
- The *second*, abbreviation *sec*, is the standard unit of time.
- The degree Celsius (centigrade) is the standard unit of temperature, where the freezing point of water is $0°$ C and the boiling point of water is $100°$ C.

To facilitate calibration, these notional definitions of the meter and the kilogram were replaced almost immediately by standard metal prototypes kept in the French national archives. But such physical models change with temperature, air pressure, and other variables. Later the definition of the meter was changed several times to make it even more precise and reproducible. As of 1983, the second is defined as $9,192,631,770$ times the period of oscillation of a certain cesium atomic clock, and the meter is defined as $1/299,792,458$ of the distance light travels in vacuum in one second. This makes the speed of light in vacuum exactly $299,792,458$ meters per second.

For scientific work, the great advantage of the metric system is the way it follows the standard way we represent numbers—the Hindu-Arabic numerals with base-10 arithmetic. A standard series of prefixes represent positive and negative powers of 10:

Table 1. Metric prefixes

prefix	abbrev.	power
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	k	10^{3}
hecto	h	10^{2}
deca	da	10^{1}
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}

As you might guess, the "hect-" in the name *hectare* comes from the multiple for 100. A hectare is 100 *ares*; the unit of area called the *are* is rarely used, though. There are standard names for even larger and even smaller powers of 10, but we will almost never need them. When we do they will be identified. Note how this system makes it possible to talk about lengths, masses, volumes, etc., over a huge range of sizes. This means that the same basic system of units applies to study both extremely small things, such as the cells in our bodies or atoms within our cells, and extremely large things, such as our solar system or the Milky Way galaxy. Moreover, the system of prefixes is set up so that if we choose the appropriate units, it is never necessary to use extremely large or extremely small numbers.

Example 1.1. Here are some examples of working with various metric units.

(1) A *kilogram* equals $10^3 = 1,000$ grams. A gram is roughly the mass of the plastic cap of a ballpoint pen; body masses of humans typically run in the range 50 to 100kg.
(2) A *microgram* equals $10^{-6} = \frac{1}{1,000,000} = .000001$ of a gram (one one-millionth of a gram). This is roughly the mass of a large human cell like the egg cells produced in the female reproductive system.
(3) A *milliliter* equals $10^{-3} = .001$ of a liter. Note or recall that a milliliter is also the same as a *cubic centimeter*, since the cube 1/10 of a meter (a decimeter) on a side that gives the liter has a side equal to 10 centimeters, so 1 liter equals $10^3 = 1000$ cubic centimeters.
(4) A *terameter* is $10^{12} = 1,000,000,000,000$ meters (roughly the distance light travels in one hour in vacuum). The distance from the Sun to the planet Saturn is about 1.4 terameters.
(5) A *megasecond* is $10^6 = 1,000,000$ seconds (a bit over 11.5 days).

Measurements of time are rarely expressed in purely metric terms when they are communicated to humans, though. $\triangle$

If you have taken physics, you may recall that in discussions of that subject a big distinction is often made between the "cgs" (= centimeters, grams, seconds) and "mks" (= meters, kilograms, seconds) versions of the metric system and what are the associated units of velocity, acceleration, work, energy, and so forth depending on whether the cgs or mks system is adopted. We will not need to stress this distinction, though, and we will generally use units chosen for convenience according to the sizes of the quantities being measured.

1.3. Conversions

Because of the "powers of 10" prefix system from Table 1, conversions *within* the metric system essentially just involve shifting decimal points in numbers. This is the main reason why the metric system is so easy and why it has been adopted virtually universally both for scientific use and for everyday measurements.

Example 1.2. Here are some examples of conversions within the metric system.

(1) A distance of 17.3 kilometers can also be expressed in terms of meters like this:

$$17.3 \text{ km} \times 1000 \text{ m/km} = 17,300 \text{ m}.$$

As usual in "dimensional analysis," the units of km in the first number cancels the km in the denominator of the conversion factor 1000 meters per kilometer.

(2) Similarly, to convert a volume of 343.2 milliliters to the equivalent number of liters, we just need to remember that a milliliter is $10^{-3} = .001$ liters, and we have

$$343.2 \text{ ml} \times .001 \text{ l/ml} = .3432 \text{ l}. \qquad \triangle$$

As of 2018, there are *only three* countries in the world that do not use the metric system for everyday measurements: the African country of Liberia, the Asian country of Myanmar (formerly known as Burma), and the United States.[2] This means that the metric units above are probably less intuitively familiar than the English units:

- lengths in inches, feet, yards, miles,
- masses or weights in ounces, pounds, tons,
- volumes in fluid ounces, quarts, gallons,
- temperatures in degrees Fahrenheit.

There are others too, of course. These are just a few of the most basic and common ones. Using these units, we first need to remember conversions within the system. For instance, rounding to three significant digits in all cases we have the following.

Definition 1.3. The following constants give conversions *within the English system.*

(1) 1 foot = 12 inches, so 1 inch = 1/12 foot $\doteq$ .0833 foot.[3] 1 yard = 3 feet = 36 inches. Then 1 mile = 5280 feet, so 1 foot = 1/5280 mile $\doteq$ .000189 mile.

(2) 1 pound = 16 oz., so 1 oz. = 1/16 pound = .0625 pound. Then 1 ton = 2000 pounds, so 1 pound = 1/2000 ton = .0005 ton.

(3) 1 quart = 32 fl. oz., so 1 fl. oz. = 1/32 quart = .03125 quart $\doteq$ .0313 quart. Then 1 gallon = 4 quarts, so 1 quart = .25 gallon.

(4) 1 acre = 1/640 square mile = .0015625 square mile $\doteq$ .00156 square mile.

Then, if we want to convert between the English system and the metric system, we need to look up or remember another set of conversion factors.

Definition 1.4. The following constants give metric to English and English to metric conversions.

(1) 1 inch $\doteq$ 2.54 centimeters, and 1 cm = 1/2.54 inch $\doteq$ .394 inch.

(2) 1 foot $\doteq 12 \times 2.54$ cm $\doteq$ 30.5 cm = .305 m.

[2] Editorial Comment: This is an example of the downside of the idea of "American exceptionalism." It can be argued that we do some things differently and better than many other nations; using the English system of units is not one of those things.

[3] We will use the symbol $\doteq$ consistently to mean that the two quantities are *approximately equal.* The exact value of the rational number $\frac{1}{12}$ is the infinite repeating decimal $.08\overline{3}$. The approximate value .0833 is thus slightly smaller than $\frac{1}{12}$.

(3) 1 mile $\doteq$ 5280 ft/mile $\times$.305 m/ft $\doteq$ 1610 m $\doteq$ 1.61 km. Hence 1 km $\doteq$ 1/1.61 mile $\doteq$ .621 mile.

(4) 1 ha $\doteq$ 2.471 acres $\doteq$ .00386 square mile.

(5) 1 pound $\doteq$ .454 kilogram, and 1 kg $\doteq$ 1/.454 pounds $\doteq$ 2.20 pounds.

(6) 1 quart $\doteq$ .946 liter, and 1 liter $\doteq$ 1/.946 quarts $\doteq$ 1.06 quarts.

(7) To convert back and forth between a temperature F in degrees Fahrenheit and the equivalent Celsius temperature C, use

$$C = \frac{5}{9}(F - 32) \qquad \text{and} \qquad F = \frac{9}{5}C + 32.$$

Let's practice using the conversion factors from Definitions 1.3 and 1.4 on several examples.

Example 1.5. Be sure you understand the thinking behind these and check the arithmetic as you are reading. Reading a mathematics book should be an active endeavor.

(1) To begin, let's ask how long a distance of 3 kilometers is in miles, yards, and feet. From the above, we see

$$3 \text{ km} \times \frac{1}{1.61} \text{ miles/km} \doteq 1.86 \text{ miles.}$$

Then to get the equivalent length in feet, we multiply by the conversion factor from miles to feet:

$$1.86 \text{ miles} \times 5280 \text{ feet/mile} \doteq 9821 \text{ feet.}$$

Finally, to get the equivalent distance in yards, we multiply by the conversion factor from feet to yards:

$$9821 \text{ feet} \times \frac{1}{3} \text{ yards/foot} \doteq 3274 \text{ yards.}$$

Comment: The tens and ones digits here will be different if you use all the digits in the calculator value for 1/1.61. I rounded that, then proceeded with the rest of the computation.

(2) Next, we ask: how much does a 1 meter by 1 meter by 1 meter cube of water weigh in tons and what is its volume in fluid ounces? There are many ways to answer the first part of the question. Probably the most direct, though, is to find the mass of the water in kilograms first, and then convert this to the equivalent weight in pounds and then tons. The reason for this approach is the fact that a cube of water with side 1/10 meter has a mass of 1 kilogram by the definition of the metric units. Since our cube has a side that is 10 times as long as this, the mass of the water is $10 \times 10 \times 10 = 1000$ kilograms. Then

$$1000 \text{ kilograms} \times 2.20 \text{ pounds/kilogram} \times \frac{1}{2000} \text{ tons/pound} \doteq 1.1 \text{ tons.}$$

This says, for instance, that just the water contained in a 3 meter by 2 meter by 1/6 meter water bed would also weigh about 1.1 tons.[4] For the second

[4] This gives some idea of what architectural engineers have to deal with to design buildings that can support the weight of all the stuff that humans put in their living and working environments.

part, we note that the volume is $10 \times 10 \times 10 = 1000$ liters, and

$$1000 \text{ liters} \times 1.06 \text{ quarts/liter} \times 32 \text{ fl. oz./quart} \doteq 33920 \text{ fl. oz.}$$

(3) Now, suppose we have a flat 1 square mile field that is flooded with water to a depth of one inch. We ask: what is the total volume of the water in cubic meters? Thinking of the water as (approximately) a rectangular solid, the volume is length $\times$ width $\times$ height. We want the volume in cubic meters, so it makes sense to convert all three dimensions to meters first, then compute the volume in cubic meters. So we have 1 mile $\doteq$ 1610 meters, and

$$1 \text{ inch} \times .083 \text{ foot/inch} \times .305 \text{ meter/foot} \doteq .0253 \text{ meter.}$$

Hence the total volume of the water is approximately

$$1610 \text{ meters} \times 1610 \text{ meters} \times .0253 \text{ meter} \doteq 65580 \text{ cubic meters.}$$

We treated the shape of the field as though it was a perfectly flat square one mile on a side. The given information says we may ignore the curvature of the Earth here and assume the field is perfectly flat (that is, contained in one plane). This is reasonable because one mile is so much smaller than the Earth's radius (approximately 4000 miles). The answer would also be the same no matter what the actual shape of the field was because a solid of uniform height over a fixed perfectly planar base always has volume equal to the area of the base times the height.

(4) Finally, a temperature of 35°C is equivalent to a Fahrenheit temperature of

$$\frac{9}{5} \cdot 35 + 32 = 95^\circ \text{ F.}$$

The (much more comfortable) Fahrenheit temperature 72° F is equivalent to a Celsius temperature

$$\frac{5}{9}(72 - 32) \doteq 22.2^\circ \text{ C.} \qquad \triangle$$

1.4. Estimation

In many circumstances, exact measurements of quantities in real-world processes may be unavailable due to the difficulty or the cost of carrying them out. In these situations, a good *estimate* or "educated guess" (with emphasis on the "educated," of course) may be the most we can hope for. Making good estimates requires both a solid understanding of the process and an intuitive grasp of the units involved. Here is a simple example.

Example 1.6. Suppose we wish to estimate the average per capita daily usage of water for a U.S. resident. First we must think of all the ways that we use water in a typical day. Most of us probably use

- about 16 gallons to take a shower lasting 8 minutes (at about 2 gallons per minute),
- about 24 gallons in bathrooms (halve that if you use efficient 1 gallon per flush toilets, but most people *do not* have them),
- about 2 gallons for drinking and cooking purposes.

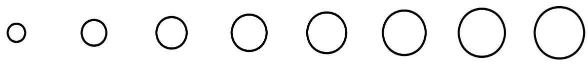

Figure 1.1. Circles with areas in arithmetic progression

That's about 42 gallons a day, but are those the only uses of water most of us have? If you think about it, you'll see that there are large water uses in addition to these:

- If you water a lawn or use a swimming pool, add 25 gallons a day.
- Add 4 gallons per day for use in washing laundry.
- Add 4 gallons per day if you use a dishwasher.

This adds up to an estimate of about 75 gallons per day (and it doesn't even include uses of water to wash the car or clean the house, etc.).

In fact, this is pretty close to the mark. The U.S. Environmental Protection Agency estimates that the average American family uses about 300 gallons of water per day.[5]

By way of contrast, per capita water usage in most other parts of the world (even in economically advanced areas such as Western Europe) is significantly lower. Even though most parts of the U.S. have adequate water resources to support our lifestyles at the present time, the same is not true in drier parts of the world. The cost of purifying water for human consumption is also increasing. △

1.5. A Feature of Human Perception

To introduce our next topic, we will discuss an interesting feature of the way our senses (vision, hearing, taste, touch, smell) deal with stimuli from the physical world. We will illustrate our point with two series of images of circles. In Figure 1.1, the areas of the circles (in suitable square units) would be numbers in the *arithmetic progression* $1, 2, 3, 4, 5, 6, 7, 8$. What this means is that the change in area from each circle to the next is always the same: 1 square unit.

Now compare this with the sequence of circles in Figure 1.2. Here, if the smallest circle at the left had area 1, then areas would be $1, 2, 4, 8, 16, 32, 64, 128$. (The scale is not the same as in the first figure because the range of areas is much greater.) In other words, each circle is twice as large in area as the one before it. We call such a sequence of number values a *geometric progression*.

Compare the two sequences of circles carefully. In the first case (Figure 1.1), you should notice that the *rate of growth* in the areas as you sweep your vision from the left to the right seems to get less and less. In fact, there is apparently very little difference between the last two circles; if presented with those two circles in

[5] According to [**1**].

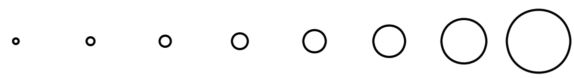

Figure 1.2. Circles with areas in geometric progression

isolation you might be hard pressed to see the difference at all (even though their areas differ by the same amount as the areas of the first two circles). On the other hand, the growth of the areas in Figure 1.2 is seemingly steadier and we have no difficulty in perceiving that the areas are increasing.

Psychologists call the phenomenon we are seeing here the *Weber-Fechner law* of perception. In rough terms, the Weber-Fechner law says that human sense perception works on a *logarithmic* scale. We will explain what this means in detail in the next sections.

1.6. Logarithms

You have probably seen logarithms in your high school algebra or precalculus classes. Recall that the idea is the following. Given a positive number $a \neq 1$, called the base of the logarithms, and any positive number x, we say that

$$\boxed{y = \log_a(x) \quad \text{if (and only if)} \quad a^y = x.} \tag{1.1}$$

In other words, the base a logarithm of x is the *exponent* to which a must be raised to yield the number x.[6]

Example 1.7. For instance

$$\log_3(81) = 4 \quad \text{since} \quad 3^4 = 81.$$

Similarly,

$$\log_{10}\left(\frac{1}{1000}\right) = \log_{10}(10^{-3}) = -3.$$

For numbers x that are not exactly equal to whole number powers of the base a, a calculator, mathematical computer software, or a table is used to find values of the logarithm $\log_a(x)$. For example by any of these methods, we find

$$\log_{10}(2) \doteq .30103.$$

This is true since

$$10^{.30103} \doteq 2.$$

[6]The reason for the restriction $a \neq 1$ can be seen if we think of trying to find y with $1^y = x$ if $x \neq 1$. In practice, values $a > 1$ are used much more commonly than values $0 < a < 1$.

Similarly, using a calculator

$$\log_{10}(.58) \doteq -.23657.$$

Because $.58 < 1$, a negative exponent y is needed to produce an equation $10^y = .58$. $\triangle$

From (1.1), if we know $y = \log_a(x)$, then recovering x is simply a matter of computing $x = a^y$. For instance, if we know $\log_{10}(x) = 4$, then $x = 10^4 = 10,000$.

Properties of exponents carry over into corresponding properties for logarithms.

Proposition 1.8. *For all $a > 0$ different from 1 and all positive x, x_1, x_2, b, the base-a logarithms satisfy the following:*

(1) $\log_a(x_1 x_2) = \log_a(x_1) + \log_a(x_2)$. *In words, "the log of a product is the sum of the logs."*

(2) $\log_a\left(\frac{x_1}{x_2}\right) = \log_a(x_1) - \log_a(x_2)$. *In words, "the log of a quotient is the difference of the logs."*

(3) $\log_a(1) = 0$.

(4) $\log_a(x^b) = b \cdot \log_a(x)$. *In words, "the log of a base to a power is the power times the log of the base."*

Proof. We are not going to be doing a lot of proofs, but we will look at these because understanding how logarithms work is important and the proofs show that. In fact, all of these statements follow from properties of exponents.

(1) For instance, if $y_1 = \log_a(x_1)$ and $y_2 = \log_a(x_2)$, then we have equations

$$a^{y_1} = x_1 \quad \text{and} \quad a^{y_2} = x_2.$$

Hence multiplying and using the fact that the exponents add when we multiply two powers of the same base a, we have

$$x_1 x_2 = a^{y_1} a^{y_2} = a^{y_1 + y_2}.$$

Therefore

$$\log_a(x_1 x_2) = y_1 + y_2 = \log_a(x_1) + \log_a(x_2),$$

since that is the exponent in the equation giving $x_1 x_2$ as a power of a.

(2) This follows in the same way since

$$\frac{x_1}{x_2} = \frac{a^{y_1}}{a^{y_2}} = a^{y_1 - y_2}.$$

(3) The equation $\log_a(1) = 0$ follows since $a^0 = 1$.

(4) If $y = \log_a(x)$, then $a^y = x$. Hence, raising both sides to the b power, we get $x^b = (a^y)^b$. But $(a^y)^b = a^{b \cdot y}$ since the exponents multiply. This gives $\log_a(x^b) = b \cdot y = b \cdot \log_a(x)$. $\square$

The idea of logarithms is usually ascribed to the Scottish mathematician John Napier (1550–1617). Napier was interested mostly in the ways parts (1), (2), and (4) of Proposition 1.8 can be used to simplify numerical calculations. The idea is that you have a table of logarithms computed for a suitably "dense" set of numbers between 1 and 10, say including $\log_{10}(2.31) \doteq .3636120$. Then you can deal with numbers of any magnitude by using scientific notation like this:

$$\log_{10}(2.31 \times 10^4) = \log_{10}(2.31) + \log_{10}(10^4) = \log_{10}(2.31) + 4 \doteq 4.3636120.$$

Most importantly, using this approach, complicated jobs of multiplying, dividing, or raising numbers to powers can be replaced by the simpler computations of addition and subtraction of the logarithms, or multiplication of the logarithm by the power. (As recently as the early 1970s, when I was in high school,[7] it was still possible to find whole semester courses devoted to these calculations in some school curricula. The textbooks usually contained 7-place log tables making up most of the book. Of course the availability of calculators has made the very idea of such courses hopelessly old-fashioned today.)

Most scientific calculators have keys for computing both $\log_{10}$, the base-10 or *common logarithm*, and ln, the so-called *natural logarithm*, where the base is a number called $e \doteq 2.71828$. The reasons for calling this odd-looking choice of base "natural" are studied in calculus classes and we will not go into the details here.

Much scientific work uses the base-10 or common logarithms pretty exclusively and we will follow that practice in this book as well. You should be aware, however, that converting back and forth between these two (and actually *any* two) systems of logarithms is very easy if you ever need to do it. The reason is a consequence of the reasoning in part (4) of Proposition 1.8.

For instance, if $e^y = x$, so $y = \ln(x)$, then using the equation $e = 10^{\log_{10}(e)}$ and substituting in for the e in $e^y = x$, we also have

$$(10^{\log_{10}(e)})^y = 10^{y \log_{10}(e)} = x,$$

so

$$\log_{10}(x) = y \log_{10}(e) = \ln(x) \log_{10}(e).$$

In other words, to convert from the natural log of x to the common log of x, you just multiply by the constant $\log_{10}(e) \doteq .43429$. To convert the other way, you divide the common log by .43429, which is the same as multiplying by $2.3026 \doteq \ln(10)$.

The most general statement along these lines is given in the following general conversion formulas. For positive numbers a, b, x, with $a, b \neq 1$:

$$\boxed{\log_b(x) = \log_a(x) \cdot \log_b(a) \quad \text{and} \quad \log_a(x) = \frac{\log_b(x)}{\log_b(a)}.} \tag{1.2}$$

Here is an example.

Example 1.9. Say $b = e$, so $\log_b = \ln$ is the natural logarithm. Also take $a = 10$. Using a calculator, we have

$$\ln(4.33) \doteq 1.4656.$$

Hence rounding to five decimal digits,

$$\log_{10}(4.33) \doteq \frac{1.4656}{\ln(10)} \doteq \frac{1.4656}{2.3026} \doteq .63650.$$

All the values are rounded here; if you compute $\log_{10}(4.33)$ directly you will get a value closer to .636488. These small differences generally do not make much

[7] I hesitate to admit this, but that was before the advent of hand-held electronic calculators.

difference when dealing with real-world data, where a measurement may only be known to within two or three significant digits due to possible experimental or observational errors. But if more precise values are required, it is always possible to use all of the decimal digits returned in the calculator values of logarithms. △

1.7. Logarithmic Scales

We say a quantity or a magnitude is defined using a *logarithmic scale* if it is computed using a logarithm of some other quantity. For instance, if instead of plotting values of a distance along the x-axis in a plot we used the logarithms of the distances, then we would be using a logarithmic scale.

Example 1.10. The Weber-Fechner law we discussed before can be stated in a more precise way by saying that the human visual system deals with information about the areas of plane regions using a logarithmic scale. That is, it is effectively the logarithm of the area of the region that "registers" more directly in our minds than the area itself. Recall that in Figure 1.1, we saw circles of areas $1, 2, 3, 4, 5, 6, 7, 8$. If we take the (common) logarithms of these values, we get the numbers in Table 2. The successive differences are decreasing as we go to the right.

Table 2. Logarithms of the areas in Figure 1.1

Area	1	2	3	4	5	6	7	8
$\log_{10}(\text{Area})$	0	.30103	.47712	.60206	.69897	.77815	.84510	.90309

For instance, the first two logs differ by .30103, while the last two differ by much less, namely approximately .06. Thus, the logarithmic scale seems to be reproducing the intuitive idea that the last two areas differed much less in visual terms than the first two.

On the other hand, in Figure 1.2, we saw circles of areas $1, 2, 4, 8, 16, 32, 64, 128$. The common logarithms of these numbers are (approximately) shown in Table 3. Here the successive differences are all $\log_{10}(2) \doteq .30103$ and this seems to match the perception that those circles are growing steadily in size. △

Table 3. Logarithms of the areas in Figure 1.2

Area	1	2	4	8	16	32	64	128
$\log_{10}(\text{Area})$	0	.30103	.60206	.90309	1.20412	1.50515	1.80618	2.10721

There are many other important examples of logarithmic scales used in science. We will discuss one of these next, and you will see others in the exercises and in future chapters.

Example 1.11. In chemistry, the pH (from "potential of hydrogen") of an aqueous (water-based) solution is a measure of its acidity or alkalinity. Acids and bases are distinguished by the level of H^+ ions present. Formally,

$$\boxed{pH = -\log_{10}(a_{H^+}) = \log_{10}\left(\frac{1}{a_{H^+}}\right)}, \tag{1.3}$$

where a_{H^+} is the *hydrogen ion activity* (essentially the concentration in units of moles per liter of solution). For instance pure water has $a_{H^+} \doteq 1.0 \times 10^{-7}$, and hence

$$pH = -\log_{10}(1.0 \times 10^{-7}) = 7.$$

Acids are aqueous solutions with $pH < 7$ and the smaller the pH, the "stronger" the acid. For instance, lemon juice has $pH \doteq 2$, while concentrated hydrochloric acid can have $pH \doteq 0$. At the other end of the scale, on average, human blood is slightly alkaline or basic ($pH \doteq 7.4$) and strongly basic solutions like extra-strength liquid drain cleaner can have $pH \doteq 14$. Because of the $\log_{10}$ in the definition (and the negative sign), if solution A has $pH = 4$ and solution B has $pH = 5$, then the hydrogen ion activity for solution A is 10 times as large as that for solution B. The a_{H^+} for solution A is 1.0×10^{-4} and the a_{H^+} for solution B is 1.0×10^{-5}. △

1.8. Project

Introduction. This project is inspired by a similar project developed for the book *Quantitative Reasoning and the Environment* by Greg Langkamp and Joseph Hull (Prentice Hall, 2007), but uses different and more recent data.

The Greenland ice sheet is estimated to contain about 1/9 of the total freshwater ice on the planet. Many recent observations, such as those reported in [**2**], indicate that melting is well under way at present. The goal of this first chapter project is for you to use information from the map[8] in Figure 1.3 to make an estimate of the total amount of ice contained in the Greenland ice sheet, and to understand where the predictions of the effects of total melting are coming from.

Because this ice is presently on land, if it should all melt and flow into the oceans, sea levels around the world would rise. This is different from the situation for the Arctic sea ice, or much of the ice in the shelves surrounding Antarctica, which rest on sea water. Melting in those cases would not appreciably change sea levels because that ice is floating on and supported by water already. This is a consequence of Archimedes' Principle.[9] According to Archimedes, *a floating object displaces a volume of the fluid that has the same mass as the object.* In Exercise 17 below, you will work out the details to show that if a chunk of ice floating in a body of water melts, the level of the water will not change. This can also be verified by an easy experiment in a pot or bathtub. This is good since there are ongoing changes in the Antarctic ice shelves. For instance, on July 12, 2017, the Larsen C ice shelf, with an area of about 44,200 square kilometers, broke off from the Antarctic coast and became what is probably the largest iceberg ever observed by humans. There

[8] See [**3**] and [**4**] for the underlying science.
[9] See Proposition 5 of his work *On Floating Bodies*

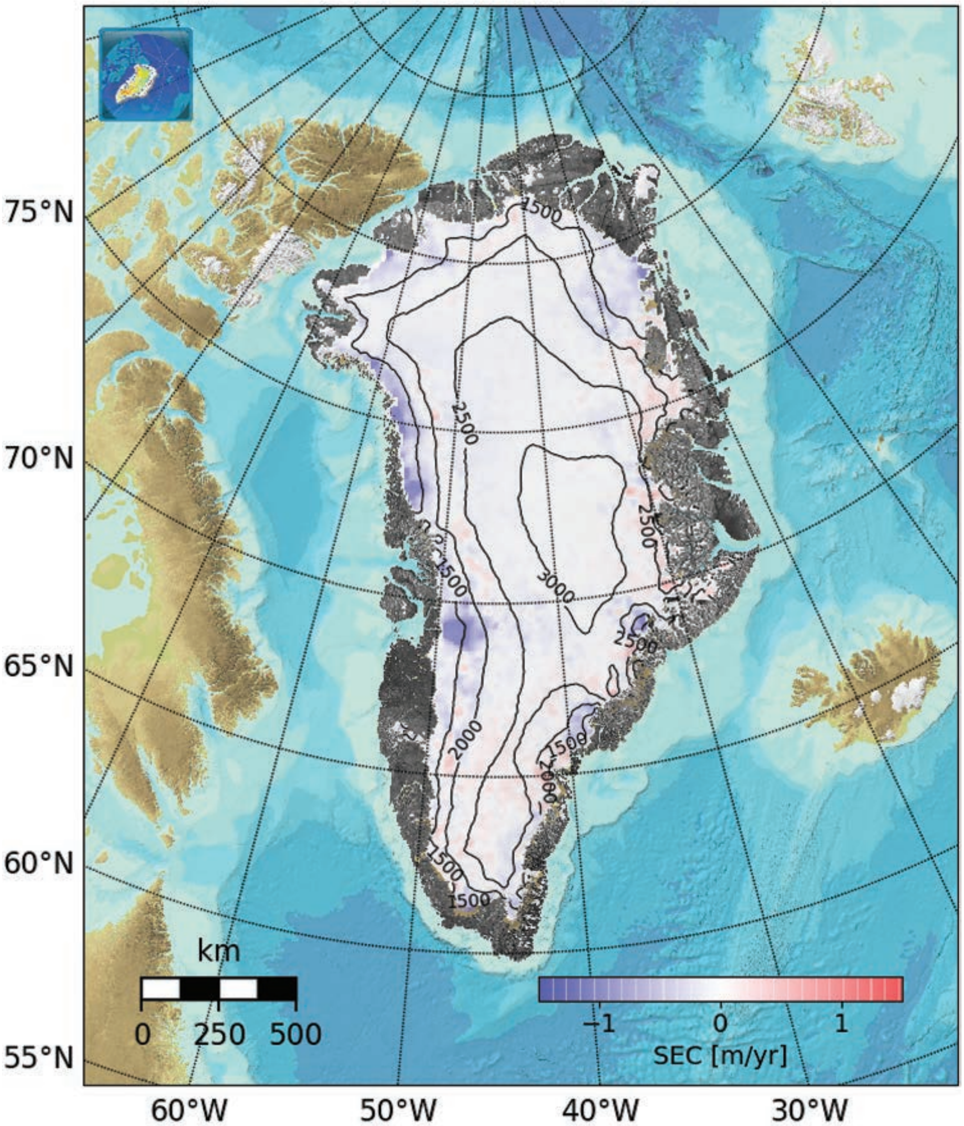

Figure 1.3. Contour plot of surface elevation of the Greenland ice sheet, with estimated changes over the period 2013–2017 derived from Cryosat-2 satellite measurements. Source: ESA CCI ice sheets Greenland project [**3**] and Simonsen et al. [**4**].

are important freshwater ice sheets covering the land mass of Antarctica as well as these sea ice formations, and those make up most of the remaining fresh water ice on the Earth.

As we will see, there is a large volume of water contained in the Greenland ice sheet. For this reason, its fate is of more than passing interest for all humans. Many of our major cities and settled coastal areas occur in regions that are close enough to the current sea level that any significant increase would cause major disruptions. For instance, the *highest* (natural) elevation of any point on the island of Manhattan in New York City is only 81 meters ($\doteq$ 265 feet) above sea level and most of the island lies much lower than that. The effects of the relatively small increase in average sea level that have already occurred were evident, for instance, in the flooding of lower Manhattan that happened during "super-storm Sandy" in October 2012.

Trying to forecast how the ice sheet will evolve over time can be intricate because weather conditions producing melting change from year to year, while the ultimate fate of the ice sheet depends on long-term trends such as the levels of carbon dioxide and other "greenhouse gases" in the atmosphere, deposition of soot onto the ice, and other factors.[10] In every year, some of the ice sheet flows into the ocean and produces (or "calves") icebergs. At the same time, new snow falls onto the ice sheet and creates new ice. While these two processes were apparently balancing each other out over a long period of time until recently, they have not been balancing in many recent years. For instance, the summer of 2012 was much warmer than the long-term average up to that time in Greenland and it produced large and well-publicized melting events for which the ice was not replaced by new snowfall. On the other hand, the years 2016 and 2017 were cooler and included larger than average snowfall amounts in parts of Greenland. These were large enough, in fact, that there might even have been a small net increase in the volume of the ice sheet those years. However, this was not evenly distributed geographically and melting also did take place those years. The National Snow and Ice Data Center (a research institute funded by NASA, the National Oceanic and Atmospheric Administration (NOAA) and the National Science Foundation) maintains a website, [**5**], that provides daily updates on key measurements related to the Greenland ice sheet and information about historical trends. The data for the 2018 season shows that there were much larger than usual surface melting areas in late May and early June, but the rest of the year so far (as of this writing on July 18) has been close to the 1980–2010 median in surface melting area.

The Data. Refer to the map of Greenland showing the surface elevation of the ice covering most of its land area in Figure 1.3. We will make use of the *surface elevation contour lines* to make our estimate. These are the black lines superimposed on the map of Greenland labeled with the numbers 1500, 2000, 2500, 3000. Each of these numbers represents a height above sea level in meters. The area included inside one of these contour lines all has surface elevation *at least* the number associated with that contour.

Our methods will yield rough estimates or approximations of the ice volume. They will be based on the following simplifying assumptions:

- Greenland is not exactly flat. Some of the areas at the highest elevation are mountains in the region on the eastern coast that are not covered by the ice sheet. The bedrock under the ice sheet is not exactly flat either. But in fact the ice sheet is so massive that it has forced large sections of interior bedrock *below* sea level, while other sections are at or above sea level. Hence, to do

[10] Since the first two of these are connected with human burning of fossil fuels for energy, there is also a political component in some public discussions of the issues here.

our estimate, we will average things out and make a simplifying assumption that the ice sheet is resting on a perfectly flat base at sea level.

- As another simplifying assumption, we will treat the ice sheet as though it is one gigantic block of ice with no cracks, no hollow spaces, etc.,
- Even though it appears as a large area on the familiar Mercator projection maps you have probably seen, that map projection distorts areas of regions near the poles and makes them look much larger than they actually are. Greenland does not make up a very large portion of the surface area of the roughly spherical Earth. As a result we will not lose too much if we simply estimate areas as though they corresponded to areas on the flat map (i.e., without trying to take the curvature of the Earth into account). The distance scale marked in the legend of the map can be used, together with a ruler, to approximate linear dimensions of regions.
- For the purposes of this estimate, let's assume that the surface elevation inside each contour line up to the next higher elevation value is *equal* to the number printed over the contour in each case. Also assume all of the white areas outside the 1500 meter contour line have ice at surface elevation 1000. The dark colored areas outside that region also have some ice, but we will neglect them.
- The violet and pink colors refer to the right hand scale at the bottom. They show an estimated *surface elevation change* in m/yr over the period 2013 to 2017, as estimated by measurements from the Cryosat-2 satellite launched by the European Space Agency.

Questions. The chapter project will involve investigating the following questions and writing up your results as directed.

(A) Estimate the ice areas with surface elevation 1000 m, 1500 m, 2000 m, 2500 m, and 3000 m. Explain how you are doing this in a clearly-written paragraph. Note: There are *many ways* to do this in a reasonable fashion and there is not just one right answer! Suggestion: Print out a paper copy of the page with the map and overlay a transparency marked with 250 km by 250 km squares. Count the equivalent number of such squares in each region to estimate the area. You don't need to get super-detailed or picky, but be as accurate as possible.

(B) Multiply each of your area estimates by the depth estimate to get a volume estimate. (Don't forget that we are assuming the ice sheet starts at an elevation of 0 m.) Add the ice volume estimates to get a total volume and express in units of *cubic kilometers*.

(C) As a "reality check" for your method, the total volume of the Greenland ice sheet is often estimated to be about $3,000,000$ cubic kilometers. How close did you come to that? Is your method systematically underestimating or overestimating the ice volume? If you cannot necessarily say either way because different aspects of what you did would tend to pull the estimate in opposite directions, explain.

(D) When ice melts, the volume of the water that is produced is slightly smaller:

$$(\text{volume of water}) \doteq (.92) \cdot (\text{volume of ice})$$

(see Exercise 17 below). Imagine that an amount of water equal to the melted form of your estimate of the volume of the ice is added to the oceans all at once. How much would sea levels rise as a result? One way to estimate that is to use the same idea as in part (3) of Example 1.5 above. What is the total surface area of the oceans on the Earth? (You should look this up online. If you find different estimates, how will you choose which one to use? Explain your thinking.) If the water from the melted Greenland ice sheet was spread evenly over that area, how deep would it be, in meters? Would the actual change in sea level be less than or greater than this estimated height? Explain.[11]

(E) Find an elevation contour map of Manhattan. Use that information to estimate what portions of that island would be under water if all the freshwater ice sheets melted.

(F) (Preview of Chapter 2.) Using the violet and pink shaded regions, estimate whether the total volume of the ice sheet increased or decreased over the period 2013 to 2017. Note that you will need to use both the estimated areas of regions, and the colors representing the surface elevation changes in m/yr to do this. Suggestion: If an area of increase balances out an area of decrease, approximately, you can ignore both of them for your estimate.

Assignment. Write up your solutions for these questions as a project report. Include all of your calculations, the version of the map you used to estimate the areas of the different regions of the ice sheet, and your answers to all the "explain" portions of the questions above.

[11] Technical Note: In case you are worried about the fact that we are ignoring the spherical shape of the Earth, this method is actually sufficient (i.e., accurate enough) for our purposes because of the fact that the depth of the water would be much smaller than the radius of the Earth. Here's one way to think about it: if there is no land area, then adding the water from the melted ice sheet is equivalent in mathematical terms to changing the radius of the spherical Earth from r to $r+\Delta r$, with Δr representing the (unknown) depth of the new water, much smaller than r itself. The volume of the added water is the difference between the volume of the larger sphere of radius $r + \Delta r$ and the volume of the original sphere of radius r:

$$\frac{4\pi(r+\Delta r)^3}{3} - \frac{4\pi r^3}{3} = 4\pi r^2 \times \Delta r + 4\pi r \times (\Delta r)^2 + \frac{4\pi(\Delta r)^3}{3}.$$

Since we assume Δr is much smaller than r, the last two terms on the right are negligible in size compared to the first term and we obtain an estimate

$$\text{volume of added water} \doteq 4\pi r^2 \times \Delta r = \text{surface area of sphere} \times \Delta r.$$

The change in sea level is then approximated by

$$\Delta r \doteq \frac{\text{volume of added water}}{\text{surface area of sphere}}.$$

The same idea works even if the water covers only a portion of the surface area of the sphere. The denominator would be replaced by the surface area of the portion of the Earth covered by water in that case.

Exercises

1. Express a volume of 5343 cubic centimeters in terms of liters and then cubic meters.
2. Express a weight of 4.3 tons in pounds and then in ounces.
3. Express the volume 8 fl. oz. in terms of liters and then milliliters.
4. Express an area of 130 square kilometers in terms of square miles, square yards, and finally square feet.
5. On July 12, 2017, the Larsen C ice shelf broke free of the rest of the Antarctic ice shelves and became a free-floating iceberg. Its total area at the time was estimated to be 44,200 square kilometers (roughly the size of the state of Delaware).
   (a) What was the area of the new iceberg in square miles?
   (b) The average thickness of the Larsen C ice shelf was estimated at 350 meters. Estimate the total volume of ice contained in the new iceberg at the time it separated, in cubic kilometers, then in cubic feet.
6. How many minutes have passed since 12:00 noon on January 20, 2017? (Don't estimate. Find a value as close to exact as possible. Your answer will depend on when you are doing this problem, of course. State what that time is!)
7. Light travels at a velocity $c = 299,792,458$ m/sec in vacuum.
   (a) What is the equivalent velocity in units of mi/hr? (Note: To get from m/sec to mi/hr, you will need to multiply by the conversion factors for mi/m and then sec/hr.)
   (b) How long is a *light year* (the distance light travels in one year), measured in km, then in mi?
   (c) At a speed of .5c, how long would it take to get from Earth to Proxima Centauri, the nearest star outside our solar system? (Look up the distance online.)
8. There is exactly one temperature where the Celsius and Fahrenheit scales give the same numerical value. What is this temperature? Show your work to determine it.
9. Estimate how many times an average human heart beats over the course of the lifetime of its "owner." Your number should be amazing if you think about it—very few things we can make with moving parts are that durable! (Hints: First estimate how many times the heart beats over a short time span like one minute. This depends on lots of things—whether the person is at rest or exercising, what the general health of the person is, how old the person is, etc. Don't worry too much about those, though. Most people spend more time in a state close to resting than in intense exercise. You'll also need to estimate the average life span of a human being. Feel free to look up any information

you need online, but think carefully about what you find and ask whether it is general enough for your purposes!)

10. Without using a calculator, compute exactly:
    (a) $\log_5(125)$.
    (b) $\log_3\left(\frac{1}{729}\right)$.
    (c) $\log_{10}(.001) \times \log_7(49)$.
    (d) $\log_2\left(\frac{16^3 \times 8}{1024}\right)$. Do this using Proposition 1.8 first, then check your work by simplifying the fraction $\frac{16^3 \times 8}{1024}$ first before taking the logarithm.
11. Using a calculator (and converting as necessary using (1.2)) find
    (a) $\log_{10}(5.34689)$, then $\log_{10}(53.4689)$, then $\log_{10}(534.689)$. Explain the pattern you are seeing using Proposition 1.8. Note that the decimal point is shifting one digit to the right each time.
    (b) $\log_7(34.333)$.
    (c) $\ln(100.3)$.
12. (Refer to (1.3).) What is the hydrogen ion activity a_{H^+} for a solution with $pH = 8$? Same question if $pH = 3$?
13. The intensities of sounds are often measured in units called *decibels*, or dB. The method involves comparing the sound pressure level measured as a result of the sound with a standard reference level corresponding to "silence," where only the resting air pressure is experienced by the measuring device. In acoustics, the following definition is used for the sound pressure level, L, measured in decibels:
$$L = 20 \times \log_{10}\left(\frac{p_m}{p_r}\right).$$
Here, p_r is the reference sound pressure $p_r = 20$ micropascals[12] in air and p_m is the measured sound pressure. From this formula, we can see that sound pressures are measured along a *logarithmic scale*, in agreement with what we said before about the Weber-Fechner law and human sense perception.
    (a) What is the sound pressure level in decibels corresponding to a measured sound pressure $p_m = 5000$ micropascals?
    (b) What is p_m if L has the value 1 dB, 2 dB, 10 dB?
    (c) A jet engine produces sound pressure at the level of 150 dB. What would the measured sound pressure p_m be then?
14. Strength of earthquakes is currently measured using the *moment magnitude* scale. The moment magnitude M is a quantity defined as
$$M = \frac{2}{3}\log_{10}(S) - 10.7,$$

[12] The pascal is a metric system unit with dimensions of force per unit area. A pressure of 1 pascal is 1 newton per square meter. The pascal has dimensions kg/(m $\times$ sec^2).

where S is the seismic moment (a quantity measured in units of force times distance and representing the energy released). This quantity is also given on a logarithmic scale. The constants $\frac{2}{3}$ and -10.7 are chosen so that the numbers generated are roughly equal to the older *Richter scale* magnitude, which used a different method.

(a) If two earthquakes have seismic moments S_1 and S_2 and moment magnitudes M_1 and M_2, show that the ratio between the two seismic moments is

$$\frac{S_1}{S_2} = 10^{\frac{3}{2}(M_1 - M_2)}.$$

(b) If one earthquake has moment magnitude $M_1 = 6$ and a second one has moment magnitude $M_2 = 5$, how much stronger is the first one in terms of the ratio between the two seismic moments? Express your ratio in decimal form and explain its meaning.

15. (For the more mathematically minded.) Look at the argument given in footnote 11. If Δr is much smaller than r, why can we say that $t_2 = 4\pi r \times (\Delta r)^2$ is negligible compared to $t_1 = 4\pi r^2 \times \Delta r$? Similarly, why can we say $t_3 = \frac{4\pi(\Delta r)^3}{3}$ is negligible compared to t_1? Hint: What are the ratios $\frac{t_2}{t_1}$ and $\frac{t_3}{t_1}$ after you simplify?

16. *Invasive species* are a problem in many habitats. If you have spent any time in the southeastern part of the U.S. (especially South Carolina, Georgia, Alabama) you have surely seen one of the most successful and persistent invasive plant species: *kudzu* vines (*Pueraria montana*), "the vines that ate the South." Like many of the most notorious examples of invasives, kudzu was intentionally introduced into its present habitat—in this case, from Japan and other parts of Asia into the U.S. in the late 1800s. The Soil Erosion Service and Civilian Conservation Corps promoted the use of kudzu as a ground cover to prevent soil erosion; other people sold the plants as ornamental ground cover. Unfortunately, the climate and environment of the South were a perfect match for kudzu and it has proliferated. The major problems caused by kudzu come from the fact that it is what has been called a "structural parasite." Rather than supporting itself, it covers and smothers both human-built structures (including power and telephone lines) and other trees and plants. It has very rapid growth and a versatile "double-barreled" reproductive strategy. Kudzu uses both sexual reproduction via flowers and seeds, and asexual self-cloning whereby a kudzu plant generates vines that root, separate from the parent plant, and grow into new genetically identical individuals.

(a) There were no kudzu plants in the U.S. before 1876. Today it covers about $3,000,000$ hectares (about $7,400,000$ acres). How much is that in square miles?

(b) (Preview of Chapter 2.) What is the average growth rate of the area covered by kudzu since 1876 in hectares per year, and also in acres per year?

(c) Another advantage of kudzu is that it has a very efficient symbiotic relationship with soil bacteria that lets the plants fix large quantities of nitrogen as ammonia (NH_3) in the soil it grows in. This lets kudzu plants essentially make their own fertilizer and grow in areas that are initially too nitrogen-poor to support other plants. According to data reported by Forseth and Irwin,[13] a stand of kudzu can fix 235 kg of nitrogen per hectare per year. How much is that in pounds per square mile per month? Not all of this ammonia stays in the soil either; it can leach into streams and lakes and cause changes in the plants and animals there as well.

17. (Preview of Chapter 2.) In the introduction to the chapter project, we stated *Archimedes' Principle* on floating objects: *an object floating in a fluid displaces a volume of the fluid that has the same mass as the object.* In this exercise, you will work out the consequences for freshwater ice floating in water. Liquid fresh water has a mass density of 1 gram/cubic centimeter. Fresh water ice is *less dense than liquid water*, at about .92 gram/cubic centimeter (at 0° C at standard atmospheric pressure). This is what makes it possible for ice to float.

    (a) One consequence is that if a quantity of fresh water freezes, its mass does not change, but it expands in volume. How much greater is the volume of ice than the volume of the water before it freezes, in percentage terms?

    (b) Suppose we have x grams of fresh water ice floating on water. What is the volume of the ice? What is the mass of the water that is displaced by the floating ice?

    (c) Now, suppose the ice melts into the water. How much space does it take up after it melts? What happens to the level of the water the ice was floating in?

[13] See [**6**].

References for Chapter 1

[1] https://www.epa.gov/watersense/how-we-use-water, consulted July 2, 2018.

[2] E. Kintisch, *Meltdown: As algae, detritus, and meltwater darken Greenland's ice, it is shrinking ever faster*, Science **24** (Feb. 2017), 788–791.

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[4] S. B. Simonsen, L. S. Sørensen, *Implications of changing scattering properties on Greenland ice sheet volume change from Cryosat-2 altimetry*, Remote Sensing of Environment **190** (2017), 207–216, ISSN 0034-4257, https://doi.org/10.1016/j.rse.2016.12.012.

[5] National Snow and Ice Data Center, http://nsidc.org/greenland-today, consulted July 17, 2018.

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