

## CHAPTER 1

# The First Day

### PROBLEM LIST

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Professor Mortimer Ignatius Blakeley walked into Room C-122 on the first day of Problem Solving Camp. Eight eager but somewhat apprehensive aspiring mathematicians were seated at their desks. Professor Blakeley already knew quite a bit about these students from having read their files during the selection process and from the reception the previous night. They had been chosen from a highly talented group of undergraduate math majors throughout the USA, and now the time had finally come to see how they would do!

The Problem Solving Camp students were the following:

- Emiliano Alvarez, whose parents moved to the Boston area from Ecuador shortly before Emiliano was born.
- Daniel Arkin, from the Milwaukee area.
- Sandra Billingsley, a resident of Cincinnati.
- Sam Dalton, from Columbia, Missouri.
- Clayton Martin, a black student from Tucson.
- Patrick O’Connell, a resident of Ireland who came to the USA for his undergraduate education.
- Jung Wook Park, whose home was in Seoul, Korea. He also came to the USA for the first time to begin his undergraduate studies.
- Fumei Yang, a student who came with her family from Guizhou Province in China to a suburb of Denver when she was 12.

“Welcome to Problem Solving Camp!” said Professor Blakeley. “I am sure you will find this a great opportunity to learn new mathematics and to enjoy working on some challenging problems. You are already familiar with the course mechanics. To quickly review, in the mornings we will discuss mathematics and work on some new problems. In the afternoon session I’ll pass out some problems related to the morning’s discussion, and we can work on them with the help of a couple of course assistants. We can also discuss progress on earlier problems that have not yet been solved. Now let us get down to serious business!

“First some basic notation that will be used throughout the course. You are all familiar with the notation  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$  for the integers, rationals, real numbers, and complex numbers. By the way, why is the letter  $\mathbb{Z}$  used for the integers? Why not  $\mathbb{I}$ ?”

“Zahl,” said Emiliano quickly.

“Exactly!” said Professor Blakeley. “Zahl is the German word for ‘number’ and Zahlen for ‘numbers.’ And who first used the letter  $\mathbb{Z}$ , in whatever font, to denote the integers?”

No one replied.

“Sad . . . , such a basic question. I believe it was first used by Nicolas Bourbaki in the 1930’s. I suppose you all know that Nicolas Bourbaki is a pseudonym for a group of mostly French mathematicians that began around 1935 trying to reformulate much of math in an extremely abstract and formal manner.

“Even if you don’t know the history of the notation  $\mathbb{Z}$ , at least you should know how to write it. It should be written in two strokes, like this:



“Whatever you do, don’t make the mistake of beginning  $Z$ . You will then get the ugly, lopsided  $\mathbb{Z}$ .”

The students were beginning to get the idea that Professor Blakeley’s lecturing style was a little eccentric.

“Well, let us move on to a little more notation. I use  $\mathbb{N}$  to denote the set of nonnegative integers and  $\mathbb{P}$  for the positive integers. Sometimes  $\mathbb{N}$  is used to denote the natural numbers, but there doesn’t seem to be universal agreement on whether 0 is a natural number. Therefore you should think  $\mathbb{N}$  for ‘nonnegative’ and  $\mathbb{P}$  for ‘positive.’ Finally, when  $n \in \mathbb{N}$  I will write  $[n] = \{1, 2, \dots, n\}$ , so in particular  $[0] = \emptyset$ . Where did the symbol  $\emptyset$  come from?”

No one volunteered, so Professor Blakeley continued, “It is a letter from the Norwegian alphabet that was suggested by André Weil. It has nothing to do with the Greek letter  $\phi$ .”

“Finally the time has arrived to do some mathematics! We should start out with a problem with important real-world applications. And what could be more important than the question of how to cut cheese? Therefore suppose that we want to cut a cube of cheese that is three inches on each side into  $27 \times 1 \times 1 \times 1$  inch cubes. You are allowed to rearrange the pieces after each cut and to cut more than one piece at once, but each cut must be a single planar slice. What is the least number of cuts necessary? Please keep quiet if you have seen this problem before.”

“The number of pieces can at most double with each cut. Since  $2^4 < 27$  it will take at least five cuts. It shouldn’t be hard to rearrange the pieces to achieve this,” said Daniel. Professor Blakeley knew from Daniel’s application that he was very clever but somewhat impulsive.

“Wait a second,” said Fumei. “The first cut is really lopsided. No matter how you cut, you’ll have one piece with nine squares and one with 18...” Several students were about to speak, but Fumei continued. “In fact, since  $2^4 < 18$  you need at least five more cuts, and obviously you can do it in six cuts.”

Professor Blakeley was impressed. Although Fumei’s solution was not the most elegant, it was a good argument that had a lot of potential for generalization. “Very good!” he said. “Does anyone who has seen the problem before know the most elegant solution?”

Jung Wook Park spoke up. “The center cube has six sides. Each needs a separate cut.”

“Exactly,” said Professor Blakeley. “Very clever, but perhaps we can get further with Daniel’s and Fumei’s approach. What about the general case: an  $n_1 \times n_2 \times \dots \times n_d$  ‘brick’ in  $d$ -dimensional space. The math may be interesting, though we lose the applicability to cheese cutting.”

Clayton said, “Not necessarily. I’ve been wondering how to cut a four-dimensional block of cheese that popped out of one of my Klein bottles.”

“I stand corrected.” Just from talking with Clayton for a few minutes at the reception, Professor Blakeley knew that he had a well-developed sense of humor. “But we can still think about how to cut an  $n_1 \times n_2 \times \dots \times n_d$  brick. Let’s mull over this, and please raise your hand if you think you have a solution or just a significant insight.”

Professor Blakeley patiently waited while the students worked furiously. After only a couple of minutes the hand of Sandra went up, soon followed by Jung Wook and Daniel.

*Amazing*, thought Professor Blakeley. “Well, Sandra, since you raised your hand first, let’s hear your thoughts. Please come to the board and explain.”

Sandra walked up to the front of the classroom and explained with the help of the chalkboard, “It requires  $\lceil \log_2(n_i) \rceil$  cuts to break up a line of  $n_i$  bricks in the  $i$ th coordinate direction. Since we can cut up only one direction at a time for each piece, a lower bound on the number of cuts is

$$(1.1) \quad \sum_{i=1}^d \lceil \log_2(n_i) \rceil.$$

It seems pretty clear this should be possible to obtain just by doing the strategy for  $d = 1$  one direction at a time.”

“Excellent!” said Professor Blakeley. “Indeed, it’s not much work to make your argument completely rigorous. This is really the way to look at this problem, and not by some special trick that works only in a handful of cases.

“There are some different kinds of problems related to cake cutting. Let’s start with a certain mathematical game played between two players, Alice and Bob (the traditional names for the players of two-player math games). This time we are cutting chocolate bars rather than cheese. Suppose we have such a bar divided by indentations into an  $m \times n$  array of squares. Alice begins and (unless  $m = n = 1$ ) breaks the bar into two pieces by cutting along one of the grid lines. The players take turns choosing one of the pieces and cutting it into two along one of the grid lines, if this piece is not just a  $1 \times 1$  square. The first player unable to move loses. This situation will occur when we reach  $mn$   $1 \times 1$  squares. For each value of  $m$  and  $n$ , who will win under optimal play? What is the correct strategy for the winner? Again raise your hand if you see the answer.”

This time hands went up very quickly. Professor Blakeley pointed to Daniel, implicitly asking him to speak.

Daniel said, “The number of pieces increases by one after each turn. Therefore the game will end after  $mn - 1$  turns. Thus Alice wins if and only if  $mn$  is even. It doesn’t matter how she plays.”

“Good, good! Probably many of you would get bored very quickly playing this game. Not only is the winner independent of how either player plays, but also the number of moves. The game is called *impartial cutcake*, ‘impartial’ because at any stage of the game, the moves available to each player are the same.

“Let’s try to make things more interesting. Rather than have the same moves available for each player, let each cut be available to just one of the players. To be specific, regard our  $m \times n$  chocolate bar to be an array with  $m$  rows and  $n$  columns. Alice can only make vertical cuts separating two adjacent columns (so on her first turn she has  $n - 1$  choices), and similarly Bob for rows. Alice goes first as before, and the first player unable to move loses. This game is called simply *cutcake*. Who wins with optimal play? Raise your hand if you have an idea.”

After a short while Emiliano raised his hand. After Professor Blakeley nodded he said, “If  $n > m$ , say, then Alice has more available moves. It seems plausible that Alice will always win.”

Fumei, who had been furiously computing some small cases, said, “That doesn’t seem correct. For the  $3 \times 2$  case, Alice will lose whether or not she has the horizontal or vertical cuts.”

Emiliano replied, “I see . . . . But it must be the case that if  $n$  is sufficiently larger than  $m$ , then Alice will win if she has the vertical cuts.”

Meanwhile Patrick had been writing some code on his PC. Professor Blakeley knew that he was a computer whiz. Patrick said, “I’ve checked that Alice wins on a  $1 \times n$  board for any  $n \geq 2$  (trivial), on a  $2 \times n$  board for  $n \geq 4$ , and on a  $3 \times n$  board for  $n \geq 8$ . I’m using the obvious fact that if Alice wins on  $m \times n$ , then she also wins on  $m \times (n + 1)$ .”

“Great!” said Professor Blakeley. “That does suggest an obvious conjecture. While you’re at it, consider the generalization where we begin with finitely many chocolate bars of various sizes. Let’s take a short break before getting back to these questions.”

Professor Blakeley was pleased to see that the students spent most of their break time thinking about chocolate bars. A couple of students were even eating them. What better way to spend a break from mathematics than doing mathematics? Also during the break Clayton approached Professor Blakeley with scissors and a piece of paper divided into a  $3 \times 5$  grid of squares. Clayton challenged the Professor to a game of impartial cutcake and said that the Professor could move first. “That’s very generous of you, Clayton,” said Professor Blakeley, “but I don’t have much of a talent for mathematical games, so you should find a stronger opponent.”

After the break Professor Blakeley said, “Let me first say some general words concerning games like cutcake. I will speak somewhat informally and not give precise definitions. A game is called *partizan* if it is always disadvantageous to move. It is not hard to see that cutcake is a partizan game. Any cut a player makes to a rectangle  $R$  simply removes that option for him- or herself while doubling the number of options for the other player on the two pieces into which  $R$  has been cut. It turns out that for any partizan game  $G$  we can assign a real number  $\nu(G)$  called the *value* of the game. We should think that  $\nu(G)$  represents the number of moves that Alice is ahead of Bob. For instance, a  $1 \times n$  chocolate bar  $R$  has value  $\nu(R) = n - 1$ . If  $R$  is part of a larger collection of chocolate bars of various sizes, then Alice has  $n - 1$  possible moves which she can take on  $R$  at her leisure, and Bob can do nothing to disrupt these  $n - 1$  moves. Similarly, an  $m \times 1$  rectangle has value  $-(m - 1)$ , since now Alice is  $m - 1$  moves behind. A  $1 \times 1$  rectangle has value 0, meaning that mover loses. If we have a set  $S = \{R_1, \dots, R_k\}$  of disjoint chocolate bars, then a move on  $S$  consists of cutting some  $R_i$  in an allowed way. We can denote this by writing  $S = R_1 + \dots + R_k$ . It seems very plausible and can be made completely rigorous that

$$\nu(S) = \nu(R_1) + \dots + \nu(R_k).$$

In particular,  $\nu(S) = 0$  if and only if mover loses (with optimal play). This is quite clear when all the  $R_i$ ’s are of sizes  $m_j \times 1$  and  $1 \times n_j$ . The player’s moves are completely independent, so it is just a question of counting how many moves each player has. One way to show that a complicated position  $S$  has a certain value  $v$  is to take a rectangle  $R$  that is known to have value  $-v$  and then to show that mover loses in the game  $S + R$ . Thus to understand cutcake completely, it suffices to determine the value  $\nu(R)$  of any rectangle.

“Let us write for instance  $2R_{a,b} + 3R_{c,d}$  to be the game with two  $a \times b$  rectangles and three  $c \times d$  rectangles. A tedious analysis of all possibilities shows that if we begin with  $R_{3,8} + R_{4,1}$ , then mover loses. Since  $\nu(R_{4,1}) = -3$  it follows that  $\nu(R_{3,8}) = 3$ . Now one can also show that  $\nu(R_{5,9}) = 1$ . Note that in general  $\nu(R_{a,b}) = -\nu(R_{b,a})$ , since

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	-1	0		1		2		3		4		5		6	
3	-2	0		1		2		3		4		5		6	
4	-3	-1		0				1				2			
5	-4	-1		0				1				2			
6	-5	-2		0				1				2			
7	-6	-2		0				1				2			
8	-7	-3		-1				0							
9	-8	-3		-1				0							
10	-9	-4		-1				0							
11	-10	-4		-1				0							
12	-11	-5		-2				0							
13	-12	-5		-2				0							
14	-13	-6		-2				0							
15	-14	-6		-2				0							

FIGURE 1.1. Cutcake values

on  $R_{a,b} + R_{b,a}$  the second player can always win by an obvious mimicking strategy. Thus we see that  $R_{3,8} + 3R_{9,5} = 0$ , i.e., mover loses on  $R_{3,8} + 3R_{9,5}$ . I hope this simple example illustrates sufficiently the power of the value function  $\nu$ .

“So the problem we need to solve is to give a simple formula for  $\nu(R_{i,j})$ , or at least a simple method to compute this number. Any ideas?”

Fumei said, “It would help to see some data so I can get some feeling for what is going on.”

“Exactly!” said Professor Blakeley. “Unless you have unusual insight and see what to do immediately, naturally you want to see some data. In fact, for this problem a sufficient amount of data leads to an obvious conjecture. One way of saying the rule is that if  $m \leq n$  and  $m$  has  $k + 1$  binary digits, then  $\nu(R_{m,n}) = v$  if and only if  $(v + 1)2^k \leq n \leq (v + 2)2^n - 1$ . In particular,  $\nu(R_{m,n}) = 0$  if and only if  $m$  and  $n$  have the same number of binary digits.”

“Here is a table of  $\nu(R_{m,n})$  for  $m, n \leq 15$ ,” continued Professor Blakeley as he displayed Figure 1.1 on the screen. “Once the value of  $\nu(R_{m,n})$  is guessed, its validity is straightforward to prove by induction. I’ll let you figure this out by yourself.

“Cutcake happens to be an especially simple partizan game because the value of every position is an integer. In more complicated partizan games we can get fractional values. For nonpartizan games the situation becomes much more complicated. I don’t need to discuss this with you, because all this and much more is

thoroughly covered in the fantastic book *Winning Ways* by Berlekamp, Conway, and Guy.<sup>1</sup> If you like this kind of mathematical game theory, then be sure to take a look at this book.

“I would be remiss if I didn’t at least mention the theory of ‘fair cake-cutting,’ although we don’t have time for proofs. I’m sure you’re familiar with the procedure of dividing some object like a piece of cake among two persons. The two persons may have different ‘valuation functions’ which tell them how much they like each piece of the cake. The goal is to share the cake so each person thinks they got at least their fair share (called *proportionality*) and neither person thinks the other got more than their fair share (called *envy-freeness*). It is easy to see that envy-freeness implies proportionality. The well-known method is for one person to divide the cake into two pieces which he or she regards of equal value, and for the second person to choose one of the pieces. We have to put some reasonable assumptions on the valuation functions in order for this procedure to work. We can regard a *valuation* as a finitely additive nonatomic probability distribution  $V$ . Thus the value of the entire cake is 1, each individual point has value 0, and if  $A, B$  are disjoint pieces of cake, then  $V(A \cup B) = V(A) + V(B)$ .

“For three persons, an envy-free algorithm was devised in 1960 by John Selfridge, later independently found by Conway. It requires as many as five slices. In 1995 Steven Brams and Alan Taylor found a finite procedure that works for any number  $n$  of persons, but the number of slices for sufficiently large fixed  $n$  is an unbounded function of the valuations. Finally in 2018 Haris Aziz and Alan Mackenzie described an envy-free algorithm that has a bounded number of slices for fixed  $n$ . If  $f(n)$  is the least number of slices which suffices for any  $n$  valuations, then it is known that  $f(n) \geq cn^2$ , while the upper bound is an exponential tower of six  $n$ ’s. Quite a difference!

“Let’s see if we can figure out an algorithm for eight persons.” Professor Blakeley then reached into his backpack and pulled out a cupcake wrapped in cling wrap and a plastic knife. He unwrapped the cupcake and said, “I am happy to have all of you share this delicious cupcake, but let’s first divide it fairly into eight pieces. Who would like to begin?”

None of the students were able to think of an algorithm that could conceivably take an exponential tower of six 8’s slices. They had their doubts about the practicality of such an algorithm, even if they could discover it. Even  $8^{8^8}$  slices would far exceed the number of atoms in the cupcake.

“Not to worry,” said Professor Blakeley, “there is a simple solution!” He reached into his backpack and pulled out seven more cupcakes. He then gave a cupcake to each student.

“But Professor Blakeley,” said Sandra, “it doesn’t seem fair that you are getting nothing.”

“A very good point!” replied the Professor. Once more he reached into his backpack, pulled out a cupcake about 50% larger than any of the others, unwrapped it, and took a big bite. The other students also began eating their cupcakes, and thus ended the morning class.

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<sup>1</sup>According to [101], Richard Guy was for several years the world’s oldest living mathematician, until his death on March 9, 2020. He was born on September 30, 1916.