

Bibliography

- [Ahlfors] L. V. Ahlfors, *Complex analysis: An introduction of the theory of analytic functions of one complex variable*, 2nd ed., McGraw-Hill Book Co., New York-Toronto-London, 1966. MR0188405
- [Apostol] T. M. Apostol, *Mathematical analysis: a modern approach to advanced calculus*, Addison-Wesley Publishing Company, Inc., Reading, Mass., 1957. MR0087718
- [Arbaugh et al.] F. Arbaugh, M. Smith, J. Boyle, and M. Steele, *We Reason & We Prove for All Mathematics*, Corwin, Thousand Oaks, CA, 2018.
- [Beck-Bleicher-Crowe] A. Beck, M. N. Bleicher, and D. W. Crowe, *Excursions into mathematics*, The millennium edition, with a foreword by Martin Gardner, A K Peters, Ltd., Natick, MA, 2000. MR1744676
- [Bers] L. Bers, *Calculus*, Holt, Rinehart and Winston, 1969. (amazon.com has inexpensive used copies.)
- [Birkhoff-MacLane] G. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, 4th ed., Macmillan, New York, NY, 1977.
- [Boas] R. P. Boas Jr., *Inequalities for the derivatives of polynomials*, Math. Mag. **42** (1969), 165–174, DOI 10.2307/2688534. MR259429
- [Boyer-Merzbach] C. B. Boyer, *A history of mathematics*, 2nd ed., edited and with a preface by Uta C. Merzbach, John Wiley & Sons, Inc., New York, 1989. MR996888
- [Buck] R. C. Buck, *Advanced calculus*, 2nd ed., with the collaboration of Ellen F. Buck, McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1965. MR0265522
- [CCSSM] *Common Core State Standards for Mathematics*, June, 2010, <http://www.corestandards.org/Math/>
- [Courant-Robbins] R. Courant and H. Robbins, *What Is Mathematics?*, Oxford University Press, New York, 1941. MR0005358
- [Ellis-Bieda-Knuth] A. B. Ellis, K. Bieda, and E. Knuth, *Essential Understanding of Proof and Proving*, National Council of Teachers of Mathematics, Reston, VA, 2012.
- [Euclid1] Euclid, *The Thirteen Books of the Elements*, Volume I, T. L. Heath, transl., Dover Publications, New York, NY, 1956.
- [Euclid2] Euclid, *The Thirteen Books of the Elements*, Volume II, T. L. Heath, transl., Dover Publications, New York, NY, 1956.
- [Feynman-HS] R. P. Feynman and A. R. Hibbs, *Quantum mechanics and path integrals*, emended ed., emended and with a preface by Daniel F. Styer, Dover Publications, Inc., Mineola, NY, 2010. MR2797644
- [Fraleigh-B] J. B. Fraleigh and R. A. Beauregard, *Linear Algebra*, 3rd ed., Addison-Wesley, Reading, MA, 1995.
- [Greenberg] M. J. Greenberg, *Euclidean and Non-Euclidean Geometries*, 4th ed., Freeman, New York, NY, 2008.
- [He] J.-H. He, *Zu-Geng's axiom vs Cavalieri's theory*, Appl. Math. Comput. **152** (2004), no. 1, 9–15, DOI 10.1016/S0096-3003(03)00529-0. MR2050049
- [Hess] A. Hess, *A highway from Heron to Brahmagupta*, Forum Geom. **12** (2012), 191–192, <http://forumgeom.fau.edu/FG2012volume12/FG201215.pdf>. MR2955633
- [HLP] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed., Cambridge, at the University Press, 1952. MR0046395
- [Hobson] E. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Dover Publications, New York, NY, 2005.
- [Kalman] D. Kalman, *Fractions with cycling digit patterns*, College Math. J. **27** (1996), no. 2, 109–115, DOI 10.2307/2687398. MR3599374

- [Katz] V. J. Katz, *A History of Mathematics*, 3rd ed., Addison-Wesley, Boston, MA, 2008.
- [Kiselev] *Kiselev's Geometry. Book II. Stereometry*, adopted from Russian by Alexander Givental, Sumizdat, 2008. <http://www.sumizdat.org>
- [Landau] E. Landau, *Foundations of Analysis*, reprint ed., AMS Chelsea Publishing, American Mathematical Society, Providence, RI, 2001.
- [Mac1] MacTutor History of Mathematics, Zu Chongzhi Biography, <http://tinyurl.com/baqrvj>
- [Mac2] MacTutor History of Mathematics, Zu Geng Biography, <http://tinyurl.com/jal5csc>
- [Maor] E. Maor, *Trigonometric delights*, Princeton University Press, Princeton, NJ, 1998. MR1614631
- [MUST] *Mathematical Understanding for Secondary Teaching*, M. K. Heid, P. S. Wilson, and G. W. Blume (eds.), National Council of Teachers of Mathematics, IAP, Charlotte, NC, 2015.
- [NCTM1989] National Council of Teachers of Mathematics, *Curriculum and Evaluation Standards for School Mathematics*, NCTM, Reston, VA, 1989.
- [NCTM2009] National Council of Teachers of Mathematics, *Focus in High School Mathematics: Reasoning and Sense Making*, NCTM, Reston, VA, 2009.
- [Newton-Poon] X. A. Newton and R. C. Poon, Mathematical Content Understanding for Teaching: A Study of Undergraduate STEM Majors, *Creative Education* 6 (2015), 998–1031, <http://tinyurl.com/glwzmd>
- [Osler] T. J. Osler, An easy look at the cubic formula, <https://tinyurl.com/y5k4dtz8>
- [Physics-notes] *Physics of Music–Notes*, <http://www.phy.mtu.edu/~suits/clarinet.html>
- [Pólya-Szegő] G. Pólya and G. Szegő, *Problems and Theorems in Analysis*, Volume I and Volume II, Springer-Verlag, Berlin and New York, 1972 and 1976.
- [Quinn] F. Quinn, *A revolution in mathematics? What really happened a century ago and why it matters today*, Notices Amer. Math. Soc. **59** (2012), no. 1, 31–37, <https://www.ams.org/notices/201201/rtx120100031p.pdf>, DOI 10.1090/noti787. MR2908157
- [Rilke] Rainer Maria Rilke, *Letters to a Young Poet*, Letter Seven, translation by Stephen Mitchell, <http://www.carrothers.com/rilke7.html>
- [Rockafellar] R. T. Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J., 1970. MR0274683
- [Rosenlicht] M. Rosenlicht, *Introduction to analysis*, reprint of the 1968 edition, Dover Publications, Inc., New York, 1986. MR851984
- [Ross] K. A. Ross, *Elementary analysis: the theory of calculus*, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1980. MR560320
- [Rudin] W. Rudin, *Principles of mathematical analysis*, 3rd ed., International Series in Pure and Applied Mathematics, McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976. MR0385023
- [Schmid-Wu] W. Schmid and H. Wu, The major topics of school algebra, 2008. Retrieved from <https://math.berkeley.edu/~wu/NMPalgebra7.pdf>
- [Simmons] G. F. Simmons, *Calculus with Analytic Geometry*, McGraw Hill, New York, NY, 1985 (available in used book market).
- [Simon] L. Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, vol. 3, Australian National University, Centre for Mathematical Analysis, Canberra, 1983. MR756417
- [Smith1] J. O. Smith, III, Introduction and Overview to *Spectral Audio Signal Processing*, <https://tinyurl.com/yafcv28t>
- [Smith2] J. O. Smith, III, Why sinusoids are important, <https://tinyurl.com/ybk4pqba>
- [Stewart] J. Stewart, *Calculus: Early Transcendentals*, 5th ed., Brooks/Cole, Belmont, CA, 2003 (any earlier edition would do; available in used book market).
- [WikiBrahmagupta] Wikipedia, Brahmagupta's formula. Retrieved from https://en.wikipedia.org/wiki/Brahmagupta%27s_formula
- [WikiBretschneider] Wikipedia, Bretschneider's formula. Retrieved from https://en.wikipedia.org/wiki/Bretschneider%27s_formula
- [WikiChebyshev] Wikipedia, Chebyshev polynomials. Retrieved from <http://tinyurl.com/y4x3dqw3>
- [WikiFermat] Wikipedia, Fermat's Last theorem. Retrieved from https://en.wikipedia.org/wiki/Fermat%27s_Last_Theorem

- [Wu1999] H. Wu, On the education of mathematics majors. In *Contemporary Issues in Mathematics Education*, E. Gavosto, S. G. Krantz, and W. G. McCallum (eds.), MSRI Publications, Volume 36, Cambridge University Press, Cambridge, England, 1999, 9–23, <http://math.berkeley.edu/~wu/math-majors.pdf>
- [Wu2006] H. Wu, How mathematicians can contribute to K–12 mathematics education, *Proceedings of International Congress of Mathematicians, 2006*, Volume III, European Mathematical Society, Madrid, 2006, Zürich, 2006, 1676–1688, <http://math.berkeley.edu/~wu/ICMtalk.pdf>
- [Wu2008] H. Wu, Fractions, decimals, and rational numbers, February 29, 2008. Retrieved from <http://math.berkeley.edu/~wu/NMPfractions4.pdf>
- [Wu2010] H. Wu, Introduction to School Algebra (draft of textbook for teachers of grades 6–8) (August 14, 2010). Retrieved from <http://math.berkeley.edu/~wu/Algebrasummary.pdf>
- [Wu2011] H. Wu, *Understanding Numbers in Elementary School Mathematics*, Amer. Math. Soc., Providence, RI, 2011, <https://bookstore.ams.org/mbk-79/>
- [Wu2014] H. Wu, Potential impact of the Common Core Mathematics Standards on the American Curriculum. In *Mathematics Curriculum in School Education*, Y. Li and G. Lappan (eds.), Springer, Dordrecht, 2014, pp. 119–142, https://math.berkeley.edu/~wu/Common_Core_on_Curriculum_1.pdf
- [Wu2015] H. Wu, Mathematical education of teachers, Part II: What are we doing about Textbook School Mathematics?, AMS Blogs, March 1, 2015. Retrieved from <https://tinyurl.com/y46wnahl>
- [Wu2016a] H. Wu, *Teaching School Mathematics: Pre-Algebra*, Amer. Math. Soc., Providence, RI, 2016, <https://bookstore.ams.org/mbk-98/>. Its **Index** is available at <http://tinyurl.com/zjugv14>.
- [Wu2016b] H. Wu, *Teaching School Mathematics: Algebra*, Amer. Math. Soc., Providence, RI, 2016, <https://bookstore.ams.org/mbk-99/>. Its **Index** is available at <http://tinyurl.com/haho2v6>.
- [Wu2018] H. Wu, The content knowledge mathematics teachers need. In *Mathematics Matters in Education—Essays in Honor of Roger E. Howe*, Y. Li, J. Lewis, and J. Madden (eds.), Springer, Dordrecht, 2018, pp. 43–91. <https://math.berkeley.edu/~wu/Contentknowledge1A.pdf>
- [Wu2020a] H. Wu, *Rational Numbers to Linear Equations*, Amer. Math. Soc., Providence, RI, 2020.
- [Wu2020b] H. Wu, *Algebra and Geometry*, Amer. Math. Soc., Providence, RI, 2020.

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