Preface

Most people have (with the help of conventions) turned their solutions toward what is the easy and toward the easiest side of the easy; but it is clear that we must trust in what is difficult; everything alive trusts in it, everything in Nature grows and defends itself any way it can and is spontaneously itself, tries to be itself at all costs and against all opposition. We know little, but that we must trust in what is difficult is a certainty that will never abandon us ....

Rainer Maria Rilke ([**Rilke**])

This volume concludes a three-volume set on the mathematics of the secondary school curriculum, the first two volumes being [**Wu2020a**] and [**Wu2020b**]. This set is intended primarily for high school mathematics teachers and mathematics educators,[1] but it may also be of interest to college math students, curious parents, and others. The present volume—the third volume of the set—gives an exposition of trigonometry and calculus that respects mathematical integrity and is also aligned with the standard high school curriculum. Its leisurely discussion of the basic concepts related to the least upper bound axiom also bridges the transition from calculus to upper division college mathematics courses where proofs become the main focus of all discussions. For this reason, this volume should benefit beginning math majors as well. Because it is the third volume of a three-volume set, there are inevitably copious references throughout to the first two volumes, [**Wu2020a**] and [**Wu2020b**]. However, to make this volume as self-contained as possible, I have collected the relevant definitions and theorems from [**Wu2020a**] and [**Wu2020b**] in an appendix (page 383ff.).

These three volumes conclude a six-volume[2] exposition of the mathematics curriculum of K–12 that is, for a change, respectful of *mathematical integrity* as well as the standard school curriculum. In slightly greater detail, *mathematical integrity* means that each and every concept in these volumes is clearly defined, all statements are precise and unambiguous to prevent misunderstanding, every claim is supported by reasoning, the mathematical topics to be discussed are not stand-alone items to be studiously memorized but are an integral part of a coherent story, and finally, this story is propelled forward with a (mathematical) purpose.[3] A more expansive discussion of the urgent need as of 2020 for such an exposition can be

[1] We are using the term "mathematics educators" to distinguish university faculty in schools of education from school mathematics teachers.

[2] The volume [**Wu2011**] treats the mathematics curriculum of K–6 and the two volumes [**Wu2016a**] and [**Wu2016b**] are about the mathematics curriculum of grades 6–8.

[3] The precise meaning of *mathematical integrity* is given on page xxv.

found at the end of this preface, but also see the preface of [**Wu2020a**] that puts this need in a broader context.

Although these six volumes are primarily intended to be used for the professional development of mathematics teachers and the mathematical education of mathematics educators, they could equally well serve as the blueprint for a textbook series in K–12. The short essay, **To the Instructor** on pp. xix ff., gives a fuller discussion of this as well as some related issues.

The first chapter of this volume is about trigonometry. Although the topics discussed are fairly standard, its emphases differ from those found in *TSM* (*Textbook School Mathematics*), i.e., the mathematics in standard school textbooks[4] and in most other professional development materials. To begin with, we make explicit the fact that the trigonometric functions can be defined *only* because we know that two right triangles with a pair of equal acute angles are similar to each other. As is well known, it is not uncommon in TSM to treat these functions as if similar triangles play no role in the definitions (see Exercises 15 and 16 starting on page 31 for two examples of this phenomenon). This chapter also pays careful attention to the extension of the domain of definition of sine and cosine from $(0, 90)$ (think "acute angles") to the number line $\mathbf{R}$. This extension is usually glossed over with hand-waving, but since the reasoning behind the extension is actually quite delicate, we feel compelled to bring this issue to the attention of teachers as well as educators for their considerations of *sense making* and *reasoning* in school mathematics.

Among other notable deviations in this chapter from TSM, we can point to its emphasis on the importance of the addition formulas of sine and cosine. To further underscore their importance, we prove later in Section 6.7—once calculus becomes available—the theorem that the sine and cosine functions are characterized essentially by these very formulas. Another deviation from TSM is the careful explanation of the need to transition from degree measurements of angles to radian measurements; in the process, it gives a detailed proof of the conversion formula between degrees and radians in Section 1.5. Again, this explanation exemplifies *sense making* and *reasoning*. (Needless to say, it has *nothing* to do with "proportional reasoning", as TSM would have you believe.) Finally, Section 1.9 gives an elementary explanation of why, in the year 2020 when we are far removed from ancient astronomers' preoccupation with "solving triangles",[5] the sine and cosine functions still deserve our serious attention.

For both teachers and educators, the content focus of this chapter is clearly an essential component of what they need to know about trigonometry in order to discharge their basic professional obligations. In addition, the precise explanation given in the appendix of Section 1.4 (pp. 46ff.) of what a trigonometric identity is and what it means to prove such an identity should be of special interest because neither topic is treated adequately in TSM and both suffer from misconceptions that are perpetuated in the education literature.

Beyond Chapter 1, the rest of this volume revolves around the concept of *limit* and its applications. Because these three volumes claim to give a grade-appropriate exposition of the mathematics curriculum of grades 9–12 and because it is well

[4]For more information about TSM, see pp. xx–xxv.

[5]Ancient astronomy created trigonometry in order to "solve triangles"; see page 6.

known that "limit" never makes an overt appearance in the K–12 curriculum, this apparent contradiction demands an explanation. In fact, we will offer not one but two such explanations.

The first is that, for the purpose of improving calculus teaching in schools and for the purpose of bringing some mathematical clarity to the discussion of proofs and reasoning in education research, we have to help teachers and educators improve their own mastery of calculus. Because calculus is, *by design,* a procedure-oriented discipline, some basic familiarity with the formulas and their basic applications has to be taken for granted. However, a narrow emphasis on procedures can easily degrade a calculus class into ritualistic incantations of unproven formulas and mindless promotions of rote memorization over reasoning. Not surprisingly, calculus classes often become nothing more than that. There can be little hope of averting such an unappealing spectacle unless teachers have some idea of the reasoning behind the formulas and educators have the knowledge about *limits* and the proper perspective to discuss the relevant mathematical issues sensibly. Given the space limitations, this volume cannot possibly give a comprehensive discussion of all the standard procedures and applications as well as the requisite reasoning. For this reason, we have chosen to concentrate on the reasoning and leave most of the procedural aspects of calculus to other textbooks. (There are a few that give sensible presentations of the procedures, e.g., [**Bers**], [**Simmons**], and [**Stewart**].)

What stands in the way of a sensible presentation of the reasoning in calculus is the fact that *analysis*—as the theory of calculus is called—is mathematically sophisticated. Such being the case, the usual solution to this instructional dilemma is to either fake the reasoning by waving at it using only analogies, metaphors, and heuristic arguments, or revel in the analytic reasoning in all its austere glory by presenting it unvarnished, thereby making it inaccessible except to future STEM majors. The latter path is, in fact, what one normally encounters in most textbooks on introductory analysis. This volume tries to steer a middle course by presenting—no surprise—an *engineered* version[6] of analysis for teachers. There is no escaping the fact that we must confront the concept of limit, but here we restrict this discussion to the number line, i.e., one-variable calculus. For this reason, standard concepts about the plane, such as the definition of a planar region or convergence in the plane, are treated on a semi-intuitive level as otherwise the associated esoterica about *open sets* and *closed sets* needed for such definitions become overwhelming. In addition, we have managed to pare the technicalities down to an absolute minimum and keep our sights unflinchingly on topics that are directly relevant to K–12. Thus we will not mention compactness or cluster points of a sequence. In particular, no *subsequences, lim sup*, or *lim inf* will be found in this volume. This simplification has been achieved at the cost of losing a bit of generality in considerations of convergence, but in exchange, the exposition gains in accessibility. Like the learning of mathematics in general, it takes effort to learn about limits and convergence,[7] but we hope this volume will at least succeed in making the introductory part of analysis more accessible to teachers and educators.

The other explanation for taking up limit extensively in this volume has to do with the nature of the mathematics of K–12. Shocking as it may seem, the fact

[6]See [**Wu2006**] for the concept of *mathematical engineering.*

[7]See Rilke's advice to a young poet on page xi.

remains that *limit* lurks almost everywhere in the grades 6–12 mathematics curriculum. It is only through artful (or not so artful) suppression that *limit* manages to be well hidden in TSM. More precisely, the middle school curriculum includes the conversion of fractions to *repeating decimals* which are infinite decimals most of the time, the *circumference formula for a circle*, the *area formula for* (the inside of) *a circle*, and the concept of the *square root* or *cube root* of a positive number. In fact, even the area of a rectangle with side lengths 5 and $\sqrt{2}$ can only be explained by using limits! While these topics are usually taught in middle school with at most a casual reference to limits, teachers cannot teach these topics—and educators cannot hope to discuss these topics—sensibly if they themselves do not have a firm grounding in limits. Furthermore, the high school curriculum includes *exponential functions* such as 2^x, the *logarithm function*, extensive computations with numbers such as $\sqrt{3}$ and π, and the *radian* measure of an angle. Again, limit is deeply embedded in every one of these concepts and skills. None of these topics will make much sense in a school classroom unless teachers are able to draw on their (solid) knowledge of limits to make the lesson both understandable *as well as* mathematically honest. Not surprisingly, these topics tend not to make much sense in school classrooms as of 2020, thanks to TSM. Sometimes no great harm is done when this happens (e.g., most students seem to have no trouble memorizing the circumference of a circle as $2\pi r$), but at other times it can be devastating.

A striking example of the latter phenomenon is the perennial debate over whether the repeating decimal $0.\overline{9}$ is equal to 1. A teacher who knows nothing about limits is likely to regard each of the 9's in $0.99999\ldots$ as a "decimal digit" and therefore conclude that this number cannot be equal to $1.00000\ldots$ because, uh, you know, two *finite* decimals are equal if and only if they agree digit by digit and, therefore, the same must be true of infinite decimals. With such a mindset, it would be difficult for teachers to convince their students—or even themselves—that $0.\overline{9} = 1$. The resulting confusion in school classrooms has spilled into the internet,[8] with the result that we get to witness the spectacle of shouting matches about mathematics in cyberspace! Now imagine an alternate scenario. Suppose all our teachers were to know that, notwithstanding one's *intuitive* feelings, $0.\overline{9}$ is not "a decimal with an infinite number of decimal digits" because this phrase has no meaning. Rather, it is a symbol that calls for *taking the limit of a sequence of numbers* 0.9, 0.99, 0.999, 0.9999, etc. Therefore, $0.\overline{9} = 1$ is the statement that *the limit of the sequence* 0.9, 0.99, 0.999, 0.9999, etc., *is equal to* 1. With this understood, there should be no difficulty in accepting that $0.\overline{9} = 1$. Wouldn't it be more pleasant, educationally as well as mathematically, if all our teachers were to possess this kind of content knowledge? This is but one small example of how we hope to move school mathematics education toward a more desirable outcome by initiating a reasonable discussion of limits in the professional development of mathematics teachers.

The preceding discussion lays bare the fact that if a mathematics educator wants to engage in any sensible discussion of the mathematics of middle school and high school, a firm mastery of limits and convergence is a sine qua non. After all, any *conceptual understanding* of infinite decimals, laws of exponents, area of a circle, etc., ultimately resides in an understanding of limits and convergence, and it would be futile to try to make recommendations on the teaching and learning

[8]Try googling "Is 0.9 repeating equal to 1?".

of these topics without a complete understanding of the key issues that lie behind them. One can only speculate whether the travesty that is TSM in the teaching of infinite decimals (described above) or the laws of exponents (as briefly described, for example, in the introduction to Chapter 4 of [**Wu2020b**]) in grades 6–12 would have materialized had there been mathematically knowledgeable educators to keep a tight rein on the curriculum and textbooks.

This volume pays meticulous attention to all the aforementioned issues related to limits that are important in K–12. These include: why every infinite decimal is always a number (Section 3.1), why the "division of the numerator by the denominator of a fraction" yields a repeating decimal *equal to the fraction itself* (Section 3.4), why $0.\overline{9} = 1$ (Section 3.2), why a positive number always has a unique positive square root, and even a unique positive n-th root (Section 2.5), why one can compute with real numbers as if they were rational numbers (Section 2.1), what the number π is (Section 4.6), the meaning of the length of a curve and why the circumference of a circle is $2\pi r$ (Sections 4.3 and 4.6), the meaning of the area of an arbitrary region (Section 4.7), why the area of a rectangle is length times width (Section 4.4, *really!*), and why the area of a disk is πr^2 (Sections 4.7 and 4.6). Above all, a main goal of this foray into limits is to make sense of arbitrary exponents of a positive number α, i.e., α^x for any *real number* x, in order to be able to *prove* the laws of exponents in full generality (see the penultimate section of this volume, Section 7.3).

On the concept of area, Chapter 4 of this volume does more than make explicit the fundamental role of limit in its definition. It also takes seriously the invariance of area under congruence—something TSM does not—and demonstrates its importance by proving three area formulas for a triangle that are generally missing in TSM. To explain these formulas, consider the ASA congruence criterion for triangles: it says that all the triangles satisfying a given set of **ASA data** (the length of a side and the degrees of the two angles at the endpoints) are congruent and therefore have the same area. Thus a set of ASA data determines *uniquely* the area of any triangle satisfying the data. It follows that if we are given a set of ASA data for a triangle, there must be an area formula for the triangle directly in terms of the ASA data. The same is true for SAS and SSS. Therefore, as soon as the trigonometric functions are available, such formulas should be routinely proved in the standard curriculum if for no other reason than that of coherence (see page xxiv) and purposefulness (see page xxiv). But in TSM they are not. In Section 4.5 on pp. 237ff., we make up for the absence of these formulas in TSM by presenting them together with their proofs *in the context of the invariance of area under congruence*. Needless to say, the formula corresponding to SSS is the classical *Heron's formula* (see page 242). At the risk of belaboring the point, we call attention to the fact that, when Heron's formula is presented in the context of the area formula in terms of a set of SSS data, it ceases to be a curiosity item and becomes something entirely natural and inevitable. Now, it clearly serves a well-defined purpose and fills a mathematical niche.

As the last volume of this series that begins with [**Wu2020a**] and continues with [**Wu2020b**], this volume ties up the major loose ends left open from the earlier volumes. Precisely, it explicitly addresses the following five topics: why any positive number has a unique square root, cube root, and, in general, n-th root (Sections 2.1 and 4.2 of [**Wu2020b**]), why the division of one finite decimal by another yields a repeating decimal (Section 1.5 in [**Wu2020a**]), why FASM (the Fundamental Assumption of School Mathematics in Section 2.7 of [**Wu2020a**])—a cornerstone of most of these three volumes and a cornerstone of the mathematics of K–12—is correct, why FTS (the Fundamental Theorem of Similarity in Section 5.1 of [**Wu2020a**]) is correct, and why *rational* exponents have to be defined the way they are (see Section 4.2 of [**Wu2020b**]). The relevant explanations are given in Sections 2.5, 3.4, 2.1, 2.6, and 7.3, respectively.

These considerations bring us back to the beginning: why we have to devote something like 2,500 pages to a complete mathematical exposition of the school curriculum that respects mathematical integrity. First of all, these five topics are among the major topics of school mathematics, yet they have been consistently presented to students entirely by rote. One can take the pulse of the state of mathematics education in K–12, for example, by noting that we ask students to believe the *division of two finite decimals* to be (generally) equal to an *infinite decimal* without explaining to them what a finite decimal is,[9] what it means to divide a finite decimal by another, and what an infinite decimal is. In other words, we have a scandalous situation in which students have to believe that two things are equal even if they have absolutely no idea what either "thing" is. The least we can do here is present a *correct and grade-appropriate mathematical explanation* for all these topics and then wait for the pedagogical debate on how to modify these mathematical presentations to create more reasonable textbooks in K–12. These six volumes are a first attempt at accomplishing the former objective. Let us hope that the latter objective will materialize soon.

On a deeper level, however, there is probably no more compelling evidence than a consideration of these five topics to expose the urgent need for a *complete and systematic* mathematical overhaul—one that respects mathematical integrity—of the standard K–12 curriculum. A prime example is the case of FASM. In a span of six grades, roughly grades 3 to 8, students have to learn to compute, first with whole numbers, then fractions, then rational numbers, and finally real numbers. All known curricula—TSM, the reform curriculum of NCTM, or the CCSSM curriculum—specify that at least two years be spent on the transition from whole numbers to fractions and about one year for the transition from fractions to rational numbers. And yet there is no mention of any need to ease students' transition from rational numbers to real numbers (i.e., how to confront irrational numbers). This transition is so abrupt that one is at a loss to explain how such a glaring defect could have stayed under the radar of curricular discussions thus far if not for the total absence of any attempt to look at the mathematics of all thirteen grades of K–12 *longitudinally.* It would appear that the basic facts about the arithmetic of *real* numbers are considered to be so routine (or so shrouded in impenetrable mystery) that there is no need for any serious explanation. This is a curricular travesty of the first order.

Had any thought been given to providing guidance to students on how to add two quotients of irrational numbers at all, e.g.,

$$\frac{x}{2x^2+3} + \frac{5}{x^3-\sqrt{2}}$$

[9] In the 1990s, a third-grade textbook from a major publisher said that a finite decimal is "a whole number with a decimal point".

where x is an *irrational* number, and on *how* the addition algorithm for ordinary fractions remains correct in this context, it would have undoubtedly raised the following question to educators and textbook authors alike: whatever has happened to their former instruction on the use of the *least common denominators* for adding fractions? Could it be that the use of the least common denominator for adding fractions is a *mistake*? (See the discussion at the end of Section 1.3 in [**Wu2020a**].) If any attempt had been made to address this and related questions about multiplication and division of quotients of real numbers, the teaching of fractions in elementary school would have been in a far better place, school mathematics education would have been in a far better place, and the concept of FASM would have emerged decades ago without waiting for these six volumes to be written. Instead, students have been forced to "make believe" that real numbers can be handled "like" whole numbers so that, for example, they can "make believe" that x, $2x^2 + 3$, and $x^3 - \sqrt{2}$ above are "like" whole numbers. Therefore, to them, school mathematics education is little more than a collection of "make-believes" rather than the training ground for reasoning and critical thinking.

Analogous comments can be made about the other four topics above: the existence of n-th roots, the division of finite decimals, a complete proof of FTS, and the rationale behind the definitions of rational and irrational exponents. One can only surmise that such horrendous oversight has been due to the lack of any attempt to look at the whole school curriculum longitudinally from a mathematical perspective. These six volumes have made a first attempt at addressing and correcting this gross curricular oversight, but we fervently hope that this first attempt will not be the last.

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[10] Since the fall of 2018, this three-semester sequence has been pared down to a two-semester sequence. A partial study of the effects of these courses on pre-service teachers can be found in [**Newton-Poon**].

[11] Sadly, Dick passed away on October 9, 2019.

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