Preface

A study of differential equations is crucial for anyone with an interest in mathematics, science, or engineering. While the traditional applications of differential equations have been in physics and engineering, recent years have seen their application greatly expand into diverse areas, consistent with a recent report from the U.S. National Academies that noted, "Mathematical sciences work is becoming an increasingly integral and essential component of a growing array of areas of investigation in biology, medicine, social sciences, business, advanced design, climate, finance, advanced materials, ..." We have been significantly influenced by this perspective in choosing applications to include in this text, and, by drawing our applications from as wide a variety of fields as possible, we hope to engage and motivate students with diverse interests and backgrounds. In particular, we have included a large number of examples and exercises involving differential equations modeling in the life sciences—biology, ecology, and medicine. This reflects both the growing importance of mathematics in these fields and the broad appeal of applications drawn from them.

We've subtitled the book "Techniques, theory, and applications", reflecting our perspective that these three components of the subject are intimately tied together and mutually reinforcing. "Techniques" include not just analytic or computational methods for producing solutions to differential equations, but also qualitative methods for extracting conceptual information about differential equations and the systems modeled by them. "Theory" helps to organize our understanding of how differential equations work and codify general and unifying principles, so that the "techniques" are not just a collection of individual tools. "Applications" show the usefulness of the subject as a whole and heighten interest in both solution techniques and theory. The organization of the book interweaves these three aspects, with each building on and complementing the others.

We seize all opportunities to choose techniques which promote conceptual understanding of the subject. This explains, for example, our strong preference for the beautiful "method of annihilating operators", rather than the "method of undetermined coefficients", to solve constant coefficient linear nonhomogeneous equations. Indeed, the method of annihilating operators could be described as "the method of undetermined coefficients, with justification". Our experience, at a variety of different types of institutions, is that students respond well to this approach, which reinforces the "operator" perspective we adopt throughout the text.

We've written the text to be read and conceive of it as a *conversation* with our readers, filling in background, motivation, and a bit of storytelling that builds a bridge to the core mathematical ideas of the text. While this approach doesn't necessarily lead to the most succinct presentation possible, it makes for an eminently readable one that should prepare students for lively discussions in both traditional and flipped classroom settings. Furthermore, we believe that learning to read technical material is a valuable goal of any mathematics course. We don't assume the typical student in a differential equations course has had a great deal of experience doing this, and one of our goals is to provide the means and opportunity to gain this experience.

Overview and core material. Chapters 1, 2 (with Sections 2.7 and 2.8.2 optional), 4, and the first two or three sections of Chapter 5 form the core material for a one-semester course, with some omissions possible from the applications. This leaves significant time to cover other topics chosen by the instructor. Chapter 1 is very short and Section 1.3 is optional; we usually cover this chapter in one lecture. In Chapter 2, there is flexibility in selecting which applications to discuss, though we highly recommend that the logistic model for population growth be included among the chosen applications. Chapter 3 (numerical methods) stands alone and can safely be omitted if desired. There are some exercises scattered elsewhere throughout the text that are intended to be done with some kind of computer algebra system (these are indicated by "(CAS)"), but with a few exceptions these problems do not require material from Chapter 3. As with Chapter 2, the choice of applications to cover in Chapter 4 (including a potpourri of introductory applications in the first section and mechanical vibrations in the last section) and in Chapter 5 (forced oscillations and pharmacology models in Section 5.3 and electrical circuits in Section 5.4) is at the discretion of the instructor.

An engineering-focused course would probably turn rather quickly to Chapter 6 on Laplace transforms and Chapter 7 on power series solutions, which may be covered in either order. Neither of these chapters is required for any later chapters. Although Laplace transforms are used briefly in solving some linear nonhomogeneous systems in Chapter 9, alternative approaches are provided there. The first four sections of Chapter 7 provide a good introduction to power series methods; this can be followed by a discussion of the Frobenius method in Section 7.5.

Linear systems are the main focus of Chapters 8 and 9. Chapter 8 concentrates on planar systems, and Sections 8.4–8.7 provide tools to solve any planar homogeneous system $\mathbf{X}' = \mathbf{AX}$ when $\mathbf{A}$ is a 2×2 real matrix. Applications based on compartment models are discussed in Section 8.8; these include models from pharmacology and ecology. The first section of Chapter 9 introduces the matrix exponential for 2×2 matrices. A novel aspect of the treatment here is the development of formulas for "$e^{t\mathbf{A}}$" that combine easily with elementary vector geometry, enabling the reader to sketch the corresponding phase portraits with confidence. In Section 9.3 matrix exponentials for larger matrices are discussed using Putzer's Algorithm, which gives an efficient way to discuss $e^{t\mathbf{A}}$ with minimal linear algebra prerequisites.

Chapter 10 on nonlinear systems is rich in applications, and one may pick and choose from these as time permits. For a course that has only limited time available for this material, Sections 10.1 and 10.2 and an application or two chosen from Sections 10.4 and 10.5 provide a coherent introduction that could immediately follow a study of linear systems in Sections 8.1–8.7.

An introduction to partial differential equations and Fourier series is found in Chapter 11. After sections presenting two-point boundary value problems and a first look at partial differential equations via the advection and diffusion equations, we discuss the elementary theory of Fourier series in Sections 11.5 and 11.6. Our treatment emphasizes not only the role of orthogonality, but also the analogy between vectors in Euclidean space and periodic functions on the line as well as that between the Euclidean dot product and the integral inner product of functions. In Sections 11.7–11.10 these ideas are applied to study the heat and wave equations in one space variable and Laplace's equation in two variables.

Exercises. There are a great number and a great variety of exercises. We have tried to provide exercises that will contain genuine appeal for students. In some cases, that means a bit of a "back story" is included in the exercise. The same applies to many of the examples in the text. While this adds some length to the text, we strongly believe the richness this provides well compensates for this additional length. When an exercise is referred to in the text, only its number is given if the exercise appears within the same section; otherwise both its number and the section in which it appears are specified.

In writing the exercises, we have taken to heart one of the lessons that came out of the recent MAA calculus study "Characteristics of successful calculus programs", namely, that asking students to work problems that require them to grapple with concepts (or even proofs) and do modeling activities is key to successful student experiences and retention in STEM programs. There are also exercises that are explicitly constructed to help students identify common misconceptions. Some of these (the "Sam and Sally problems") present two reasonable sounding, but contradictory, approaches to a question and ask which is right (and why).

Projects: Informal and formal. Throughout the text you will find exercises that are suitable for more in-depth study and analysis by students, working either individually or in groups. While the difficulty of these vary, all give scope for bringing the ideas of the course to bear on interesting questions of either an applied or theoretical nature.

The text website www.ams.org/bookpages/mbk-125 also includes a more formal collection of projects, identified with the prerequisite text material. Many of these provide students with opportunities (and motivation) to explore applications of differential equations in depth, and they will often include parts intended to be completed with the use of a computer algebra system. To facilitate this, each such project is accompanied by a "starter" *Mathematica* notebook or *Maple* worksheet that illustrates the relevant commands. Students can complete the CAS-based part of the project directly in the starter notebook or worksheet. Some of the web projects also provide an opportunity to test differential equations models against real-world data (employing, for example, least-squares analyses). We believe that most students benefit from working in groups and hence suggest that students be permitted to work with a partner from their class on these web projects (submitting for evaluation one project report for the partnership).

Prerequisites. The prerequisite for the vast majority of the text is a standard year-long course in single variable calculus. Some ideas from multivariable calculus make an appearance in a few places (e.g., partial derivatives), but any needed background is provided. We have taken pains to include a review of certain topics from single variable calculus—as they appear—that experience has told us some students will need. We have particularly wanted to minimize our expectations of student familiarity with linear algebra. In the material before Chapter 8, the only matrix algebra that makes an essential appearance is the determinant of a 2×2 matrix. In Chapter 8, where systems are introduced, we devote several subsections to presenting "from the ground up" relevant matrix algebra topics.

The role of "proof". A fair number of theorems are stated and discussed in the text. We carefully explain what each theorem says (and doesn't say), illustrating with examples that clarify the content. Our perspective on "proof" is that it is first and foremost an explanation of why something is true. When a formal proof will aid the student's understanding, it is included. At times we will instead give a "justification" of the theorem, which is a more informal argument that contains the ideas of the proof in a more conversational format. Similarly, while some exercises do ask the student to provide a proof of some statement, more are designed to draw out the *ideas* that would go into a formal proof.

Modularity and sample syllabi. After the core material, described above, from Chapters 1, 2, 4, and 5, the text offers a great deal of flexibility in terms of material that may be covered as well as the order in which it may be presented. For a semester-long course, some sample syllabi that have been used are:

- Chapters 1, 2, 4, and 5 (with some choices made as to applications included), Sections 8.1–8.7 on planar linear systems (possibly including an introduction to the matrix exponential in Section 9.1), some compartment model applications from Section 8.8, Sections 10.1 and 10.2 on nonlinear planar systems, followed by applications chosen from Sections 10.4 (interacting populations), 10.5 (epidemiology), and 10.7 (pendulums). As time permits, an introduction to power series solutions may be included (Sections 7.1–7.3).
- A course that is more focused on engineering students might begin as above with Chapters 1 and 2, then turn to numerical methods in Chapter 3, followed by Chapters 4, 5, and 6 (Laplace transforms). If Chapter 6 will be covered, then many omissions are possible in Chapter 5. There should also be time for some discussion of power series solutions and/or planar systems (e.g., Sections 7.1–7.3 and Sections 8.1–8.7).
- At the University of Virginia, we have occasionally offered a special section of our differential equations course for physics students. This includes Chapters 1, 2, 4, and 5, some discussion of planar linear systems (through Section 8.7), and power series solutions (through Section 7.3) and then culminates in an introduction to partial differential equations and Fourier series (the first part of Chapter 11, through Section 11.7 or 11.8).

For a two-quarter or two-semester sequence, there is, of course, scope for quite a bit of additional material to be covered. We want to emphasize that once the core material has been discussed, the remaining topics (numerical methods, Laplace transforms, power series solutions, linear and nonlinear systems, partial differential equations, and Fourier series) are largely independent of each other, and one can pick and choose from these in whatever order and with whatever emphasis are desired.

Supplementary resources. A complete and detailed solution manual, prepared by the authors, is available to instructors. It contains solutions to all exercises. A student solution manual with solutions to all odd numbered exercises is also available. Further information about the solution manuals can be found at the text website www.ams.org/bookpages/mbk-125. In collaboration with Cengage, many exercises are available through WebAssign. Details can be found at the text website.

To the student. This text is meant to be read. Reading the text will deepen your understanding of the material; moreover, learning to read technical material is a worthy goal with wide applicability beyond just this particular subject. We've written the text to be a conversation with you, the reader, and scattered throughout you will find some nonmathematical stories that set the stage for the mathematics that follows. If an exercise looks "long", it is probably because it is designed to talk you through the problem, much as we would if we were talking to you during office hours.

We hope you will come away from this text with the strong sense that differential equations are lurking around every corner in everyday life. There are dozens of examples and exercises here that are motivated by recent news articles, blog posts, movies and TV shows, and current pop culture. No matter what interests you have outside of mathematics, we hope you will find in the myriad applications of differential equations discussed here something that connects with these interests. And we'd love to hear from you if you find an interesting application of differential equations you'd like to see included in a future edition.

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