Now (2.54) has equilibrium solution $y(t) = 0$ by "inspection". This corresponds to the tank being empty for all time. If we solve the differential equation by separating variables, we see that

$$\frac{1}{2\sqrt{y}}\,dy = -dt,$$

or

$$\sqrt{y} = -t + c. \tag{2.55}$$

We can go one step further and solve for y explicitly as $y = (c-t)^2$; however, this requires one bit of caution. To see why, let's substitute the function $y = (c-t)^2$ into the differential equation of (2.54). The result is

$$\begin{aligned} -2(c-t) &= -2\sqrt{(c-t)^2} \\ &= -2|c-t| \end{aligned}$$

since $\sqrt{a^2} = |a|$. This is satisfied only if $c - t \geq 0$, or $t \leq c$. Thus we have solutions to $\frac{dy}{dt} = -2\sqrt{y}$ of the form

$$y(t) = (c-t)^2 \quad \text{for } t \leq c.$$

Imposing the initial condition $y(1) = 0$ gives $c = 1$, and $y(t) = (1-t)^2$ for $t \leq 1$ solves our initial value problem. We can extend this solution to be defined for all times $-\infty < t < \infty$ by setting

$$y(t) = \begin{cases} (1-t)^2 & \text{if } t \leq 1, \\ 0 & \text{if } t > 1. \end{cases}$$

The graph of this function is shown in Fig. 2.22.

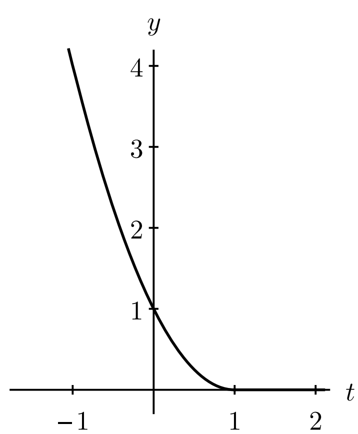

Figure 2.22. Graph of $y(t)$.

In Exercise 12 you are asked to verify that this extension is in fact differentiable at every point and satisfies the differential equation everywhere. In our physical set-up, this solution gives the height of the water in a tank that *just empties* at time $t = 1$ and stays empty for all times $t > 1$. This perspective suggests how we might find many other solutions to (2.54), namely

$$y = \begin{cases} (c-t)^2 & \text{if } t \leq c, \\ 0 & \text{if } t > c, \end{cases} \tag{2.56}$$

for *any choice* of $c \leq 1$. Now the tank just empties at time $t = c$ and stays empty for later times. As long as $c \leq 1$, the height of water at time $t = 1$ is 0, so that the initial condition $y(1) = 0$ is met. Fig. 2.23 shows the graphs of several of these solutions to (2.54).

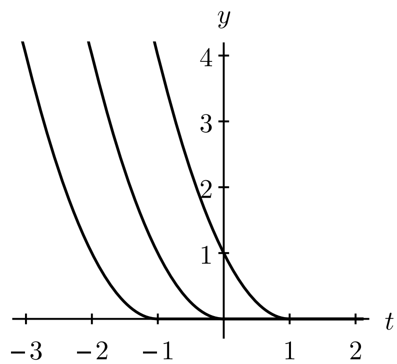

Figure 2.23. Several solutions to the initial value problem in (2.54).

Summarizing, the initial value problem in (2.54) does not have a *unique* solution. In fact we have found infinitely many solutions: the equilibrium solution and the solutions in (2.56) for any $c \leq 1$. We even gave a physical interpretation to the problem in which these infinitely many solutions are each meaningful.

Existence and uniqueness for linear equations. We begin with an example of a *linear* initial value problem.

Example 2.4.2. Consider the equation

$$\frac{dy}{dx} + y = xe^{-x}. \tag{2.57}$$

This is a linear equation with independent variable x, in standard form, having coefficient functions $p(x) = 1$ and $g(x) = xe^{-x}$. These functions are continuous on $-\infty < x < \infty$. Multiplying by the integrating factor e^x we can rewrite the equation equivalently as

$$\frac{d}{dx}(e^x y) = x \tag{2.58}$$

so that

$$e^x y = \frac{x^2}{2} + c,$$

or

$$y = \frac{x^2 e^{-x}}{2} + ce^{-x}, \tag{2.59}$$

for arbitrary constant c.

This is a solution to (2.57) for all x in $(-\infty, \infty)$. Before we impose an initial condition, let's pause to observe that the functions in (2.59) form the *complete* family of all solutions to (2.57). How do we know this? Equation (2.57) is equivalent to equation (2.58), meaning that they have exactly the same solutions. From calculus (or more precisely, from the Mean Value Theorem) we know that the only functions with derivative x are the functions $x^2/2 + c$ for c an arbitrary constant, so that if y solves (2.58), it must appear in the family (2.59).

Now suppose we add an initial condition:

$$y(x_0) = y_0, \tag{2.60}$$

where x_0 and y_0 can each be any real number. There will be exactly one choice of the constant c in (2.59) so that the resulting function y satisfies our initial condition. You can see this by substituting the values and solving for c:

$$y_0 = \frac{x_0^2 e^{-x_0}}{2} + ce^{-x_0}$$

if and only if

$$c = y_0 e^{x_0} - \frac{x_0^2}{2}.$$

In summary, the initial value problem consisting of (2.57) and (2.60) has *exactly one* solution, given by (2.59) with c as just determined. This solution is valid for $-\infty < x < \infty$.

The next example also uses a linear differential equation, but unlike the last example, the coefficient function $p(x)$ has a discontinuity.

Example 2.4.3. Every solution to the linear equation

$$\frac{dy}{dx} + \frac{1}{x}y = 0$$

is a solution to the equation obtained by multiplying both sides by x:

$$x\frac{dy}{dx} + y = 0.$$

Since this latter equation is exactly

$$\frac{d}{dx}(xy) = 0,$$

its complete family of solutions is given by

$$xy = c$$

for c an arbitrary constant. So long as we choose a value of $x_0 \neq 0$ we may find a function in the family

$$y = \frac{c}{x}$$

that satisfies any desired initial condition $y(x_0) = y_0$; just choose $c = x_0 y_0$. Summarizing, for $x_0 \neq 0$, the initial value problem

$$\frac{dy}{dx} + \frac{1}{x}y = 0, \quad y(x_0) = y_0,$$

has unique solution

$$y = \frac{x_0 y_0}{x}.$$

The interval of validity for this solution is the largest (open) interval containing x_0 and not containing 0.

What we saw in the last two examples—the existence and uniqueness of a solution to a *linear* first-order initial value problem on an interval on which the coefficient functions $p(x)$ and $g(x)$ are continuous—typifies all linear initial value problems. The next result summarizes this situation.

Theorem 2.4.4. *Consider the linear equation*

$$\frac{dy}{dx} + p(x)y = g(x)$$

with initial condition $y(x_0) = y_0$*. If the functions* p *and* g *are continuous on an open interval* (a, b) *containing the point* $x = x_0$*, then there exists a unique function* $y = \varphi(x)$ *defined for* $a < x < b$ *that solves the differential equation*

$$\frac{dy}{dx} + p(x)y = g(x)$$

and also satisfies the initial condition

$$y(x_0) = y_0,$$

where y_0 is an arbitrary prescribed value.

One way to structure a proof of Theorem 2.4.4 is to think through the algorithm we have for solving first-order linear initial value problems and see that under the hypotheses of the theorem this algorithm will always produce a solution, and no other solution is possible. This approach is outlined in Exercise 30.

Existence and uniqueness for nonlinear equations. An important feature of Theorem 2.4.4 is the upfront guarantee that the solution exists for all x in the interval (a, b) on which p and g are continuous. Things are not so nice when we look at *nonlinear* initial value problems. The next result is an "existence and uniqueness" theorem that can be used for nonlinear initial value problems. The notion of a partial derivative for a function of more than one variable appears in the statement. If $f(x, y)$ is a function of the two variables x and y, the partial derivative $\frac{\partial f}{\partial y}$ is computed by holding x constant and differentiating "with respect to y". For example, if $f(x, y) = x^2y + 2y^3e^x + x$, then $\frac{\partial f}{\partial y} = x^2 + 6y^2e^x$. You'll find more examples below, and you can find further discussion of partial derivatives in Section 2.8.2.

Theorem 2.4.5. *Consider the initial value problem*

$$\frac{dy}{dx} = f(x, y), \; y(x_0) = y_0. \tag{2.61}$$

(a) **Existence.** *Suppose that the function f is continuous in some open rectangle $\alpha < x < \beta$, $\gamma < y < \delta$ containing the point (x_0, y_0). Then in some open interval containing x_0 there exists a solution $y = \varphi(x)$ of the initial value problem* (2.61)*.*

(b) **Uniqueness.** *If in addition, the partial derivative $\frac{\partial f}{\partial y}$ exists and is continuous in the rectangle $\alpha < x < \beta$, $\gamma < y < \delta$, then there is an open interval I containing x_0 and contained within (α, β) on which the solution to* (2.61) *is uniquely determined. (Thus if $y_1(x)$ and $y_2(x)$ are solutions to* (2.61) *on I, then $y_1(x) = y_2(x)$ for all x in I.)*

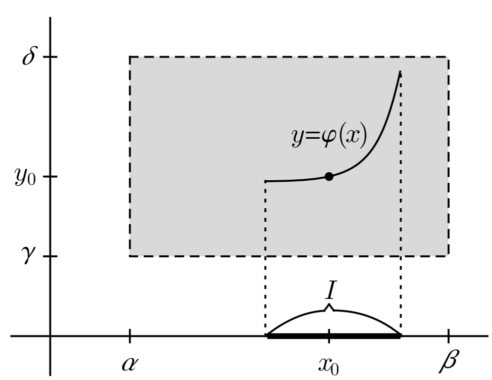

Figure 2.24. Theorem 2.4.5: If $f = f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on the shaded region, then the initial value problem (2.61) has, on some open interval I containing x_0, a solution $y = \varphi(x)$ that is uniquely determined.

The next example illustrates some of the features of this theorem.

Example 2.4.6. Consider the nonlinear initial value problem

$$\frac{dy}{dx} = y^2, \; y(0) = y_0$$

for any real number y_0. Note the initial value of x here is $x_0 = 0$. To apply Theorem 2.4.5 we first find an open rectangle containing $(0, y_0)$ on which both $f(x, y) = y^2$ and $\frac{\partial f}{\partial y} = 2y$ are continuous functions. Since these functions have no discontinuities, the rectangle can be chosen to be the whole plane, $-\infty < x < \infty$, $-\infty < y < \infty$. Theorem 2.4.5 allows us to conclude that once we choose a value y_0, there will exist *some* interval containing $x_0 = 0$ and contained in $(-\infty, \infty)$ on which a unique solution exists. It makes no a priori guarantee about the size of this interval. To further illustrate this, observe that if $y_0 = 0$, this unique solution is the equilibrium solution $y(x) \equiv 0, -\infty < x < \infty$. For other values of y_0 we first solve the differential equation by separating variables:

$$y^{-2}\, dy = \, dx,$$

so that

$$\int y^{-2} dy = \int dx.$$

Integrating gives $-y^{-1} = x + c$, or

$$y = -\frac{1}{x + c}.$$

Solving for c by plugging in $x = 0$ and $y = y_0$, we get that $c = -\frac{1}{y_0}$. So for each initial value y_0, there is a unique solution

$$y = \frac{1}{\frac{1}{y_0} - x} = \frac{y_0}{1 - y_0 x}.$$

Since the denominator $1 - y_0 x$ is 0 at $x = 1/y_0$, the interval of validity for this solution is the largest open interval containing $x_0 = 0$ but not containing $x = 1/y_0$.

Let's examine our solution for some different initial values y_0.

y_0	$y(x)$	interval of validity
1	$\frac{1}{1-x}$	$x < 1$
2	$\frac{2}{1-2x}$	$x < \frac{1}{2}$
-1	$\frac{-1}{1+x}$	$x > -1$
-2	$\frac{-2}{1+2x}$	$x > -\frac{1}{2}$
0	0	$(-\infty, \infty)$
$\frac{1}{2}$	$\frac{1}{2-x}$	$x < 2$

Thus even though the functions

$$f(x, y) = y^2$$

and

$$\frac{\partial f}{\partial y} = 2y$$

are defined and continuous for all x and y, the solution $y(x)$ to our initial value problem may have its interval of validity restricted to some subinterval of the x-axis, and in fact the subintervals change as the value of y_0 changes. Furthermore, suppose we pick one of the solutions we found, for example,

$$y(x) = \frac{1}{1 - x}.$$

This solution was determined by the initial condition $y(0) = 1$, so its graph passes through the point $(x_0, y_0) = (0, 1)$. We say the domain of this solution is the interval $-\infty < x < 1$, even though it's also defined on $1 < x < \infty$ and satisfies the differential equation there, because the x-coordinate (namely 0) of our initial value point $(x_0, y_0) = (0, 1)$ lies in $-\infty < x < 1$ but not in $1 < x < \infty$; see Fig. 2.25. Note that $x = 1$ is a vertical asymptote of the solution passing through $(0, 1)$. In general, $x = \frac{1}{y_0}$ is a vertical asymptote of the solution passing through $x = 0$, $y = y_0$, provided $y_0 \neq 0$.

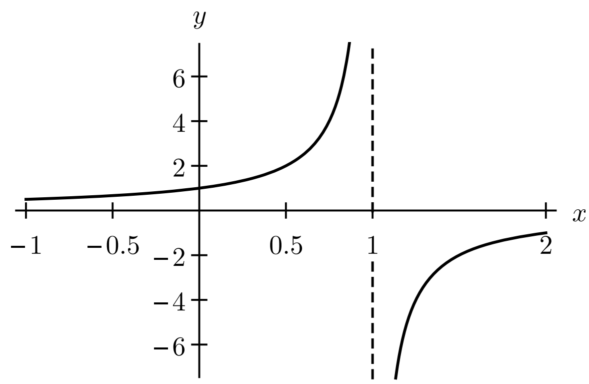

Figure 2.25. Graph of $y(x) = (1 - x)^{-1}$.

Example 2.4.7. Consider the initial value problem

$$\frac{dy}{dx} = 2x - 2\sqrt{x^2 - y}, \ y(1) = 1.$$

You can verify by substitution that both $y_1(x) = x^2$ and

$$y_2(x) = \begin{cases} x^2 & \text{if } x < 1, \\ 2x - 1 & \text{if } x \geq 1 \end{cases}$$

are solutions to this initial value problem on $(-\infty, \infty)$. (For more on where these solutions came from, see Exercise 31.) Thus this initial value problem does not have a unique solution. Why does this not contradict Theorem 2.4.5? Let's check the hypotheses of the theorem. Set

$$f(x, y) = 2x - \sqrt{x^2 - y},$$

and notice that $f(x, y)$ is continuous only whenever $x^2 - y \geq 0$ (it is not even defined if $x^2 - y < 0$). Furthermore,

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x^2 - y}},$$

which is continuous in that part of the xy-plane where $x^2 - y > 0$. Fig. 2.26 shows the part $\mathcal{S}$ of the xy-plane on which both f and $\frac{\partial f}{\partial y}$ are continuous. There is no open rectangle containing our initial condition point $(1, 1)$ and contained in $\mathcal{S}$. Therefore the theorem is not applicable, and there is no contradiction.

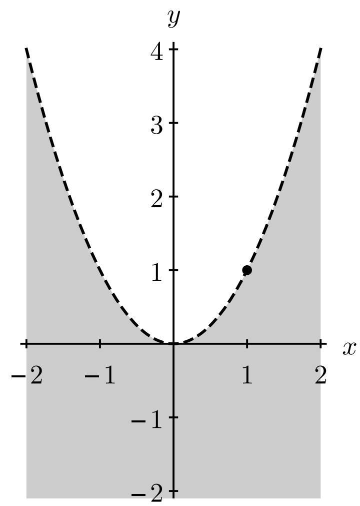

Figure 2.26. The region $\mathcal{S}$ in Example 2.4.7.

Solution curves can't intersect. As a consequence of Theorem 2.4.5, we have the following important observation.

Theorem 2.4.8 (No Intersection Theorem)**.** *Suppose that f and $\partial f/\partial y$ are continuous in an open rectangle $\mathcal{R} = \{(x, y) : \alpha < x < \beta, \gamma < y < \delta\}$ and that I is a open subinterval of (α, β). Suppose also that $y_1(x)$ and $y_2(x)$ are two solutions of $\frac{dy}{dx} = f(x, y)$ having common domain I whose graphs are both contained in $\mathcal{R}$. If $y_1(x)$ and $y_2(x)$ are not identical, then their graphs cannot intersect.*

For example, consider the differential equation $\frac{dy}{dx} = y(1 - y)$. Here $f(x, y) = y(1 - y)$, and this function as well as $\partial f/\partial y = 1 - 2y$ are continuous in $-\infty < x < \infty, -\infty < y < \infty$. The constant functions $y(x) = 0$ and $y(x) = 1$ are solutions to this differential equation on the interval $-\infty < x < \infty$; their graphs are horizontal lines as shown in Fig. 2.27. The graph of any other solution cannot cross either of these lines, since if (x_0, y_0) were such an intersection point, the initial value problem

$$\frac{dy}{dx} = y(1 - y),\ y(x_0) = y_0$$

would have more than one solution on some interval containing x_0, in contradiction to Theorem 2.4.5.

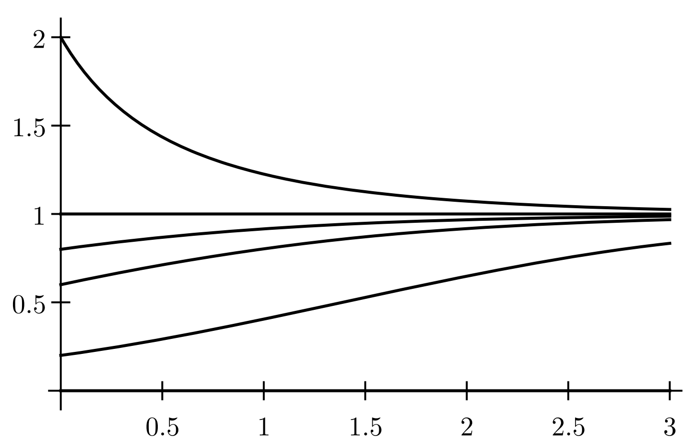

Figure 2.27. Some solutions of $\frac{dy}{dx} = y(1 - y)$.

2.4.1. Comparing linear and nonlinear differential equations. Here we summarize several key qualitative differences between first-order linear and nonlinear differential equations.

Existence and uniqueness for initial value problems. Theorem 2.4.4 assures us that if $p(x)$ and $g(x)$ are continuous on an open interval containing x_0, then the *linear* initial value problem

$$\frac{dy}{dx} + p(x)y = g(x), \quad y(x_0) = y_0,$$

has a unique solution on that interval. Any restriction on the interval of validity is apparent from the discontinuities of the coefficients $p(x)$ and $g(x)$ in the equation. By contrast, for a nonlinear initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0, \tag{2.62}$$

even when $f(x, y)$ is continuous for all x and y, there may be a restriction on the interval of validity (see Example 2.4.6), or there may be more than one solution (see Exercise 13). While Theorem 2.4.5 gives conditions under which a unique solution to (2.62) exists, we do not have a simple way of determining the interval of validity from looking at the differential equation itself.

General solutions. The method we learned in Section 2.1 for solving a linear differential equation

$$\frac{dy}{dx} + p(x)y = g(x) \tag{2.63}$$

on an interval I on which p and g are continuous produces an explicit solution with one arbitrary constant. This one-parameter family of solutions to (2.63) on I is the **general solution**: Every solution to the equation on I is included in the family.

For nonlinear equations

$$\frac{dy}{dx} = f(x, y),$$

even when we are able to produce a one-parameter family of solutions, it may not be complete. For example, we saw in Example 2.2.2 that separation of variables leads to the solutions

$$y = -1 + \left(\frac{x^2}{6} - x + C\right)^3 \tag{2.64}$$

for the equation

$$\frac{dy}{dx} = (x - 3)(y + 1)^{2/3}.$$

However, we also have (for example) the equilibrium solution $y = -1$, which is not obtained from (2.64) by any choice of C. For nonlinear equations it is also possible for nonequilibrium solutions to be missing from a one-parameter family of solutions; you can see a number of examples in the exercises, such as Exercise 13 or Exercise 31. Thus, in comparison with linear equations, knowing when we have obtained all solutions to a first-order nonlinear differential equation is a more difficult problem. Some positive results when the nonlinear equation is separable are discussed below, in Section 2.4.2. Finally, we note that the methods we have for solving nonlinear equations (separation of variables in Section 2.2 and other methods we will develop in Sections 2.7 and 2.8) typically produce *implicit* solutions, while linear equations can be solved explicitly.

2.4.2. Revisiting separable equations. We will use the No Intersection Theorem to help answer the following question: When can we know we have found *all* solutions of a separable equation

$$\frac{dy}{dx} = g(x)h(y)? \tag{2.65}$$

Recall that we have procedures to produce two general types of solutions:

- Separate variables to write $\frac{dy}{h(y)} = g(x)\,dx$ and obtain implicit solutions of the form
$$H(y) = G(x) + c$$
where $G'(x) = g(x)$, $H'(y) = \frac{1}{h(y)}$, and c is an arbitrary constant.
- Find the equilibrium solutions $y = b_j$ by determining the constants b_j satisfying $h(b_j) = 0$.

Now we ask when we can be assured that *any* solution of equation (2.65) will be one of these two types. An answer is provided by the next result.

Theorem 2.4.9. *Suppose that $g = g(x)$ is continuous on $(-\infty, \infty)$ and that $h = h(y)$ is differentiable with continuous derivative on $(-\infty, \infty)$. Suppose further that $h(y) = 0$ precisely for the values $y = b_1, b_2, \ldots, b_n$. Let G be an antiderivative of g on $(-\infty, \infty)$ (so that $G'(x) = g(x)$) and let H be an antiderivative of $\frac{1}{h}$ at all points except where $h(y) = 0$ (so that $H'(y) = 1/h(y)$ for all y not in the set $\{b_1, b_2, \ldots, b_n\}$). Then if $y = \varphi(x)$ is any solution whatsoever of*

$$\frac{dy}{dx} = g(x)h(y),$$

then either y is an equilibrium solution $y(x) = b_j$ for some j in $\{1, 2, \ldots, n\}$ or there is a constant c such that

$$H(\varphi(x)) - G(x) = c \quad \textit{for all } x \textit{ in the domain of } y = \varphi(x).$$

To see this theorem in action, we revisit two earlier examples. First, recall the separable equation $\frac{dy}{dx} = y^2$, which we solved in Example 2.4.6. Note that the hypotheses of Theorem 2.4.9 apply, since $g(x) = 1$ is continuous on $(-\infty, \infty)$ and $h(y) = y^2$ is differentiable with continuous derivative $2y$ on $(-\infty, \infty)$. The function $h(y) = y^2$ is 0 only at the point $b_1 = 0$, so that $y = 0$ is the only equilibrium solution. An antiderivative for g is $G(x) = x$, and an antiderivative for $\frac{1}{h}$ is $H(y) = -y^{-1}$. Separating variables gives the implicit solutions $H(y) - G(x) = c$ or $-y^{-1} - x = c$, for any constant c. Some algebra lets us write these implicit solutions as $y = -\frac{1}{x+c}$, a function having domain all real numbers except $-c$. Theorem 2.4.9 guarantees that we have found all solutions of $\frac{dy}{dx} = y^2$.

By contrast, let's look again at the equation $\frac{dy}{dx} = (x-3)(y+1)^{2/3}$ considered in Example 2.2.2. Here $g(x) = x - 3$ is continuous on $(-\infty, \infty)$ with antiderivative $G(x) = \frac{x^2}{2} - 3x$, but $h(y) = (y+1)^{2/3}$, while having a continuous derivative on $(-\infty, -1)$ and $(-1, \infty)$, fails to be differentiable at $y = -1$. Thus Theorem 2.4.9 does not apply. Moreover, while we have equilibrium solution $y = -1$ and the separation of variables solutions $3(y + 1)^{1/3} - (\frac{x^2}{2} - 3x) = c$ (or equivalently $y = -1 + (\frac{x^2}{6} - x + c)^3$), we have many other solutions as well, including

$$y(x) = \begin{cases} -1 + \left(\frac{x^2}{6} - x\right)^3 & \text{if } x \le 0, \\ -1 & \text{if } x > 0 \end{cases}$$

and

$$y(x) = \begin{cases} -1 + \left(\frac{x^2}{6} - x + \frac{5}{6}\right)^3 & \text{if } x \le 1, \\ -1 & \text{if } x > 1. \end{cases}$$

These are nonequilibrium solutions which are not simply obtained by some choice of constant in the implicit solution $3(y+1)^{1/3} - (\frac{x^2}{2} - 3x) = c$.

Proof of Theorem 2.4.9. The hypotheses on the functions g and h guarantee that we can apply the No Intersection Theorem, Theorem 2.4.8, on the rectangle $\mathcal{R}$ consisting of the entire plane $-\infty < x < \infty$, $-\infty < y < \infty$, since $f(x,y) = g(x)h(y)$ and $\frac{\partial f}{\partial y} = g(x)h'(y)$ are continuous in $\mathcal{R}$. The antiderivatives $G(x)$ and $H(y)$ exist by the Fundamental Theorem of Calculus.

Now suppose that $y = \varphi(x)$ is a nonequilibrium solution to $\frac{dy}{dx} = g(x)h(y)$ on some open interval I. Note that this means $\varphi'(x) = g(x)h(\varphi(x))$ on I. The No Intersection Theorem tells us that φ never takes the values $b_1, b_2, \ldots, b_n$, since the graph of φ cannot intersection the graph of any equilibrium solution $y = b_j$. This means that for x in I, the values of $\varphi(x)$ are in the domain of $H(y)$ and we compute by the chain rule that

$$\frac{d}{dx}[H(\varphi(x)) - G(x)] = H'(\varphi(x))\varphi'(x) - G'(x) = \frac{1}{h(\varphi(x))}\varphi'(x) - g(x) = \frac{\varphi'(x) - g(x)h(\varphi(x))}{h(\varphi(x))} = 0$$

for all x in I. Thus there is a constant c so that

$$H(\varphi(x)) - G(x) = c$$

for all x in I, as desired.

2.4.3. Exercises.

For the equations in Exercises 1–4, find the largest open interval on which the given initial value problem is guaranteed, by Theorem 2.4.4, to have a unique solution. Do not solve the equation.

1. $\frac{dy}{dx} + y = \tan x,\ y(1) = 2.$
2. $\frac{dy}{dx} + y = \sqrt{\sin^2 x + 1},\ y(3) = 2.$
3. $x\frac{dy}{dx} + (\cos x)y = \ln(x+2),\ y(-1) = 2.$
4. $\ln(x+3)\frac{dy}{dx} + (\sin x)y = e^x,\ y(0) = 2.$

In Exercises 5–10, determine whether Theorem 2.4.5 ensures existence of a solution to the initial value problem. If so, determine whether the theorem guarantees that there is some open interval containing the initial value of the independent variable on which the solution is unique.

5. $\frac{dy}{dx} = x^2 + y^2,\ y(1) = 2.$
6. $\frac{dy}{dx} = y^{1/3},\ y(2) = 0.$
7. $\frac{dy}{dx} = \sqrt{y},\ y(1) = 0.$
8. $\frac{dy}{dx} = \frac{x}{y^2 - 4},\ y(2) = 1.$
9. $\frac{dy}{dx} = \frac{x}{y^2 - 4},\ y(1) = 2.$
10. $\frac{dP}{dt} = P - 2P^2,\ P(0) = 1.$
11. Show that the initial value problem
$$x\frac{dy}{dx} = 2y, \quad y(0) = 0,$$
has infinitely many solutions. Note that the differential equation is linear. Why does this example not contradict Theorem 2.4.4?
12. Consider the initial value problem
$$\frac{dy}{dt} = -2\sqrt{y}, \quad y(1) = 0, \tag{2.66}$$
which we discussed in Example 2.4.1.

(a) Pick any number c and show that the function

$$y(t) = \begin{cases} (c-t)^2 & \text{if } t \le c, \\ 0 & \text{if } t > c \end{cases}$$

is differentiable at every real number t. Pay particular attention to the point $t = c$; you may find it helpful to talk about "left-hand" and "right-hand" derivatives at this point.

(b) The direction field for the equation

$$\frac{dy}{dt} = -2\sqrt{y}$$

is shown in Fig. 2.28. Explain how you can tell from this picture that the function $y = (c-t)^2$ cannot solve this equation for all $-\infty < t < \infty$.

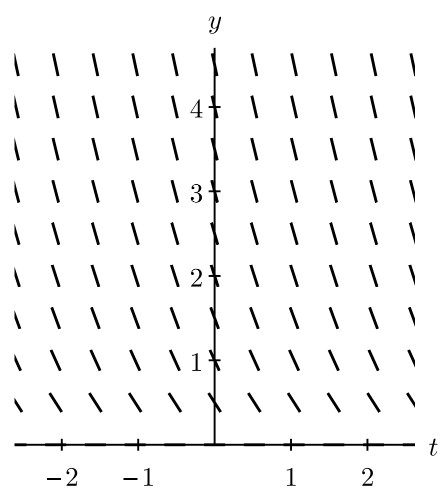

Figure 2.28. Direction field.

13. (a) If the equation $\frac{dy}{dx} = 3y^{2/3}$ is solved by separating variables, what one-parameter family of solutions is obtained?

(b) Consider the function

$$y = \begin{cases} (x-a)^3 & \text{if } x \le a, \\ 0 & \text{if } a < x < b, \\ (x-b)^3 & \text{if } x \ge b, \end{cases}$$

where a is an arbitrary negative number and b is an arbitrary positive value. Sketch the graph of this function. Is y differentiable at $x = a$? Is it differentiable at $x = b$? If your answer is yes, what are $y'(a)$ and $y'(b)$?

(c) Show that the function y defined in (b) solves the initial value problem $\frac{dy}{dx} = 3y^{2/3}$, $y(0) = 0$. Notice that as a ranges over all negative numbers and b ranges over all positive numbers, this gives infinitely many solutions (in fact, a two-parameter family's worth) to this initial value problem. Why does this not contradict Theorem 2.4.5?

(d) Does Theorem 2.4.9 apply to the differential equation $\frac{dy}{dx} = 3y^{2/3}$?

14. Find two different solutions to the initial value problem

$$4\frac{dy}{dx} = 5y^{1/5}, \quad y(1) = 0,$$

on $(-\infty, \infty)$. Why doesn't this contradict Theorem 2.4.5? Hint: You may want to define one of your solutions in a piecewise manner.

15. The two initial value problems

$$\frac{dy}{dt} = 1 - y^2, \quad y(0) = 0,$$

and

$$\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0,$$

look superficially quite similar. Solve each of these initial value problems, showing that the first has solution

$$y = \frac{e^{2t} - 1}{e^{2t} + 1}$$

and the second has solution $y = \tan t$. What is the interval of validity for each of these solutions?

16. Various solution curves of the differential equation

$$\frac{dy}{dx} = 2x^2 - xy^2$$

are shown in Fig. 2.29. Which of the following statements is correct? Note: If two curves are sometimes closer together than the width of the instrument that draws them, the curves may appear to be the same in that part of the picture.

(a) Any two distinct solution curves shown have exactly one point of intersection.
(b) No two distinct solution curves shown have a point of intersection.
(c) Any two of the distinct solution curves shown have more than one point of intersection.
(d) It is impossible to know anything about points of intersection without first solving the differential equation.

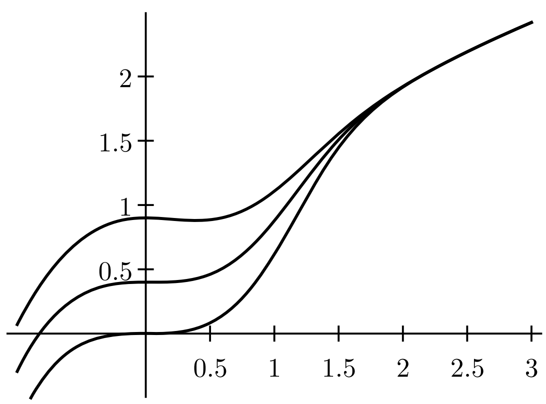

Figure 2.29. Some solutions of $\frac{dy}{dx} = 2x^2 - xy^2$.

17. (a) Show that

$$y(x) = 1 \quad \text{and} \quad y(x) = \frac{(x-1)^3}{27} + 1$$

both solve the initial value problem

$$\frac{dy}{dx} = (y-1)^{2/3}, \quad y(1) = 1.$$

(b) Suppose we define

$$g(x) = \begin{cases} \frac{(x-1)^3}{27} + 1 & \text{for } x < 1, \\ 1 & \text{for } x \geq 1 \end{cases}$$

and

$$h(x) = \begin{cases} 1 & \text{for } x < 1, \\ \frac{(x-1)^3}{27} + 1 & \text{for } x \geq 1. \end{cases}$$

Do g and h also solve the initial value problem in (a)? In answering this, check whether g and h are differentiable at $x = 1$.

18. Show that the differential equation

$$\left(\frac{dy}{dt}\right)^2 + y^2 + 1 = 0$$

has no solutions. Why doesn't this contradict Theorem 2.4.5?

19. Though we think of Theorem 2.4.5 as an existence and uniqueness theorem for *nonlinear* initial value problems, the hypotheses do not *require* this.

(a) If a linear equation

$$\frac{dy}{dx} + p(x)y = g(x)$$

is written in the form

$$\frac{dy}{dx} = f(x, y),$$

what is $f(x, y)$?

(b) Show that if $p(x)$ and $g(x)$ are continuous on an interval (a, b), then $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on the infinite rectangle $a < x < b$, $-\infty < y < \infty$.

(c) What conclusion does Theorem 2.4.5 let you draw about solutions to an initial value problem $\frac{dy}{dx} + p(x)y = g(x)$, $y(x_0) = y_0$, and how does this compare with the conclusion if you instead apply Theorem 2.4.4?

20. Suppose $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous in $-\infty < x < \infty, -\alpha < y < \alpha$ for some $\alpha > 0$, and suppose that $f(x, 0) = 0$ for all x. Can there be a solution $y = \varphi(x)$ (defined on an interval) to

$$\frac{dy}{dx} = f(x, y)$$

which is positive for some values of x and negative for other values of x? Explain.

21. The purpose of this problem is to explain why we focus on the existence and uniqueness of solutions on *intervals* rather than some other kind of set. Suppose we wish to solve the initial value problem

$$\frac{dy}{dt} + \frac{1}{t}y = 0, \quad y(1) = 2,$$

on the set $(-\infty, 0) \cup (0, \infty)$ (this is the set of all real numbers not equal to 0).

(a) Show that

$$y(t) = \begin{cases} 2/t & \text{if } t > 0, \\ c/t & \text{if } t < 0, \end{cases}$$

where c is an arbitrary constant solves this initial value problem. Conclude that relative to the set $(-\infty, 0) \cup (0, \infty)$ this initial value problem has infinitely many solutions.

(b) Show that if we restrict to any open *interval* I containing $t = 1$ on which the function $p(t) = \frac{1}{t}$ is continuous, then the initial value problem has the unique solution $y(t) = \frac{2}{t}$.

22. (CAS) Sally relies on the computer algebra system *Maple* to solve initial value problems numerically. When she used *Maple*'s default numerical solution command to solve

$$\frac{dy}{dx} = x^2 y + \frac{x+1}{x+2}, \quad y(1) = 1,$$

and then evaluate the solution at $x = 13$, this is what happened:

$$F := dsolve\left(\left\{\text{diff}(y(x), x) = x^2 \cdot y(x) + \frac{(x+1)}{x+2}, y(1) = 1\right\}, type = numeric\right);$$

proc(*x_rkf45*) ... **end proc**

$F(13)$

Error, (in F) cannot evaluate the solution further right of 12.788291, probably a singularity

When *Maple* says the solution probably has a singularity, it is suggesting there is a number $x_0 \approx 12.788291$ such that the solution of Sally's initial value problem is not defined for any x exceeding x_0. Sally knows that *Maple* is wrong. That is, she knows that the solution to her initial value problem should exist at $x = 13$. Give a thorough explanation of why Sally is right and thus *Maple* is wrong.

In Exercises 23–26, (a) Find all equilibrium solutions. (b) By separating variables, find a one-parameter family of implicit solutions, and then write the solutions explicitly. (c) Does Theorem 2.4.9 apply to guarantee that you have found all solutions? (d) If your answer to (c) is no, try to find a solution not obtained in (a) or (b).

23. $\frac{dy}{dx} = e^{2x-y}$.
24. $\frac{dy}{dx} = (y+3)^2 \cos x$.
25. $\frac{dy}{dx} = 1 + y^2$.
26. $\frac{dy}{dx} = xy^{1/3}$.

27. Consider the initial value problem

(2.67)
$$\frac{dy}{dx} = \frac{-\epsilon x}{\sqrt{x^2 + y^2 + \epsilon y}}, \quad y(0) = 1,$$

where ϵ is a (small) positive constant.

(a) Suppose that g is a positive function such that $y = g(x)$ satifies the initial value problem (2.67) on some interval centered at the origin, say $-2 < x < 2$. Show that $y = g(-x)$ also satisfies the same initial value problem on $(-2, 2)$.

(b) Show how the No Intersection Theorem, Theorem 2.4.8, can be applied to conclude that g must be an even function ($g(-x) = g(x)$ for all x in $(-2, 2)$).

Remark: We'll revisit the initial value problem (2.67) in Section 3.3, where it will be used to model the "apparent curve" for an optical illusion called the Hering illusion. The symmetry present in the illusion requires that any such apparent curve be the graph of an even function. Thus, this exercise may be regarded as a plausibility check on the model (2.67) for the Hering illusion.

28. Based on your work for Exercise 27, describe properties of a function f of x and y that will ensure that there is a open interval $(-\kappa, \kappa)$ containing 0 such that

(2.68)
$$\frac{dy}{dx} = f(x, y), \quad y(0) = y_0,$$

has a unique solution on $(-\kappa, \kappa)$ that is necessarily an even function (on $(-\kappa, \kappa)$).

29. Let $f(x, y) = -6xy^{2/3}$ and observe $f(-x, y) = -f(x, y)$ for all real numbers x and y. Find a function φ such that

(i) $y = \varphi(x)$ is a solution of

$$\frac{dy}{dx} = -6xy^{2/3}, \quad y(0) = 1,$$

on $(-\infty, \infty)$ and

(ii) φ is not an even function on $(-\infty, \infty)$.

30. In this problem we outline a proof of Theorem 2.4.4. You should keep in mind the Fundamental Theorem of Calculus, which says that if f is continuous on an open interval I and if x_0 is any point in I, then for every x in I, the function F defined by

$$F(x) = \int_{x_0}^{x} f(u)\,du$$

exists and is a differentiable function, with $F'(x) = f(x)$ (for any x in I).

(a) First consider a homogeneous linear equation

$$\frac{dy}{dx} + p(x)y = 0 \tag{2.69}$$

with initial condition $y(x_0) = y_0$, where p is continuous on some open interval I containing x_0. Show that a solution to this initial value problem *exists* by showing that

$$y(x) = Ce^{-P(x)} \tag{2.70}$$

solves (2.69) if P is defined on I by

$$P(x) = \int_{x_0}^{x} p(u)\,du$$

and checking that there is a choice of C so that the initial condition is satisfied. Show that the solution is *unique* by showing that the original differential equation is equivalent to

$$\frac{d}{dx}(e^{P(x)}y) = 0$$

and that every solution to this equation appears in the family (2.70) and finally that there is only one choice of C so that (2.70) satisfies a given initial condition.

(b) Now consider a nonhomogeneous equation

$$\frac{dy}{dx} + p(x)y = g(x) \tag{2.71}$$

with initial condition $y(x_0) = y_0$, where p and g are continuous on an open interval I containing x_0. Show that there *exists* a solution to this initial value problem on the interval I by verifying that

$$y = e^{-P(x)}\left(\int_{x_0}^{x} e^{P(u)}g(u)\,du + y_0\right),$$

where P is as defined in (a), satisfies (2.71), and has $y(x_0) = y_0$. Finally, show this is the unique solution by arguing that if y_1 and y_2 solve this initial value problem, then $y_1 - y_2$ solves the initial value problem

$$\frac{dy}{dx} + p(x)y = 0, \quad y(x_0) = 0.$$

Use part (a) to explain why this tells you that $y_1 = y_2$.

31. In this problem we will work with the family of lines having equations

$$y = 2cx - c^2 \tag{2.72}$$

for c an arbitrary constant.

(a) Show that any line in the family (2.72) is a tangent line to the parabola $y = x^2$ at some point of the parabola. Describe this point in terms of c. Fig. 2.30 illustrates this relationship.

(b) Verify that each of the functions in the one-parameter family in (2.72) solves the differential equation

$$4y = 4x\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2.$$

(c) Does the function $y = x^2$ also solve this differential equation? Is it obtained by any choice of c in (2.72)? This "missing solution" is called a **singular solution**.

(d) Show that if

$$\frac{dy}{dx} = 2x - 2\sqrt{x^2 - y},$$

then

$$\left(\frac{dy}{dx}\right)^2 + 4y = 4x\frac{dy}{dx}.$$

This shows how the solutions in Example 2.4.7 can be obtained.

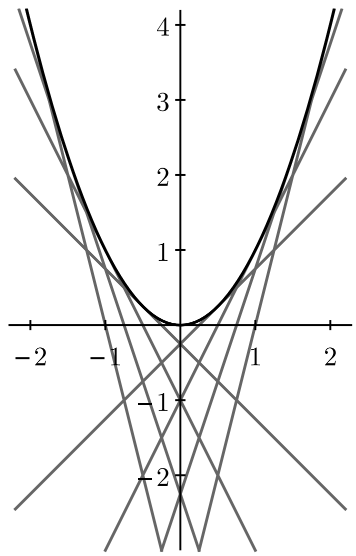

Figure 2.30. The parabola $y = x^2$ and some lines $y = 2cx - c^2$.

32. The differential equation

$$y = x\frac{dy}{dx} - \frac{1}{4}\left(\frac{dy}{dx}\right)^2 \tag{2.73}$$

from Exercise 31(b) is an example of a **Clairaut equation**. We saw in that exercise that it had the family of lines $y = 2cx - c^2$ as well as the parabola $y = x^2$ as solutions. The family of lines is the collection of tangent lines to the parabola, and the parabola is called the envelope of the family of lines. More generally, a differential equation which can be written as

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right), \tag{2.74}$$

for some nonconstant twice-differentiable function f, is called a Clairaut equation.

(a) If equation (2.73) is written in the form of (2.74), what is the function f? In (2.74), the notation $f\left(\frac{dy}{dx}\right)$ means f evaluated at $\frac{dy}{dx}$.

(b) Show that equation (2.74) has the line solutions $y = mx + b$ for any constant m for which $f(m)$ is defined and an appropriate choice of the constant b. Identify b in terms of m and f.

(c) Show that the curve described by the parametric equations

$$x = -f'(t), \quad y = f(t) - tf'(t) \tag{2.75}$$

is also a solution to (2.74). Hint: Use the fact that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

(d) Find the equation of the tangent line to the curve described by the parametric equations (2.75) at the point on the curve with coordinates $(-f'(m), f(m) - mf'(m))$. Check that this is one of the line solutions to (2.74), as described in (b).

(e) Find the solutions to the Clairaut equation

$$y - 2 = (x+1)\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$$

described in (b) and (c).

33. The uniqueness part of Theorem 2.4.5 says that if two solutions to

$$\frac{dy}{dt} = f(t, y)$$

start out the same (with value y_0 at time t_0), then they stay the same, at least on some interval of t's containing t_0. There is an extension of this statement which quantifies how "close" two solutions stay to each other if they start out "close" at time t_0. In this problem we'll explore this extension.

Suppose $\mathcal{R}$ is a rectangle as in the hypothesis of Theorem 2.4.5, and suppose we have a number M so that

$$-M \leq \frac{\partial f}{\partial y} \leq M$$

at all points of $\mathcal{R}$. Suppose further that y_1 and y_2 are solutions to

$$\frac{dy}{dt} = f(t, y)$$

with

$$y_1(t_0) = a \quad \text{and} \quad y_2(t_0) = b,$$

where (t_0, a) and (t_0, b) are in $\mathcal{R}$. The following fact can be shown: As long as $(t, y_1(t))$ and $(t, y_2(t))$ continue to belong to $\mathcal{R}$, we have the estimate

$$|y_1(t) - y_2(t)| \leq |a - b|e^{M|t-t_0|}. \tag{2.76}$$

(a) Explain how the uniqueness part of Theorem 2.4.5 is a special case of this result.

(b) Now the exponential function

$$e^{M|t-t_0|}$$

can certainly grow quickly as t moves away from t_0, so by (2.76) the values of our two functions $y_1(t)$ and $y_2(t)$ can be far apart when $t \neq t_0$ even if a and b are very close. To see this with a concrete example, solve the initial value problem

$$\frac{dy}{dt} = 3y, \quad y(0) = a,$$

and call the solution y_1, and solve

$$\frac{dy}{dt} = 3y, \quad y(0) = b,$$

and call the solution y_2. Show directly that

$$|y_1(t) - y_2(t)| = |a - b|e^{3t}.$$

Explain how this gives an example where we have equality in the inequality (2.76).

(c) In the initial value problems of (b), how large must t be so that $|y_1(t) - y_2(t)|$ is more than 100 times larger than $|y_1(0) - y_2(0)|$?

2.5. Population and financial models

The problem of estimating population is fundamental. For example, accurate models for population growth allow for rational public policy decisions of many sorts.

In the simplest situation, we are interested in population models for a single species (human, animal, bacterial, etc.) in isolation, and we think of the population as homogeneous (we ignore individual variations like gender, age, etc.). More complicated models might look at two interacting populations (for example, a predator species and its prey), or divide the population into subgroups (by age, for instance) and model each separately.

2.5.1. Malthusian model. Given all the potential complexities in population growth, it may be surprising that we can give a useful model based on one of the simplest first-order differential equations. This goes back to the English economist Thomas Malthus in 1798. He proposed that if the population under study had size $P(t)$ at time t, then the rate of change of P with respect to t should be proportional to P itself:

$$\frac{dP}{dt} = kP \tag{2.77}$$

for some positive constant k. Notice that if we measure population in numbers of individuals, P will always be integer-valued, but when we write a differential equation for P we're assuming it's a differentiable function and therefore continuous. So we are tacitly assuming that we "smooth out" our population function to be differentiable. This is a reasonable assumption if the population P is large, for then when we look at its graph as a function of time, we do so "from a distance" so as to see the whole picture. From this vantage point, individual changes in P (by a birth or death) are barely noticeable, leaving the impression of a smooth function.

Equation (2.77) is readily solved (by separating variables, for example) and we can write the solution as

$$P(t) = Ce^{kt}.$$

Since $P(0) = C$, we'll rewrite this as

$$P(t) = P_0 e^{kt}$$

where P_0 denotes the population at whatever we call time $t = 0$. Given the simplicity of this model and the complexity of population demographics in general, it is perhaps surprising that the Malthusian equation $P(t) = P_0 e^{kt}$ is of any use. Notice that if we want to fit this model to an actual population, we need two data points to start with: one to determine P_0 and one to determine k. Let's try this with the US population. The population in 1910 was 91.972 million. In 1920 it was 105.711 million. If we set our timeline so that $t = 0$ corresponds to 1910 and measure t in years, then when $t = 10$ we have $P(10) = 105.711$, in units of millions of people. We determine k from

$$105.711 = 91.972e^{10k},$$

or

$$1.149 = e^{10k}.$$

Taking the natural logarithm gives $\ln 1.149 = 10k$ so that $k = 0.0139$. What does this model predict for the populations in 1930, 1940, and 2000? We compute

$$P(20) = 91.972e^{(0.0139)(20)} = 121.448,$$

$$P(30) = 91.972e^{(0.0139)(30)} = 139.559,$$

and

$$P(90) = 91.972e^{(0.0139)(90)} = 321.335$$

as estimates for the US population, in millions of individuals, for these three dates. The actual population in these years was, respectively, 122.775, 131.669, and 282.434 million. So we can see that error in our 1930 prediction was approximately 1 percent, and our 1940 prediction is about 6 percent above the actual value. The prediction for 2000 is not so good; the predicted value is more than 13 percent higher than the actual value of 282.434 million. By the year 2914, this model gives a US population of approximately 105,800,000 million, which is about the same as the total surface area in square feet of the US. So clearly this model eventually becomes unrealistic.

Doubling time in the Malthusian model. One noteworthy feature of a population that grows according to a Malthusian model is the concept of **doubling time**: It takes a fixed amount of time for the population to go from any specific value to twice that value. To see this, let's determine the time for the population to increase from the value P_1 to the value $2P_1$ by solving the equations

$$P_1 = P_0e^{kt}$$

and

$$2P_1 = P_0e^{kt}$$

for t. In the first equation we get

$$t = \frac{1}{k}\ln\frac{P_1}{P_0} \tag{2.78}$$

and in the second we have

$$t = \frac{1}{k}\ln\frac{2P_1}{P_0}. \tag{2.79}$$

Using properties of the logarithm we rewrite the second solution as

$$t = \frac{1}{k}\left(\ln 2 + \ln\frac{P_1}{P_0}\right)$$

so that the difference between the values in (2.78) and (2.79), which is the time to double the population from P_1 to $2P_1$, is

$$\frac{\ln 2}{k}.$$

For example, according to our Malthusian model of US population growth, the population of the United States should double every

$$\frac{\ln 2}{0.0139} \approx 50 \text{ years.}$$

2.5.2. The logistic model. A difficulty with the Malthusian model for population growth is obvious—no population can grow exponentially indefinitely. Eventually competition for available resources needed to support the population must begin to limit the rate of growth. To develop a model that allows for such consideration, we consider a differential equation of the form

$$\frac{dP}{dt} = rP$$

where we now think of r as not necessarily a constant (as in the Malthusian model) but rather a *function* $r(P)$ of P. What sort of properties should this function (called the **proportional growth rate**) have? The basic idea is that, at least when the population is large enough, it should decrease as the population increases. This reflects the fact that as the population increases, there is increased competition for resources. The simplest form we might assume for the function $r(P)$ that fits this desired property is

$$r(P) = a - bP,$$

for positive constants a and b, so that $r(P)$ is a linear function that decreases as P increases; see Fig. 2.31. The constant a is sometimes called the intrinsic growth rate. This leads to the differential equation

$$\frac{dP}{dt} = (a - bP)P = aP - bP^2 \tag{2.80}$$

for the population. This should make a certain amount of intuitive sense, especially if we assume that the constant a is much larger than the constant b, so that when the population is small, relatively speaking, the term aP on the right-hand side is more significant, allowing something like Malthusian growth, but as the population gets larger, the term $-bP^2$ becomes progressively more important and acts to slow the growth of the population. We can think of the term $-bP^2$ as representing the interaction or competition among members of the population. When P is small, the population density is small and interaction is limited. As P increases, population density increases and an individual competes with others for resources. To see why the competition term should be proportional to P^2, imagine rhinos gathering at a watering hole in a time of drought, competing for space to drink. For each of the P animals in the population, there are $P - 1$ potential competitors to pick a fight with, leading to $P(P-1)/2$ possible hostile encounters. For a large population, $P \approx P - 1$, so the number of deaths resulting from hostile encounters should be (approximately) proportional to P^2 and thus having form $-bP^2$.

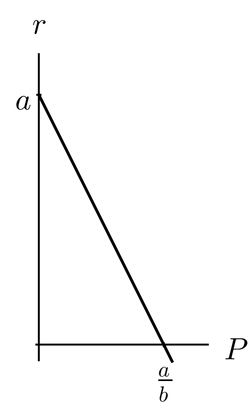

Figure 2.31. Graph of $r(P) = a - bP$.

Solving the logistic equation. Equation (2.80), which is called the **logistic equation**, is nonlinear. It is separable, and shortly we will solve it analytically. But first, we discuss some qualitative properties of its solutions and use these to give a rough sketch of the graphs of these solutions. The logistic equation is an example of an **autonomous equation**; this means that the independent variable t does not explicitly appear on the right-hand side of (2.80), so that the dependent variable P alone governs its own rate of change.

We first ask whether the logistic equation has any *equilibrium solutions*. In other words, does (2.80) have any *constant* solutions? If $P = c$ is a constant solution, then $\frac{dP}{dt}$ is 0, so we have

$$0 = (a - bP)P.$$

This tells us that there are two equilibrium solutions, $P = 0$ and $P = a/b$; shown in Fig. 2.32.

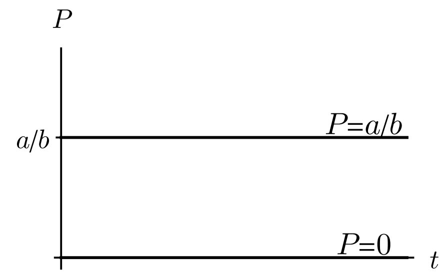

Figure 2.32. Equilibrium solutions to the logistic equation (2.80).

The meaning of the first of these is obvious, although rather uninteresting. If the population is 0, it stays 0 for all future times. The meaning of the second equilibrium solution a/b will become clear shortly. Notice that if a is much larger than b, as we assume is the typical case, a/b is large. Whenever P is between 0 and a/b, equation (2.80) tells us that $\frac{dP}{dt}$ is positive (why?) which in turn tells us that P is an increasing function of t.[16] This means that if we sketch the graph of t vs. P for solutions of (2.80), any curve in the horizontal strip between $P = 0$ and $P = a/b$ will be increasing from left to right. Whenever $P > a/b$, $\frac{dP}{dt}$ is negative (see Fig. 2.33), and the solution curves will be decreasing in this range. We can also determine the concavity of the solution curves by analyzing the second derivative

$$\frac{d^2P}{dt} = \frac{d}{dt}\left(\frac{dP}{dt}\right) = \frac{d}{dt}\left(aP - bP^2\right) = a\frac{dP}{dt} - 2bP\frac{dP}{dt} = (a - 2bP)(a - bP)P.$$

The second derivative is 0 when $P = 0, P = a/(2b)$, or $P = a/b$. Fig. 2.34 shows the sign for this second derivative in the ranges $0 < P < a/(2b)$, $a/(2b) < P < a/b$, and $a/b < P$; we ignore $P < 0$ since this has no physical meaning for our model. Thus solution curves are concave up for $0 < P < a/(2b)$ and for $P > a/b$, while they are concave down for $a/(2b) < P < a/b$. The No Intersection Theorem, Theorem 2.4.8, says that no two solution curves intersect (because $f(t, P) = aP - bP^2$ and $\frac{\partial f}{\partial P}(t, P) = a - 2bP$ are everywhere continuous). We use all of this information to sketch some of the solutions to the logistic equation in Fig. 2.35. The equilibrium solution $P = 0$ is said to be **unstable** since if $P(0)$ is close to but not equal to 0, it does *not* stay close to 0 for all later times. On the other hand, the equilibrium solution $P = a/b$ is **stable**. If $P(0)$ is near a/b (either larger or smaller), then for all later times $P(t)$ will be near a/b.

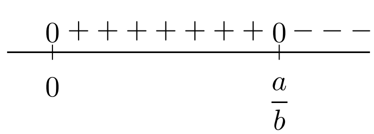

Figure 2.33. Sign of dP/dt.

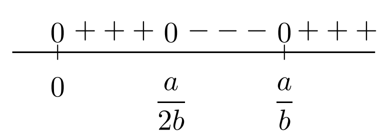

Figure 2.34. Sign of d^2P/dt^2.

[16] Recall that a function $g(t)$ is increasing on the interval $a < t < b$ if whenever $a < t_1 < t_2 < b$ we have $g(t_1) \leq g(t_2)$. The function is strictly increasing on (a, b) if we have $g(t_1) < g(t_2)$ whenever $a < t_1 < t_2 < b$. Decreasing/strictly decreasing are defined analogously.

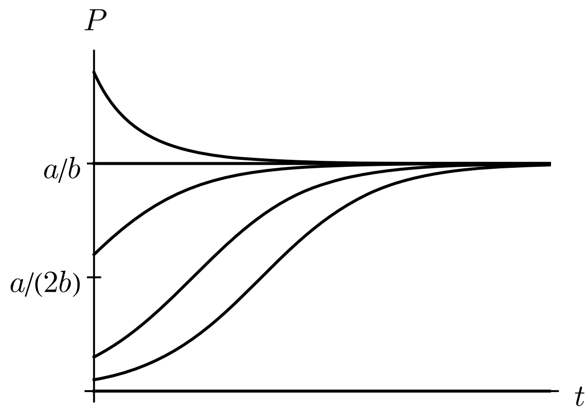

Figure 2.35. Some solutions to the logistic equation.

To solve (2.80) analytically we begin by separating the variables:

$$\int \frac{dP}{P(a-bP)} = \int dt. \tag{2.81}$$

To proceed, we determine a partial fraction decomposition on the left-hand side by writing

$$\frac{1}{P(a-bP)} = \frac{c_1}{P} + \frac{c_2}{a-bP}. \tag{2.82}$$

We determine values for c_1 and c_2 so that this decomposition holds. If we find a common denominator on the right-hand side of equation (2.82) and then equate numerators on the left and right, we find that

$$1 = c_1(a-bP) + c_2P.$$

If we set $P = 0$ and $P = a/b$ in the last equation, we determine that $c_1 = \frac{1}{a}$ and $c_2 = \frac{b}{a}$. Thus

$$\int \frac{dP}{P(a-bP)} = \frac{1}{a}\int \frac{1}{P}\,dP + \frac{b}{a}\int \frac{1}{a-bP}\,dP$$

and from equation (2.81) we have the implicit solution

$$\frac{1}{a}\ln|P| - \frac{1}{a}\ln|a-bP| = t + k$$

for an arbitrary constant of integration k. Because of the physical meaning of P as population, we can replace $\ln|P|$ with $\ln P$. Multiplying by a and using properties of logarithms we write

$$\ln \frac{P}{|a-bP|} = at + k' \tag{2.83}$$

where k' is an arbitrary constant.

It will be useful to have an explicit solution for P in terms of t. If we exponentiate both sides of equation (2.83), we obtain

$$\frac{P}{|a-bP|} = Ce^{at} \tag{2.84}$$

where $C = e^{k'}$ is an arbitrary *positive* constant. We can remove the absolute value bars on the left-hand side of (2.84) by allowing C to be an arbitrary *real* constant, so that

$$\frac{P}{a-bP} = Ce^{at}, \tag{2.85}$$

for C real. The choice $C = 0$ corresponds to the equilibrium solution $P = 0$ that we've already noted. Notice that C is related to the population P_0 at time $t = 0$ by

$$\frac{P_0}{a - bP_0} = C. \tag{2.86}$$

We can solve (2.85) explicitly for P as follows:

$$\begin{aligned} P &= Ce^{at}(a - bP), \\ P + Cbe^{at}P &= aCe^{at}, \\ P &= \frac{aCe^{at}}{1 + bCe^{at}} = \frac{aC}{e^{-at} + Cb}. \end{aligned}$$

Recalling (2.86) we can write the population function as

$$P(t) = \frac{a\frac{P_0}{a-bP_0}}{e^{-at} + \frac{P_0}{a-bP_0}b} = \frac{aP_0}{(a - bP_0)e^{-at} + bP_0}. \tag{2.87}$$

We can analyze (2.87) to draw some interesting conclusions about populations that grow according to the logistic equation. First, observe that (2.87) does give the expected equilibrium solutions in case $P_0 = 0$ or $P_0 = a/b$. Next, let's assume $P_0 > 0$ and look at long-time behavior; that is, we consider

$$\lim_{t\to\infty} \frac{aP_0}{(a - bP_0)e^{-at} + bP_0}.$$

Since $a > 0$,

$$e^{-at} \to 0 \text{ as } t \to \infty.$$

This means

$$P(t) \to \frac{aP_0}{bP_0} = \frac{a}{b}$$

as $t \to \infty$. This represents a significant qualitative difference from the Malthusian population model. No longer can the population grow without bound. The ratio a/b is called the **carrying capacity** of the environment. Our remark that a is typically much larger than b is consistent with the ratio a/b being large.

The logistic model for population growth was first proposed by the Belgian sociologist and mathematician Pierre-Francois Verhulst in 1837. Verhulst lacked access to population data that could have provided support for this theory, so his work did not get much attention until it was rediscovered 80 years later by Raymond Pearl and Lowell Reed, who are acknowledged as being the first to apply the logistic model to study the population growth of the United States.

Example 2.5.1. Let's use the logistic equation to model world population. The natural value of a, or the intrinsic growth rate for humans when there are plenty of resources and thus no competition, is about $a = 0.029$. In 1979, the world population was 4.38×10^9 and was increasing at 1.77% per year. The population increasing at 1.77% per year translates into $\frac{1}{P}\frac{dP}{dt} = 0.0177$ because $\frac{dP}{dt}$ is the population's rate of increase, and dividing by the population P gives the fractional rate of change for the population. This gives

$$0.0177 = 0.029 - b(4.38 \times 10^9),$$

and therefore

$$b = 2.580 \times 10^{-12}.$$

The carrying capacity is therefore $\frac{a}{b} = 11.24 \times 10^9$, or 11.24 billion. So our model predicts that the population of earth over time will approach 11.24 billion people. The world population in July

2005 was 6.45 billion people, which is larger than half of the carrying capacity. Thus we are above the inflection point and on the stressful "concave down" part of the population curve.[17]

In principle if we know the population at three points in time, we can determine the unknowns a, b, and P_0 in (2.87).[18] The computations are somewhat involved algebraically and are simplest if these three points are equally spaced in time. Exercise 15, which analyzes the post-Katrina population of New Orleans, shows how to manage the algebra in this case.

Other differential equation models that have features in common with the logistic model are often used. One of these, called the Gompertz equation, is discussed in the exercises. However, none of the models described in this section allow for oscillating populations, a phenomenon that is quite common in nature. For example, oscillations may occur if a population is subject to epidemics when it gets too large, or if the environment varies seasonally in its ability to support the population. A modification of the logistic model to allow for such seasonal fluctuations is given in Exercise 23 of Section 2.7.

Logistic models can be made more accurate over longer periods of time by periodically re-evaluating the coefficients a and b in the logistic equation. In particular, as technological advances allow the population to use resources more productively, we should decrease the value of b.

2.5.3. Financial models. When money is invested in a savings account that pays interest at a rate of $r\%$ per year, compounded continuously, the amount A in the account after t years is determined by the differential equation

$$\frac{dA}{dt} = rA \tag{2.88}$$

where r is expressed as a decimal. This is analogous to a Malthusian population model; now our "population" is the amount of money in the account, and the money grows at a rate proportional to the current size of the account. Equation (2.88) has solution $A(t) = A_0e^{rt}$ where A_0 is the account balance at time $t = 0$. This simple model assumes a constant interest rate r, which may not be realistic. It also assumes no withdrawals or deposits during the time period under consideration. In the next examples we show how to adjust for steady withdrawals or deposits; this is analogous to allowing for constant emigration or immigration in a population model.

Example 2.5.2. You open a savings account that pays 4% interest per year, compounded continuously, with an initial deposit of \$1,000. If you make continuous deposits at a constant rate that totals \$500 per year, what is your balance after 5 years? Notice that your total contribution is $\$1{,}000 + 5(\$500) = \$3{,}500$. The linear differential equation for the dollar amount A in the account after t years is

$$\frac{dA}{dt} = 0.04A + 500, \quad A(0) = 1{,}000.$$

We solve this using the integrating factor $e^{-0.04t}$ and obtain

$$Ae^{-0.04t} = -\frac{500}{0.04}e^{-0.04t} + C.$$

The initial condition $A(0) =\$1{,}000$ determines $C = 13{,}500$, so that

$$A(t) = -12{,}500 + 13{,}500e^{0.04t}.$$

After 5 years the account balance is $A(5) =\$3{,}989$.

[17] A recent United Nations report gave a prediction of a world population of 10.1 billion by the year 2100, considerably higher than earlier predictions. One factor cited was a lessened *demographic* significance of the AIDS epidemic in Africa. The new report predicts the population in Africa to triple in this century.

[18] More realistically, we might have a set of data points that we conjecture lie on a logistic curve. Choosing different sets of 3 points from our data would typically give different parameters for the corresponding logistic equation. The challenge then is to determine a logistic curve that "best fits" the entire set of data.

Example 2.5.3. Suppose a retirement account earns 5% per year, continuously compounded. What should the initial balance in the account be to permit steady withdrawals of $50,000 per year for exactly 35 years? If we let $A(t)$ be the account balance in thousands of dollars at time t measured in years, we interpret this question as: What should $A(0)$ be so that $A(35) = 0$, assuming

$$\frac{dA}{dt} = 0.05A - 50?$$

Check that this linear (and separable) equation has solutions

$$A(t) = 1{,}000 + Ce^{0.05t}.$$

We want $A(35) = 0$ so that

$$1{,}000 + Ce^{1.75} = 0$$

and $C \approx -173.77$. This tells us that

$$A(t) = 1{,}000 - 173.77e^{0.05t}$$

and $A(0) \approx 826.23$. Thus we need an initial account balance of approximately $826,230. The next example assesses how easy this will be to achieve.

Example 2.5.4. Suppose that starting at age 25, you make steady contributions to a retirement account (with initial balance 0). What should your yearly contribution be if you want to have a balance of $826,230 after 40 years? Assume your account will earn 5% interest, compounded continuously. Denote by $R(t)$ the amount, in thousands of dollars, in the retirement account at time t in years. We have

$$\frac{dR}{dt} = 0.05R + d$$

where d is our deposit rate (in thousands of dollars). We have $R(0) = 0$ and we want $R(40) = 826.23$. The differential equation has solution

$$R(t) = -\frac{d}{0.05} + \frac{d}{0.05}e^{0.05t}$$

and upon setting $t = 40$ and $R = 826.23$ we have

$$826.23 = -\frac{d}{0.05} + \frac{d}{0.05}e^{2}.$$

This says

$$d = \frac{(826.23)(0.05)}{e^2 - 1} \approx 6.466,$$

so our desired deposit rate is about $6,466 dollars per year.

More realistic models of long-term retirement planning usually assume that you will increase your withdrawal rate to take inflation into account. Suppose we plan to increase our withdrawal continuously by 3% per year, so

$$\frac{dW}{dt} = 0.03W.$$

The withdrawal rate at time t (in years) is then

$$W(t) = W_0e^{0.03t},$$

where W_0 is the initial withdrawal rate. Let's take W_0 to be $50,000 as in Example 2.5.3, so that our withdrawal rate in thousands of dollars is

$$W(t) = 50e^{0.03t}.$$

Now the balance A in our account earning 5% per year (continuously compounded) is determined by

$$\frac{dA}{dt} = 0.05A - 50e^{0.03t},$$

where we continue to measure A in thousands of dollars. Solving this linear equation gives

$$A(t) = 2{,}500e^{0.03t} + Ce^{0.05t}.$$

We can describe C in terms of the initial account balance A_0 as

$$C = A_0 - 2{,}500,$$

so that

$$A(t) = 2{,}500e^{0.03t} + (A_0 - 2{,}500)e^{0.05t}. \tag{2.89}$$

What should our initial account balance be to permit 35 years of withdrawals? We want $A(35) = 0$; substituting this into (2.89) gives $A_0 = 1{,}258.5$, or approximately \$1,258,500, more than one and a half times the amount given by the model of Example 2.5.3.

2.5.4. Exercises.

1. In 2000 a man living in Maryland purchased two live adult Northern snakehead fish from a market in Chinatown, NY, initially intending to prepare an Asian dish with them. Snakeheads are an invasive species whose introduction into lakes and rivers in the US is much feared because of the potential negative impact on native species.[19] However instead of cooking the fish, the man kept them as pets for a while and later released the pair into a Crofton, Maryland, pond. Beginning in early summer 2002, snakeheads started being seen in the pond by fishermen. In September of 2002, in a desperate attempt to eradicate the population from this pond before it spread to the nearby Little Patuxent River, the pond was poisoned with rotenone and 1,006 dead snakeheads were recovered. Assuming the snakehead population grew according to the Malthusian model
$$\frac{dP}{dt} = kP$$
and that the initial pair was introduced into the pond in June 2000, estimate the value of k and predict what the population would have been 1 year later (in September 2003) had not the State biologists intervened. The SciFi Network has aired two movies inspired by this incident: *Snakehead Terror* and *Frankenfish*.

   A possible criticism of this model is the fact that, since we approximate the (integer-valued) population by a differentiable function, it should be used only for populations that are "large".

2. In the period 1790–1840 US population doubled about every 25 years. If population growth during this period is modeled by a Malthusian equation $dP/dt = kP$, what should k be?

3. Yasir Arafat, a long-time Palestinian leader, died in November 2004 under mysterious circumstances. Traces of the toxic radioactive isotope polonium 210 were found on his toothbrush and on clothes he wore shortly before falling ill. In November 2012 Arafat's body was exhumed to search for further evidence of ^{210}Po poisoning. What percentage of ^{210}Po would be present after 8 years, given that it has a half-life of 138 days? (This means that it decays at a rate proportional to the amount present, and the proportionality constant is such that after 138 days only half the initial amount is left.)

[19] In addition to their voracious appetites, the fact that they can breath air and crawl considerable distances across land adds to their "monster fish" reputation.

4. A bacterial culture is growing according to the Malthusian equation
$$\frac{dM}{dt} = kM$$
where M is the mass of the culture in grams. Between 1:00 PM and 2:00 PM the mass of the culture triples.
   (a) At what time will the mass be 100 times its value at 1:00 PM?
   (b) If the mass at 3:00 PM is 11 grams, what was the mass at 1:00 PM?
5. Suppose a population is growing according to the Malthusian model
$$\frac{dP}{dt} = 0.015P$$
where we measure P in millions of people and t in years. Modify this equation to include a constant emigration rate of 0.2 million people per year. Assume the population on January 1 is 8.5 million. Solve your modified equation and estimate the population 6 years later.
6. You borrow $25,000 to buy a car. The interest rate compounded continuously on your loan is 6%. Assume that payments of k dollars per year are made continuously. What should k be if your loan is for 48 months? How much do you pay in total over the 48-month period of the loan?
7. (a) You borrow $10,000 at an interest rate of 4%. You will make payments of k dollars per year continuously, and the period of the loan is 4 years. What is the total amount of interest paid (above the $10,000 principal) over the course of the loan?
   (b) If you borrow $20,000 instead, but the interest rate and loan period are unchanged, do you end up paying exactly twice as much in total interest?
   (c) If you borrow $10,000, but now at an interest rate of 8%, still with a loan period of 4 years, do you end up paying exactly twice as much in total interest as in (a)?
8. In a 2009 article in *The Washington Post*[20] regarding depleted 401(k) accounts, the following statement is made: "'A $100,000 account that has declined by roughly a quarter can return to its original level in two years, if a worker contributes as much as he or she can to the 401(k) while the employer matches half of that amount and the market gains 8% each year,' said Steve Utkus, director of Vanguard's Center for Retirement Research." Determine the dollar amount of the (continuously made) contributions of the worker and the employer if this statement is correct.
9. (a) If $1,000 is invested at an interest rate of 5%, how much is in the account after 8 years?
   (b) How much should be invested initially if you want to have an account balance of $20,000 after 10 years?
   (c) Suppose we start with an initial deposit of $5,000 and make deposits of $1,000 per year in a continuous fashion. What is the account balance after 8 years? Assume an interest rate of 5%.
   (d) Starting with an initial deposit of $5,000, we plan to make withdrawals of k dollars per year in a continuous fashion. If $k = \$750$, how long will the money last? Is there a value of $k > 0$ so that the account balance never gets to 0? If so, what is the maximum such k? Assume an interest rate of 5%.
10. Redo Example 2.5.4 assuming an interest rate of 8%.
11. A recent front-page story in the New York Times entitled *In the Latest London Blitz, Pets Turn Into Pests*[21] described the rapidly increasing population of rose-ringed parakeets in London and the surrounding suburbs. It is believed that escapees from pet cages fueled this burgeoning

[20] Nancy Trejos, *Toll on 401(k) Savings Adds Years More of Toil*, The Washington Post, January 29, 2009.
[21] Elisabeth Rosenthal, The New York Times, May 14, 2011.

population of tropical birds and that perhaps slightly warmer temperatures have facilitated their breeding. Though they are beautiful birds with exuberant personalities, their presence in large numbers in disruptive.

Assume that the wild parakeet population in Britain follows a Malthusian model. The article gives a population in 1995 of 1,500 birds, and of 30,000 “a few years ago”; interpret this as being the 2010 population. What is the projected population for 2020? When was the population 100?

12. A fish population grows according to the Malthusian model
$$\frac{dP}{dt} = 0.05P.$$
When fishing is allowed, the population is determined by
$$\frac{dP}{dt} = 0.05P - 0.65P = -0.6P.$$
Suppose when the fish population is P_0, a year of fishing takes place. Assuming that fishing is then suspended, how long does it take for the population to recover from the fishing (that is, how long to return to a population of P_0)?

13. An intuitive approach to the logistic equation is to suppose that the rate of change $\frac{dP}{dt}$ is jointly proportional to (i.e., proportional to the product of) the population and to the “unused fraction of the population resource” which is $1-\frac{P}{K}$ where K is the upper limit of sustainable population. Show that this assumption does lead to the logistic equation.

14. In both the Malthusian and logistic models, we set the timeline so that the initial value of $P(t)$ occurs at $t = 0$ (so $P(0) = P_0$ is the initial value). Sometimes it is convenient to have the initial value P_0 occur at another time $t = t_0$, so that $P_0 = P(t_0)$.
   (a) Show that the solution of the initial value problem
   $$\frac{dP}{dt} = kP, \quad P(t_0) = P_0,$$
   is
   $$P(t) = P_0 e^{k(t-t_0)}.$$
   (b) Show that the solution of the initial value problem
   $$\frac{dP}{dt} = aP - bP^2, \quad P(t_0) = P_0,$$
   is given by
   $$P(t) = \frac{aP_0}{(a - bP_0)e^{-a(t-t_0)} + bP_0}.$$

15. The population of New Orleans in July 2005, just before Hurricane Katrina, was 455,000. The population at the first, second, and third year anniversaries of Katrina was, respectively, 50,200, 273,600, and 327,600. In this problem we will show how to fit this data to a logistic model and determine the coefficients a and b in equation (2.80). Recall that the solution to
$$\frac{dP}{dt} = aP - bP^2$$
is
$$P(t) = \frac{1}{\frac{b}{a} + \left(\frac{1}{P_0} - \frac{b}{a}\right) e^{-at}}$$
where $P_0 = P(0)$. We'll measure t in years and take $t = 0$ to correspond to July 2006, so that $P_0 = 50{,}200$.

(a) Set

$$r = \frac{b}{a}, \quad s = \frac{1}{P_0} - \frac{b}{a}, \quad \text{and} \quad x = e^{-a}.$$

Show that if you know P_0 and have determined x and s, then you can easily compute a and b. Also verify that

$$\frac{1}{P(t)} = r + sx^t.$$

(b) If we set $P_1 = P(1)$ and $P_2 = P(2)$, check that

$$\frac{1}{P_0} = r + s, \quad \frac{1}{P_1} = r + sx, \quad \frac{1}{P_2} = r + sx^2.$$

From the given data, we have $P_1 = 273{,}600$ and $P_2 = 327{,}600$, but we'll defer using these numerical values until we have carried out some algebraic simplifications.

(c) Show that

$$\frac{\frac{1}{P_1} - \frac{1}{P_2}}{\frac{1}{P_0} - \frac{1}{P_1}} = x$$

and

$$\frac{\frac{1}{P_0} - \frac{1}{P_1}}{1 - x} = s.$$

(d) Now use the numerical values of P_0, P_1, and P_2 and determine x, s, r, a, and b.

(e) What is the predicted population of New Orleans in July 2009?

(f) What does this model give for the carrying capacity of New Orleans? As of July 2008, had the inflection point on the logistic curve been passed?

16. Total sales of a new product sometimes follow a logistic model. If $S(t)$ denotes the total number of a product sold by time t, then $dS/dt = aS - bS^2$ for some constants a and b. In thousands of units, the total number of iPods sold by Apple by the end of 2005 was 4,580, by the end of 2006 it was 18,623, and by the end of 2007 it was 39,689. Using the results of the last exercise, use these three data points to determine a and b. At the start of 2008 were iPod sales on the concave up or concave down portion of the logistic sales curve?

17. Two populations with the same initial size P_0 and same carrying capacity $K = a_1/b_1 = a_2/b_2$ are modeled by the differential equations

$$\frac{dP}{dt} = a_1 P - b_1 P^2$$

and

$$\frac{dP}{dt} = a_2 P - b_2 P^2.$$

The graphs of the two populations are shown in Figs. 2.36 and 2.37. Which is larger, a_1 or a_2? Hint: For any solution $P(t)$ to the logistic equation, determine the slope of the graph of $P(t)$ at its inflection point in terms of a and the carrying capacity $\frac{a}{b}$. You do not need to solve the differential equation to do this!

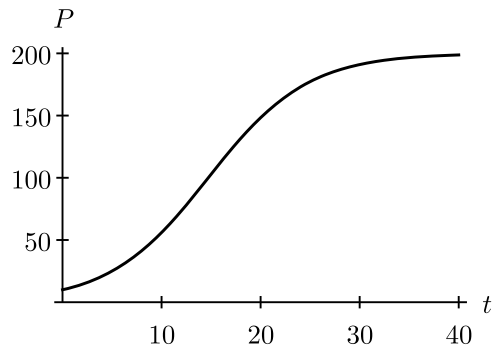

Figure 2.36. Coefficients a_1, b_1.

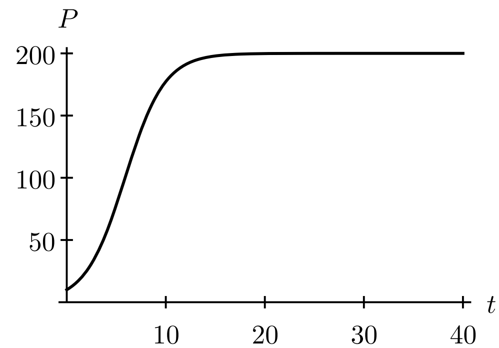

Figure 2.37. Coefficients a_2, b_2.

18. Suppose that in the logistic model we choose $t = 0$ to correspond to the time when the population is one-half of its carrying capacity a/b. Show that then we can write (2.87) as

$$P(t) = \frac{a/b}{1 + e^{-at}}.$$

19. For positive constants a and b, an equation of the form

$$\frac{dP}{dt} = aP^2 - bP$$

is sometimes called an explosion/extinction equation. This model is appropriate for a situation where births occur at a rate proportional to the square of the population size, while deaths occur at a rate proportional to the population.

(a) Solve this differential equation in the case that $a = b = 1$. Since $P(t)$ represents a population, think of units of P as millions of individuals, so that the equilibrium at $P = 1$ represents one million individual organisms (whatever they are).

(b) Find the solution with $P(0) = 2$. At what time t does the population "explode" (become infinite)?

(c) Find the solution with $P(0) = \frac{1}{2}$. What happens to the population as time tends to infinity?

20. Suppose we propose to model population growth by the equation

$$\frac{dP}{dt} = aP + bP^2$$

where a and b are positive constants.

(a) Explain how you know without doing any calculations at all that the population will grow without bound, assuming $P(0) > 0$.

(b) Show, by solving the equation exactly, that there is a finite time t_0 so that

$$\lim_{t \to t_0} P(t) = \infty.$$

This model is sometimes called the doomsday model, with t_0 being doomsday.

21. The **Gompertz differential equation** is

$$\frac{dP}{dt} = aP - bP \ln P$$

where a and b are positive constants. Show that it has the form

$$\frac{dP}{dt} = r(P)P$$

for some function $r(P)$ (which you should identify explicitly). Sketch the graph of P vs. $r(P)$. The Gompertz equation is used to model population growth and also the closely related problem of tumor growth. Tumor cells grow in a confined environment with restricted availability of nutrients, a situation that has parallels with a population growing in a fixed space with limited resources.

22. Solve the Gompertz equation (see Exercise 21) as a separable equation. A substitution will be helpful to carry out the necessary integration.

    What is the long-term behavior of a (positive) population that grows according to this equation? That is, determine

$$\lim_{t\to\infty} P(t).$$

23. Suppose the population P at time t (in years) of a city is modeled by a logistic equation

$$\frac{dP}{dt} = \frac{1}{3}P - \frac{1}{18 \times 10^4}P^2.$$

    How would you modify this equation if additionally 5,000 people per year move out of the city (in a continuous fashion)? Find any equilibrium solutions of your modified equation.

24. Sally is cramming for a history exam. She has an amount of material M to memorize. Let $x(t)$ denote the amount memorized by time t (in hours).
    (a) Assume that the rate of change of the amount of material memorized is proportional to the amount that remains to be memorized. The proportionality constant is a measure of natural learning ability, and in Sally's case set it to be 0.2. Suppose that $x(0) = 0$. Solve the resulting initial value problem, and determine how long it takes her to memorize half of the material.
    (b) How long does it take to memorize 90 percent of the material?
    (c) More realistically, while she is memorizing new material, she is also forgetting some of what she has already memorized. Assume that the rate of forgetting is proportional to the amount already learned, with proportionality constant 0.02. Modify your differential equation from (a) to take forgetting into account, and determine the value of t for which $x(t) = M/2$. Can she get everything memorized? If not, what's the best she can do?

25. A spherical mass of bacteria is growing in a Petri dish. The mass grows uniformly in all directions (so it continues to be spherical in shape) and only the bacteria on the surface of the mass reproduce.
    (a) The differential equation

$$\frac{dN}{dt} = kN^{2/3}$$

    (k a positive constant) is proposed to model the number N of bacteria at time t. Solve this equation with initial condition $N(0) = N_0$.
    (b) Using the formulas

$$S = 4\pi r^2 \quad \text{and} \quad V = \frac{4}{3}\pi r^3$$

    for the surface area S and the volume V of a sphere in terms of the radius r, explain where the power $2/3$ comes from in the equation in (a).

26. **Retrieval from long-term memory.** Suppose you are asked to list all items you can think of in a particular category; for example list animals, or types of flowers. If we assume you have some finite number M of items in the specified category stored in your long-term memory, one model of how these items are retrieved hypothesizes that the rate of recollection is proportional to the difference between M and the number you have already named. Translate this into a differential equation for $n(t)$, the number of items you have listed by time t, and then solve the differential equation assuming $n(0) = 0$. What happens to $n(t)$ as $t \to \infty$?

27. A simple blood test measuring β-HCG (human chorionic gonadotropin) levels in the blood is used to assess the health of a pregnancy in the earliest weeks, before ultrasound technology

becomes useful. The rate of increase of β-HCG is more predictive than the absolute numbers. In the early weeks of a normally progressing pregnancy β-HCG doubles about every 48 hours, and a doubling time of greater than 72 hours is considered abnormal. Serial β-HCG measurement is routine in couples undergoing any kind of infertility treatment, as well as in many other cases.

(a) If β-HCG is 80 units when measured at 14 days past ovulation (DPO) and 195 at 18 DPO, is this within normal limits? Assume the blood is drawn at the same time each day and that the amount A of HCG grows according to the differential equation $\frac{dA}{dt} = kA$. Hint: If a normal doubling time is anything less than or equal to 72 hours, what does this tell you about a "normal" value of k?

(b) If β-HCG is 53 units when measured at 9:00 AM on 13 DPO and 100 when measure at 5:00 PM on 15 DPO, is this within normal limits?

(c) In cases of in-vitro fertilization (IVF) where multiple embryos are implanted, it is typical to measure β-HCG 4 or 5 times over a period of several weeks before the first ultrasound can be done at approximately 40 DPO. Here the phenomenon known as a "vanishing twin" is not uncommon. This means one or more embryos that begin early development (and hence initially contribute to the production of β-HCG) cease development before they are visualized on ultrasound. The typical β-HCG pattern in such a situation may show some measurements that do not meet the "normal" doubling time of at most 3 days before settling into a normal rate of increase (indicating healthy development of the remaining embryos). Suppose the β-HCG measurements for an IVF pregnancy are as shown below. Assume the blood is drawn at the same time each day. Do any two consecutive measurements fall within normal limits?

DPO	14	16	19	26	38
β-HCG	227	310	570	2,800	45,000

28. Suppose we have a collection of data points for population as shown below. We want to decide if this set of data is reasonably "logistic", and if so, to give estimates for the parameters a and b in the logistic equation $\frac{dP}{dt} = aP - bP^2$.

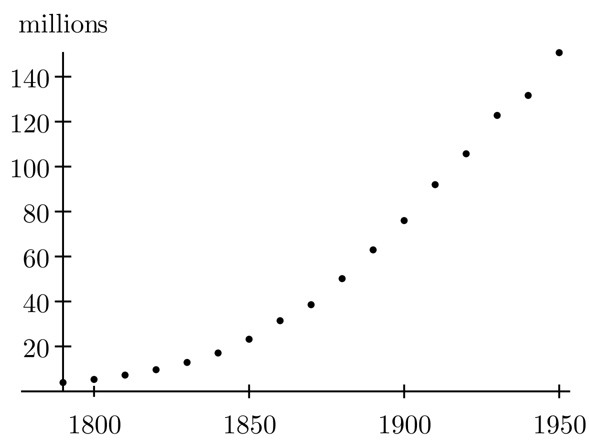

Figure 2.38. US population 1790–1950.

(a) Imagine that the points are spaced in time by 1 unit; in our example one unit of time will be a decade, and $t = 0$ will correspond to the year 1790. For $t = 1, 2, 3, 4, \ldots$, imagine calculating the ratios

$$\Delta_t = \frac{P(t+1) - P(t-1)}{2}.$$

Explain why this is a (very) rough approximation to $P'(t)$.

(b) Now imagine computing the ratios $\frac{\Delta_t}{P(t)}$ for $t = 1, 2, 3, 4, \ldots$. Explain why this is an approximation for the proportional growth rate

$$\frac{\frac{dP}{dt}}{P}.$$

(c) If you make a graph of P (horizontal axis) vs. $\frac{\Delta_t}{P(t)}$ (using your calculations from (b)), what should this graph look like if the population growth is logistic? If it is logistic, how would you estimate the parameters a and b from this graph?

(d) Using any computer software at your disposal, calculate Δ_t and $\Delta_t/P(t)$ as in parts (a)–(b) for $t = 1, 2, \ldots, 15$ using the US population data given below. Graph the points $(P(t), \Delta_t/P(t))$ you obtained. They should (roughly!) look like they lie on a line. Estimate the parameters a and b from this line.

(e) Plot on the same coordinate grid both the raw population data and the solution of the logistic equation with the a and b values you found in part (d) and $P(0) = 3{,}930{,}000$.

t	Year	Population
0	1790	3,930,000
1	1800	5,310,000
2	1810	7,240,000
3	1820	9,640,000
4	1830	12,870,000
5	1840	17,070,000
6	1850	23,190,000
7	1860	31,440,000
8	1870	38,560,000
9	1880	50,160,000
10	1890	62,950,000
11	1900	75,990,000
12	1910	91,970,000
13	1920	105,710,000
14	1930	122,780,000
15	1940	131,670,000
16	1950	150,700,000

2.6. Qualitative solutions of autonomous equations

Recall that the logistic differential equation is an example of an *autonomous* differential equation. A first-order equation is said to be autonomous if it can be written in the form

$$\frac{dy}{dt} = f(y), \tag{2.90}$$

so that the independent variable t does not appear explicitly on the right-hand side of the equation. Autonomous equations have several noteworthy features that make them particularly amenable to qualitative analysis. The field marks for the direction field of any autonomous equation have the same slope along any *horizontal* line; see Fig. 2.39.

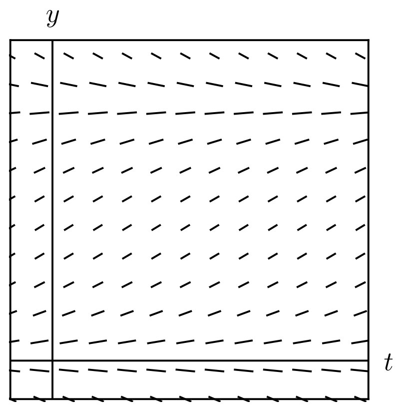

Figure 2.39. Direction field for an autonomous equation.

So if we calculate the field marks at points along one *vertical* line, we can fill out a fuller picture of the direction field by just shifting these field marks left and right.

This should make the following assertion believable: If you have a solution $y_1(t) = \varphi(t)$ of equation (2.90) and you translate this solution (to the right or the left) by constructing the function

$$y_2(t) = \varphi(t - c)$$

for some constant c, then the resulting new function is also a solution of (2.90). For example, the function $y(t) = 2e^{3t}$ solves the autonomous equation $\frac{dy}{dt} = 3y$, and so does the function $y(t) = 2e^{3(t-2)} = 2e^{-6}e^{3t}$.

Short of sketching solutions using the direction field, we can often get a useful picture of the solution curves to an autonomous equation $\frac{dy}{dt} = f(y)$ by determining the equilibrium solutions and the increasing/decreasing behavior of the other solutions. We illustrate how this goes with the equation

$$\frac{dy}{dt} = f(y) = (y-2)(y-3)(y+1)^2. \tag{2.91}$$

Equilibrium solutions. To find the equilibrium solutions we determine all constants k for which $f(k) = 0$. Corresponding to each such constant k is a horizontal-line solution $y = k$ in the ty-plane, and we begin our qualitative solution by sketching these solutions. In equation (2.91), the equilibrium solutions are $y = 2$, $y = 3$, and $y = -1$, as shown in Fig. 2.40.

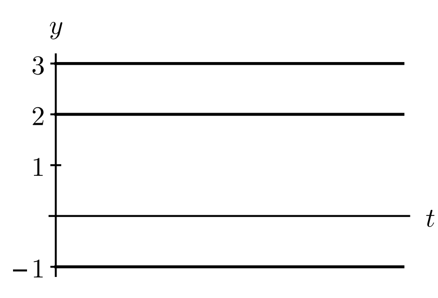

Figure 2.40. Equilibrium solutions for $dy/dt = (y-2)(y-3)(y+1)^2$.

Increasing/decreasing properties. Continuing with equation (2.91), we next determine for each of the ranges of values $-\infty < y < -1$, $-1 < y < 2$, $2 < y < 3$, and $3 < y < \infty$ whether

$\frac{dy}{dt}$ is positive or negative. Continuity of f will guarantee that $f(y)$ is either always positive or always negative in each of these ranges. We'll keep track of the positive or negative sign of $f(y)$ on a number line, which for later convenience we orient vertically; see Fig. 2.41. For example, when $y > 3$, each factor $(y-2)$, $(y-3)$, and $(y+1)^2$ is positive, ensuring that $\frac{dy}{dt} > 0$ whenever $y > 3$.

What does this number line in Fig. 2.41 tell us? Since the sign of $\frac{dy}{dt}$ determines the increasing/decreasing property of a solution $y = y(t)$, we see that solution curves in the horizontal strips $y > 3$, $-1 < y < 2$, or $y < -1$ are increasing (since $\frac{dy}{dt}$ is positive) while any solution curve in the strip $2 < y < 3$ is decreasing. We summarize this information in Fig. 2.42 using "up" and "down" arrows along our vertical line to indicate whether solution curves in the corresponding horizontal strips increase or decrease. This up/down line is called a **phase line** for the differential equation (2.91).

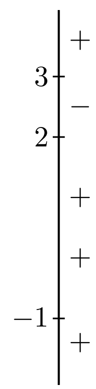

Figure 2.41. Sign of $dy/dt = (y-2)(y-3)(y+1)^2$.

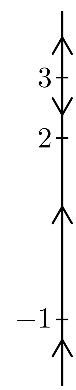

Figure 2.42. Phase line for $dy/dt = (y-2)(y-3)(y+1)^2$.

No two distinct solution curves can intersect (by the No Intersection Theorem, Theorem 2.4.8). Thus, in particular, none can cross the horizontal lines $y = 2$, $y = 3$, and $y = -1$ (the equilibrium solutions). So, as suggested by the phase line, the equilibrium solutions break the ty-plane up into horizontal strips on which the solution curves are either increasing or decreasing.

Moreover, any solution to an autonomous differential equation $dy/dt = f(y)$, with f and f' continuous, either

- tends to ∞ as t increases or
- tends to $-\infty$ as t increases or
- tends to an equilibrium solution as t increases.

A key ingredient in a formal proof of these claims is the topic of Exercise 23, which outlines how to show that if $y(t)$ is any solution to an autonomous equation $dy/dt = f(y)$ and if

$$\lim_{t\to\infty} y(t) = L$$

for some finite value L, then L must be an equilibrium value: $f(L) = 0$. Moreover, if at time $t = t_0$ a solution to $dy/dt = f(y)$ (with f and f' continuous) is between two consecutive equilibrium values k_1 and k_2 with $k_1 < k_2$, then either

- the solution tends to the larger equilibrium value k_2 as $t \to \infty$ if the solution curves are increasing in the range $k_1 < y < k_2$ or
- the solution tends to the smaller equilibrium value k_1 as $t \to \infty$ if the solution curves are decreasing in the range $k_1 < y < k_2$.

Using these ideas, we show qualitatively some solutions for the equation

$$dy/dt = (y-2)(y-3)(y+1)^2.$$

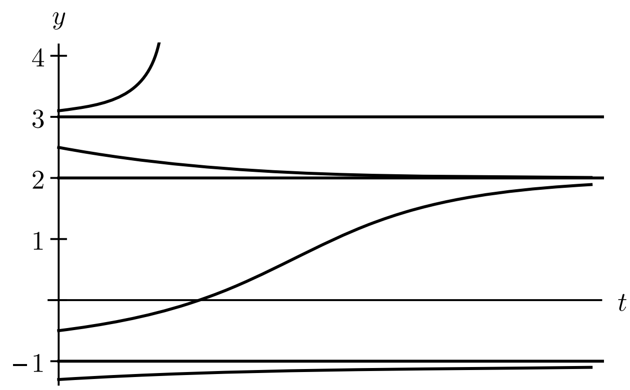

Figure 2.43. Qualitative solutions for $dy/dt = (y-2)(y-3)(y+1)^2$.

Stability of equilibrium solutions. Fig. 2.43 also illustrates the notion of stability for an equilbrium solution. The equilibrium solution $y = 2$ is said to be **(asymptotically) stable** since any solution $y(t)$ whose initial value $y(0)$ is sufficiently close to the equilibrium value 2 tends to this equilibrium value as $t \to \infty$. By contrast, the equilibrium solution $y = 3$ is **unstable**. Any solution $y(t)$ whose initial value is close to 3, but not equal to 3, will ultimately move away from 3, either approaching 2 as $t \to \infty$ or tending to ∞. A third possibility is illustrated by the equilibrium solution $y = -1$, which is called **semistable**. Any solution whose initial value is a little bigger than -1 will move away from $y = -1$, eventually approaching 2 as $t \to \infty$. On the other side though, a solution whose initial value is less than -1 approaches the equilibrium value -1 as t increases. In terms of the phase line, an equilibrium solution $y = k$ is stable if both arrows adjacent to k point towards k, and it is unstable if the adjacent arrows both point away from k. The semistable classification occurs when one adjacent arrow points towards k and the other points away from k.

Blowing up in finite time. One caution in interpreting the "solution tends to ∞" or "solution tends to $-\infty$" conditions above: These can happen in finite time! For example, let's look at the equation

$$\frac{dy}{dt} = y + y^2, \tag{2.92}$$

whose phase line is shown in Fig. 2.44.

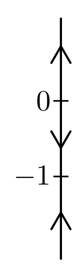

Figure 2.44. Phase line for $dy/dt = y + y^2$.

Explicit solutions to (2.92) can be found by separating variables, and we find solutions

$$y = \frac{Ce^t}{1 - Ce^t} \tag{2.93}$$

for arbitrary constant C (see Exercise 9). You can check that the value of C is related to the value y_0 of y at $t = 0$ by

$$C = \frac{y_0}{1 + y_0}. \tag{2.94}$$

In particular, the solution with value $y = 1$ at $t = 0$ has $C = \frac{1}{2}$:

$$y = \frac{\frac{1}{2}e^t}{1 - \frac{1}{2}e^t} = \frac{e^t}{2 - e^t}. \tag{2.95}$$

By the phase line sketch, we expect this solution to be increasing, and since there are no equilibrium values larger than 0, any solution with $y(0) > 0$ must tend to infinity. Notice that our solution in (2.95) has (left-hand) limit ∞ at $t = \ln 2$:

$$\lim_{t \to \ln 2^-} \frac{e^t}{2 - e^t} = \infty,$$

so that this solution tends to infinity in finite time, not as $t \to \infty$. Other solutions with $y(0) > 0$ also blow up in finite time, since

$$\lim_{t \to \ln \frac{1}{C}^-} \frac{Ce^t}{1 - Ce^t} = \infty.$$

Equation (2.94) says that if $y(0) = y_0$ is positive, then C is less than 1, so that $\ln \frac{1}{C} > 0$. We cannot tell from the phase line alone if a solution tends to infinity in finite time.

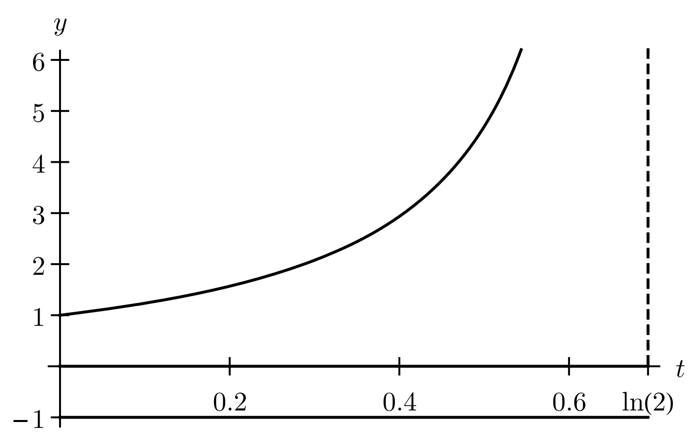

Figure 2.45. $y = e^t/(2 - e^t)$ and equilibrium solutions $y = 0$, $y = -1$.

Sometimes the information at our disposal will only allow us to find the equilibrium solutions and phase line *qualitatively*. The next example shows how this might be done.

Example 2.6.1. We will sketch some solutions to the equation

$$\frac{dy}{dt} = f(y), \tag{2.96}$$

where the graph of $f(y)$ is as shown in Fig. 2.46.

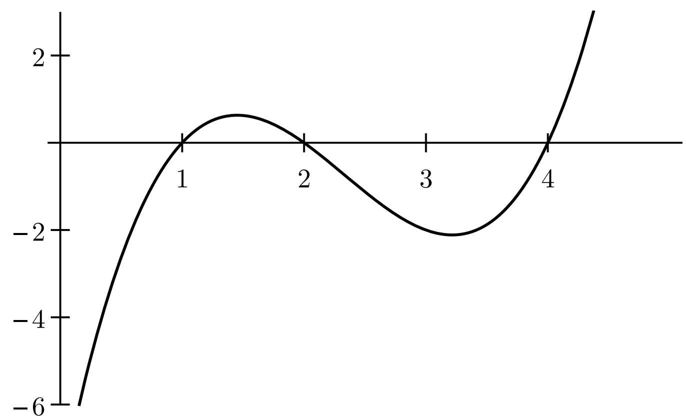

Figure 2.46. Graph of $f(y)$.

The equilibrium solutions take the form $y = k$ where $f(k) = 0$ (and hence $\frac{dy}{dt} = 0$). From the graph, we see the equilibrium solutions are approximately $y = 1$, $y = 2$, and $y = 4$. The phase line shows the equilibrium values 1, 2, and 4. Next we want to decide if $\frac{dy}{dt}$ is positive or negative in each of the intervals $-\infty < y < 1$, $1 < y < 2$, $2 < y < 4$, and $y > 4$. From the graph of $f(y)$ we see that for $y < 1$, $f(y)$ is negative. For y between 1 and 2, $f(y)$ is positive; between 2 and 4, $f(y)$ is negative; and finally for $y > 4$, $f(y)$ is positive. The phase line for equation (2.96) is as shown in Fig. 2.47.

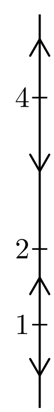

Figure 2.47. Phase line for $dy/dt = f(y)$.

Using the phase line, we give a rough sketch in Fig. 2.48 of some solutions curves for (2.96). Begin the sketch by showing the equilibrium solutions, $y = 1$, $y = 2$, and $y = 4$, as horizontal lines. Next use the phase line to determine the increasing and decreasing behavior of solution curves between the equilibrium solutions. For example, solutions in the horizontal strip between $y = 1$ and $y = 2$ are increasing, while those that lie in the strip between $y = 2$ and $y = 4$ are decreasing. Our discussion above also tells us about long-term behavior: If $y(0)$ is between 1 and 2 or between 2 and 4, then $y(t)$ tends to 2 as $t \to \infty$. Solutions with $y(0) > 4$ tend to infinity, while those with $y(0) < -1$ tend to $-\infty$.

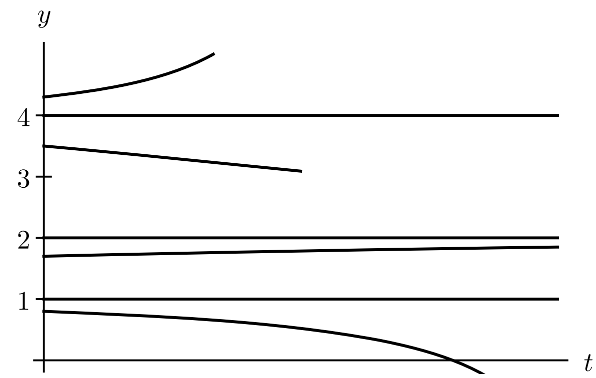

Figure 2.48. Some approximate solution curves.

2.6.1. Bifurcations. Many of the differential equations we have used in applications contain one or more **parameters**; these are *constants* which may take different values in different particular situations. The Malthusian equation $\frac{dP}{dt} = kP$ has one parameter k while the logistic equation $\frac{dP}{dt} = aP - bP^2$ has two parameters a and b. We expect the value of these parameters to depend on the particular population we wish to study; moreover, we typically have only approximate values for the parameters.

In this section we will consider autonomous first-order differential equations containing one real parameter b. The equilibrium solutions and the qualitative behavior of the nonequilibrium solutions may change as the value of this parameter changes. We might hope that a small change in b corresponds to a small change in the solution curves, but this is not always the case. The next example illustrates this.

Example 2.6.2. Consider the differential equation

$$\frac{dy}{dt} = by - y^3 \tag{2.97}$$

which depends on the parameter b. The equilibrium solutions are found by solving $y(b - y^2) = 0$, and when $b > 0$ we have three equilibrium solutions: $y = 0$, $y = \sqrt{b}$, and $y = -\sqrt{b}$. The phase line for the case $b > 0$ is shown in Fig. 2.49. Solutions with $y(0) < -\sqrt{b}$ are increasing to $-\sqrt{b}$ as $t \to \infty$ and solutions with $-\sqrt{b} < y(0) < 0$ are decreasing to $-\sqrt{b}$ as $t \to \infty$. If $0 < y(0) < \sqrt{b}$, then the solution increases to $\sqrt{b}$ as $t \to \infty$, and if $\sqrt{b} < y(0)$, the solution decreases to $\sqrt{b}$ as $t \to \infty$. The equilibrium solution $y = 0$ is unstable, while the solutions $y = \sqrt{b}$ and $y = -\sqrt{b}$ are stable.

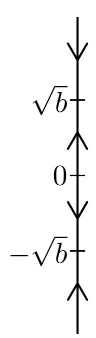

Figure 2.49. Phase line for $dy/dt = by - y^3$ when $b > 0$.

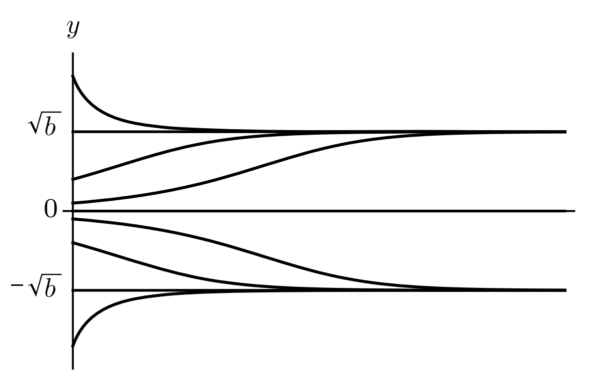

Figure 2.50. Typical solution curves for $\frac{dy}{dt} = by - y^3$ when $b > 0$.

By contrast, if $b \le 0$, there is only one equilibrium solution, $y = 0$, which is stable. Solutions with $y(0) > 0$ decrease to 0 as $t \to \infty$, while solutions with $y(0) < 0$ increase to 0 as $t \to \infty$.

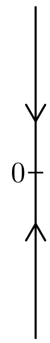

Figure 2.51. Phase line for $dy/dt = by - y^3$ when $b < 0$.

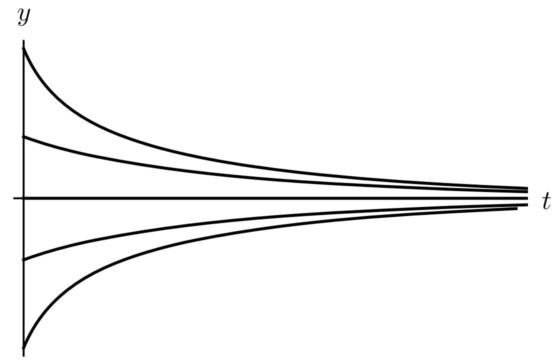

Figure 2.52. Typical solution curves for $\frac{dy}{dt} = by - y^3$ when $b < 0$.

Thus the number of equilibrium solutions and the long-term behavior of nonequilibrium solutions are dramatically different for $b > 0$ than for $b < 0$ and we say that a **bifurcation** occurs at the parameter value $b = 0$.

The graph of the equation $by - y^3 = 0$ in the by-plane (with the b-axis horizontal) is called a **bifurcation diagram**. Shown in Fig. 2.53, it consists of the line $y = 0$ and the parabolic curve $b - y^2 = 0$. This particular bifurcation diagram is sometimes called a **pitchfork**. A point (b, k) lies on the bifurcation diagram if and only if $y = k$ is an equilibrium solution of equation (2.97).

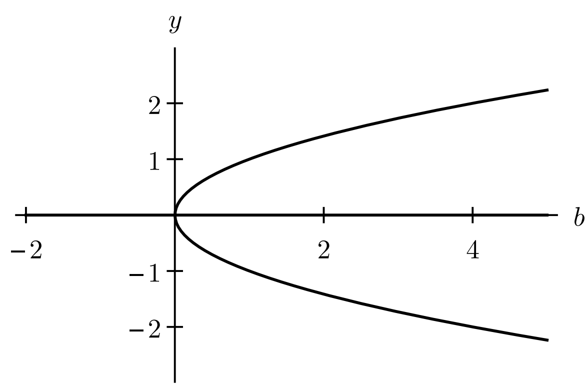

Figure 2.53. Bifurcation diagram for $\frac{dy}{dt} = by - y^3$.

You can see from the bifurcation diagram that for each value of b less than the bifurcation value 0 there is only one equilibrium solution (each vertical line to the left of 0 intersects the pitchfork in only one point), while for each value of b greater than 0 there are three equilibrium solutions (each vertical line to the right of 0 intersects the pitchfork in three points).

In general, for a differential equation that depends on a real parameter, a bifurcation value is a real number b_0 such that for parameter values near b_0 but on opposite sides, solutions to the corresponding equations behave quite differently, qualitatively speaking. When a differential equation with a parameter appears in an application, the value of the parameter is often controlled by external conditions. Its size relative to any bifurcation values may have a dramatic influence on the long-term behavior of the solutions. Exercise 20 explores this in the context of a fish farm.

2.6.2. Exercises.

For the autonomous equations in Exercises 1–8, (a) find all equilibrium solutions, (b) draw a phase line, (c) based on the phase line, describe the stability of equilibrium solutions, and (d) using the phase line, determine $\lim_{t\to\infty} y(t)$, given that y satisfies the equation and $y(0) = 1$.

1. $\frac{dy}{dt} = y^2 - 3y - 4$.
2. $\frac{dy}{dt} = (2 - y)^2$.
3. $\frac{dy}{dt} = y^2(3 - y)$.
4. $\frac{dy}{dt} = k(A - y)$, where $k > 0$ and $A > 1$.
5. $\frac{dy}{dt} = y(y - 2)$.
6. $\frac{dy}{dt} = -y^3 + 3y^2 - 2y$.
7. $\frac{dy}{dt} = (2 - y)\ln(1 + y^2)$.
8. $\frac{dy}{dt} = y(e^y - 3)$.
9. Solve
$$\frac{dy}{dt} = y + y^2$$
by separating variables, obtaining (2.93) as a one-parameter family of solutions.
10. Consider the differential equation
$$\frac{dy}{dt} = k(1 - y)^2$$
where k is a positive constant.
 (a) What is the equilibrium solution?
 (b) Explain why you know from the differential equation that every (nonequilibrium) solution is an increasing function of t.
 (c) Show that
$$\frac{d^2y}{dt^2} = -2k^2(1 - y)^3,$$
and describe the concavity of the solution curves.
 (d) Set $k = 1$ and solve the differential equation with the initial condition $y(0) = 1/2$ and with the initial condition $y(0) = 3/2$. Sketch the first solution for $t > -2$, and the second solution for $t < 2$.
11. In this problem you will give a qualitative solution to
$$\frac{dy}{dx} = (y + 2)(y - 3). \tag{2.98}$$
 (a) Find all equilibrium solutions, and sketch them in the xy-plane.
 (b) Draw the phase line, showing where solutions are increasing and decreasing.
 (c) What is $\frac{d^2y}{dx^2}$? Determine where this second derivative is zero, where it is positive, and where it is negative. Explain what this tells you about the concavity of solution curves for equation (2.98).
 (d) Using the information you have just determined, sketch solution curves with $y(0) = 0$, $y(0) = 1$, $y(0) = 2$, $y(0) = 4$, $y(0) = -1$, and $y(0) = -3$. Show a portion of each of these solution curves with both positive and negative values of x.
 (e) Classify each equilibrium solution as either stable, unstable, or semistable.
12. Consider the differential equation
$$\frac{dy}{dt} = 4 - y^2. \tag{2.99}$$
 (a) What are the equilibrium solutions of (2.99)?
 (b) Draw a phase line for (2.99) and describe the long-term behavior of the solution of (2.99) satisfying $y(0) = 1$.

(c) Solve the initial value problem consisting of (2.99) and $y(0) = 1$ exactly and check that your analysis of its long-term behavior in part (b) is correct.

13. A differential equation of the form

$$\frac{dP}{dt} = P \cdot f(P),$$

where the graph of $f(P)$ is as shown in Fig. 2.54, is proposed to model the population of a certain species.

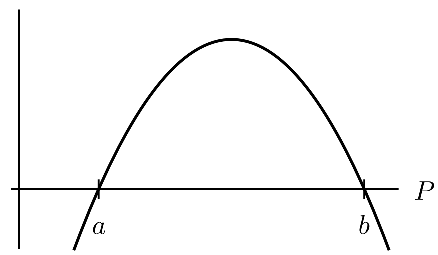

Figure 2.54. Graph of $f(P)$.

(a) Show that there are 3 equilibrium populations.
(b) Are either of the positive equilibrium populations stable? If so, which one(s)?
(c) Show that the limiting value of the population can be different depending on the initial (positive) value of the population. Explain why this is different from a logistic population model.

14. Consider the differential equation

$$\frac{dy}{dt} = f(y)$$

where the graph of $f(y)$ is as shown in Fig. 2.55.
(a) What are the equilibrium solutions?
(b) Sketch the phase line for $\frac{dy}{dt}$.
(c) Using the phase line, sketch some solution curves in the ty-plane.

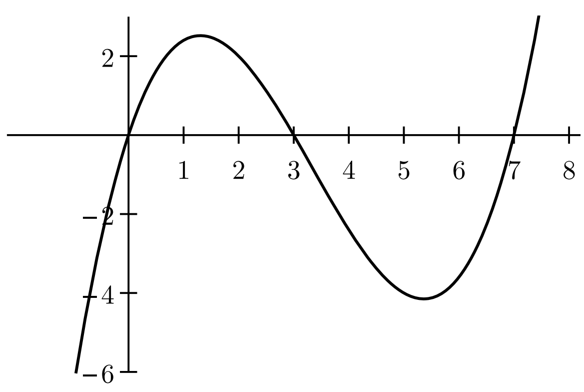

Figure 2.55. Graph of $f(y)$.

15. Consider the initial value problem

$$\frac{dy}{dt} = \frac{(y-5)^2(y-14)}{y^2+1}, \quad y(0) = 10.$$

Identify each of the following statements as true or false, and explain your reasoning. Hint: Start by determining all equilibrium solutions of the differential equation.

(a) $y(10)$ must be between 5 and 14.
(b) $y(15)$ cannot be equal to 0.
(c) $y(15)$ cannot be equal to 15.

16. Consider the initial value problem

$$\frac{dy}{dt} = y(y-1)^2(y-2)(y-3), \quad y(0) = y_0.$$

Describe the behavior of the solution $y(t)$ as t increases if
(a) $y_0 = 1/2$,
(b) $y_0 = 3/2$,
(c) $y_0 = 5/2$,
(d) $y_0 = 7/2$,
(e) $y_0 = 3$.

17. Suppose we have a differential equation

$$\frac{dy}{dt} = f(y)$$

where f is a continuous function. If we also know that $f(1) = 2$ and $f(3) = -3$, which of the following statements is correct?
(a) There is an equilibrium solution $y = k$ for some value of k between 1 and 3.
(b) There is an equilibrium solution $y = k$ for some value of k between -3 and 2.
(c) You can't conclude anything about equilibrium solutions without further information.
Suggestion: If you don't remember the Intermediate Value Theorem, look it up in a first-year calculus book.

18. Based on a phase line for

$$\frac{dy}{dt} = y^{2/3} \tag{2.100}$$

Sally says that if y satisfies (2.100) and $y(0) = -1$, then $\lim_{t\to\infty} y(t)$ must be 0. Sam disagrees. Who is right?

19. The fish population $P(t)$ in a large fish farm changes according to the differential equation

$$\frac{dP}{dt} = 3P - P^2 - 2,$$

where P is in millions of fish and t is in years. Notice that we have modified the logistic equation by assuming that the fish are being harvested at a constant rate of 2 million per year.
(a) Find all equilibrium solutions of the differential equation.
(b) For what range of P values is the population increasing with time, and for what P values is it decreasing?
(c) If the initial population is $P(0) = 1.5$, can we continue to harvest at a rate of 2 million fish per year indefinitely? Answer the same question if $P(0) = 0.8$.
(d) What is the smallest initial population which permits harvesting 2 million fish per year indefinitely?
(e) Find the solution $P(t)$ with $P(0) = 1.5$.

(f) Using your solution in (e), determine approximately how many years must pass before the fish population reaches 1.8 million, which is 90% of the fish farm's carrying capacity.

20. In this exercise we consider the fish farm population model of Exercise 19, assuming a constant harvesting rate h, the value of which will vary. Thus, we have

$$\frac{dP}{dt} = 3P - P^2 - h \tag{2.101}$$

where P is in millions of fish, t is in years, and $h \geq 0$ is the constant harvest rate in millions per year.

(a) If $h = 0$, what is the fish farm's carrying capacity?

(b) For $h = 0, h = 1, h = 9/4$, find equilibrium solutions of (2.101) and describe their stability.

(c) Show that if $h > 9/4$, the equation (2.101) has no equilibrium solutions.

(d) What value of h is a bifurcation value of (2.101)? For positive values of h less than the bifurcation value, describe the long-term behavior of the fish population assuming it is initially at least 1.5 million. Do the same for parameter values exceeding the bifurcation value.

(e) Give the bifurcation diagram for (2.101); that is, graph the equation

$$3P - P^2 - h = 0 \tag{2.102}$$

in the hP-plane (the horizontal axis is the h-axis).

(f) Recall that a point (h, k) lies on the graph of (2.102) if and only if $y = k$ is an equilibrium solution of (2.101). Your bifurcation diagram of part (e) is a parabola. Which part of the parabola corresponds to stable equilibrium solutions? What terminology applies to the first coordinate of the parabola's vertex?

21. **Harvesting renewable resources.** In this problem we look at the effect of fishing on a fish population that grows logistically.

(a) Suppose our fish population P is growing according to the equation

$$\frac{dP}{dt} = aP - bP^2 \quad (a > 0, b > 0)$$

and we begin to remove fish from the population. Suppose first that the fish are removed at a rate proportional to the size of the population, so now

$$\frac{dP}{dt} = aP - bP^2 - cP \tag{2.103}$$

for some positive constant c (sometimes called the effort coefficient). Show that if $a > c$, this new equation has two equilibrium values, $p_1 = 0$ and $p_2 = \frac{a-c}{b}$.

(b) Still assuming $a > c$, what happens to the solutions to (2.103) as $t \to \infty$, if $P(0) > 0$?

(c) The product of the stable equilibrium p_2 and the effort coefficient c is a *sustainable yield*, $Y = cp_2$. When fishing occurs at this level it can be continued indefinitely. Give a formula for Y in terms of a, b, and c. What choice of c will maximize Y (assuming a and b are constant)? What is the corresponding maximal sustainable yield?

(d) Alternately, assume that fish are harvested at a constant rate so that

$$\frac{dP}{dt} = aP - bP^2 - h \tag{2.104}$$

for some positive constant h. Show that if

$$0 < h < \frac{a^2}{4b},$$

this differential equation has two positive equilibrium solutions. Classify each as stable or unstable.

(e) If $h = \frac{a^2}{4b}$ in (2.104), show there is one equilibrium value $p = \frac{a}{2b}$. Explain why if $P(0) > \frac{a}{2b}$, the population will stay above $\frac{a}{2b}$ for all $t > 0$, so that this level of fishing can be maintained indefinitely, and there is a sustainable yield of $\frac{a^2}{4b}$.

(f) Show that when $h > \frac{a^2}{4b}$ there is no equilibrium solution to (2.104).

22. In this problem,[22] we take the simplest possible view of the average temperature $T(t)$ of the earth at time t and assume that the two factors that control $T(t)$ are radiation from the sun and radiation leaving the earth. We model this with the differential equation

$$\frac{dT}{dt} = r_i(T) - r_o(T) \tag{2.105}$$

where $r_i(T)$ and $r_o(T)$ are functions of the temperature T. For example, a higher average temperature T may cause a decrease in the amount of ice and snow near the poles. Since ice and snow are more reflective than, say, water, this causes an increase in the absorption of solar energy. Also, as T increases, the radiation leaving the earth increases too, as a warm body radiates more than a cold one. Equation (2.105) is an autonomous differential equations since the right-hand side is a function of T but not explicitly of t. Suppose the functions $r_i(T)$ and $r_o(T)$ have graphs as shown in Fig. 2.56.

(a) On what intervals of T is $r_i(T) - r_o(T)$ positive? Negative?

(b) What are the equilibrium solutions of equation (2.105) if $r_i(T)$ and $r_o(T)$ are as shown? Sketch these in a t vs. T graph as horizontal lines.

(c) Using your answers to (a) and (b), give a qualitative solution to equation (2.105). Just concentrate on the increasing and decreasing behavior of the solution curves.

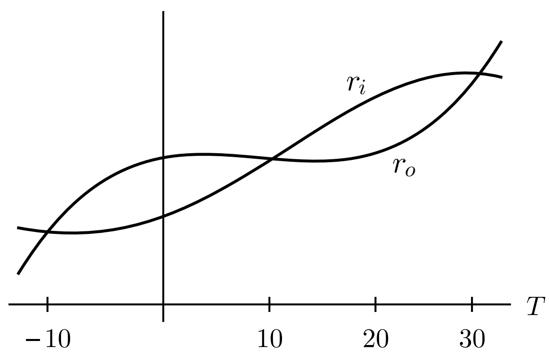

Figure 2.56. Graphs of $r_i(T)$ and $r_o(T)$.

23. Suppose that $y_1(t)$ is a solution on $[0, \infty)$ to the autonomous differential equation

$$\frac{dy}{dt} = f(y), \tag{2.106}$$

where f is a continuous function. Show that if the limit

$$\lim_{t\to\infty} y_1(t) = L$$

exists (where L is a finite number), then $y = L$ must be an equilibrium solution of the equation (2.106). Hints:

(a) Explain why $y_1(t+1) - y_1(t) \to 0$ as $t \to \infty$.

(b) Using a result from calculus, show that for each $t \geq 0$ there is a number $s(t)$ between t and $t+1$ satisfying $y_1(t+1) - y_1(t) = y_1'(s(t))$.

[22]Adapted from Clifford Taubes, *Modeling Differential Equations in Biology*, 2nd edition, Cambridge University Press, New York, 2008.

(c) Using parts (a) and (b) show that $f(L) = 0$. Conclude that $y = L$ is an equilibrium solution.

24. The following differential equation has been proposed to model grazing by a herd of cows on vegetation in a certain field:[23]

$$\frac{dV}{dt} = \frac{1}{3}V - \frac{1}{75}V^2 - H\frac{0.1V^2}{9+V^2}. \tag{2.107}$$

In this equation time t is measured in days, $V(t)$ is the amount of vegetation in the field, in tons, at time t, H is the (constant) size of the herd, and

$$\frac{0.1V^2}{9+V^2}$$

is the amount of vegetation eaten per cow in one day. An initial condition for this model specifies the amount of vegetation at time 0, $V(0) = V_0$. The term $H\frac{0.1V^2}{9+V^2}$, which is sometimes referred to as the "predation term" has a graph whose general shape is shown below. It starts out small, grows slowly up to a certain point, then increases sharply until it begins to level off. This leveling off represents a saturation—the cows can only eat so much per day, no matter how abundant the vegetation.

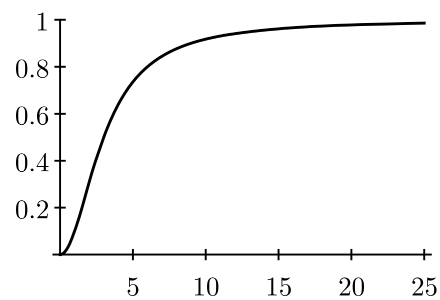

Figure 2.57. Graph of $V^2/(V^2+9)$.

(a) Show that if $H = 0$, the vegetation grows logistically, and find the carrying capacity.
(b) Figs. 2.58–2.60 show the graphs of

$$y = \frac{1}{3}V - \frac{1}{75}V^2 \text{ (dashed)} \quad \text{and} \quad y = H\frac{0.1V^2}{9+V^2} \text{ (solid)} \tag{2.108}$$

for three different herd sizes: $H = 10$, $H = 20$, and $H = 30$. For each of these three herd sizes, use the information available in these graphs to predict that there is one positive equilibrium solution to equation (2.107) when $H = 10$, three positive equilibrium solutions when $H = 20$, and one positive equilibrium solution when $H = 30$.
(c) For the smallest herd size, $H = 10$, approximately what is the positive equilibrium solution? Predict what happens as $t \to \infty$ to those solutions that have $V(0)$ greater than this equilibrium value. Predict what happens to those solutions with $V(0)$ less than the equilibrium value.
(d) Now focus on the intermediate herd size $H = 20$. Using the information in the graphs in Figs. 2.58–2.60, give a rough sketch of several solutions to the differential equation, showing V as a function of t, with various initial conditions. Begin by showing the equilibrium solutions (as horizontal lines) in your sketch. Show that if $V(0) = 6$, then the amount of vegetation decreases for $t > 0$, but if $V(0) = 9$, the amount of vegetation increases for $t > 0$.

[23] R. M. May, *Stability and Complexity in Model Ecosystems*, Princeton University Press, Princeton, NJ, 1974.

(e) If your goal is to ensure that V eventually stays above 10, is it better to have $H = 20, V(0) = 9$ or $H = 30, V(0) = 16$?

(f) (CAS) Using your computer algebra system's ability to plot implicitly, plot a bifurcation diagram for equation (2.107) in the region $0 \le H \le 40$, $-5 \le V \le 30$ of the HV-plane. Approximately what value of H in the interval $[0, 40]$ is a bifurcation value?

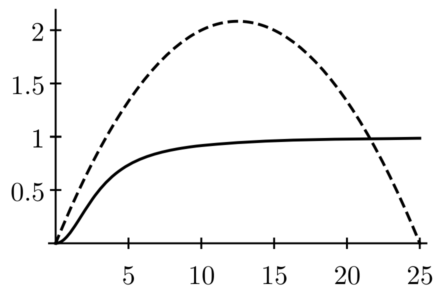

Figure 2.58. $H = 10$.

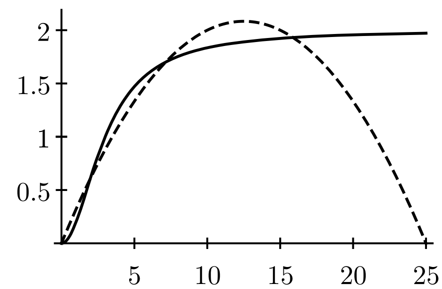

Figure 2.59. $H = 20$.

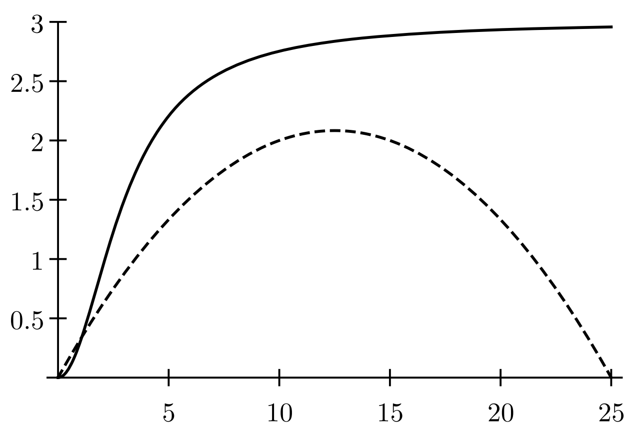

Figure 2.60. $H = 30$.

2.7. Change of variable

In this section we will look at some first-order differential equations for which a change of variable can be used to convert the equation to one that is either separable or linear. Choosing useful substitutions is something of an art form; nevertheless, we can single out certain types of equations for which there are general principles.

2.7.1. Homogeneous equations.

Example 2.7.1. Consider the equation

$$\frac{dy}{dx} = \frac{y^3 - x^3}{y^2 x}$$

and convince yourself that this equation is neither separable nor linear. However, notice that the right-hand side can be written as

$$\frac{y^3 - x^3}{y^2 x} = \frac{y^3}{y^2 x} - \frac{x^3}{y^2 x} = \frac{y}{x} - \left(\frac{x}{y}\right)^2,$$