To the Instructor

These three volumes (the other two being [**Wu2020a**] and [**Wu2020b**]) have been written expressly for high school mathematics teachers and mathematics educators.[1] Their goal is to revisit the high school mathematics curriculum, together with relevant topics from middle school, to help teachers better understand the mathematics they are or will be teaching and to help educators establish a sound mathematical platform on which to base their research. In terms of mathematical sophistication, these three volumes are designed for use in upper division courses for math majors in college. Since their content consists of topics in the upper end of school mathematics (including one-variable calculus), these volumes are in the unenviable position of straddling two disciplines: mathematics and education. Such being the case, these volumes will inevitably inspire misconceptions on both sides. We must therefore address their possible misuse in the hands of both mathematicians and educators. To this end, let us briefly review the state of school mathematics education as of 2020.

The phenomenon of TSM

For roughly the last five decades, the nation has had a de facto national school mathematics curriculum, one that has been defined by the standard school mathematics textbooks. The mathematics encoded in these textbooks is extremely flawed.[2] We call the body of knowledge encoded in these textbooks **TSM** (**Textbook School Mathematics**; see page xix). We will presently give a superficial survey of some of these flaws,[3] but what matters to us here is the fact that institutions of higher learning appear to be oblivious to the rampant mathematical mis-education of students in K–12 and have done very little to address the insidious presence of TSM in the mathematics taught to K–12 students over the last 50 years. As a result, mathematics teachers are forced to carry out their teaching duties with all the misconceptions they acquired from TSM intact, and educators likewise continue to base their research on what they learned from TSM. So TSM lives on unchallenged.

These three volumes are the conclusion of a six-volume series[4] whose goal is to correct the universities' curricular oversight in the *mathematical education of*

[1] We use the term "mathematics educators" to refer to university faculty in schools of education.

[2] These statements about curriculum and textbooks do not take into account how much the quality of school textbooks and teachers' content knowledge *may* have evolved recently with the advent of **CCSSM** (Common Core State Standards for Mathematics) ([**CCSSM**]) in 2010.

[3] Detailed criticisms and explicit corrections of these flaws are scattered throughout these volumes.

[4] The earlier volumes in the series are [**Wu2011**], [**Wu2016a**], and [**Wu2016b**].

teachers and educators by providing the needed mathematical knowledge to break the vicious cycle of TSM. For this reason, these volumes pay special attention to *mathematical integrity*[5] and transparency, so that every concept is precisely defined and every assertion is completely explained,[6] and so that the exposition here is as close as possible to what is taught in a high school classroom.

TSM has appeared in different guises; after all, the NCTM reform (see [**NCTM1989**]) was largely ushered in around 1989. But beneath the surface its essential substance has stayed remarkably constant (compare [**Wu2014**]). TSM is characterized by a lack of clear definitions, faulty or nonexistent reasoning, pervasive imprecision, general incoherence, and a consistent failure to make the case about why each standard topic in the school curriculum is worthy of study. Let us go through each of these issues in some detail.

(1) Definitions. In TSM, *correct* definitions of even the most basic concepts are usually not available. Here is a partial list:

> fraction, multiplication of fractions, division of fractions, one fraction being bigger or smaller than another, finite decimal, infinite decimal, mixed number, ratio, percent, rate, constant rate, negative number, the four arithmetic operations on rational numbers, congruence, similarity, length of a curve, area of a planar region, volume of a solid, expression, equation, graph of a function, graph of an inequality, half-plane, polygon, interior angle of a polygon, regular polygon, slope of a line, parabola, inverse function, etc.

Consequently, students are forced to work with concepts whose mathematical meaning is at best only partially revealed to them. Consider, for example, the concept of division. TSM offers no precise definition of division for whole numbers, fractions, rational numbers, real numbers, or complex numbers. If it did, the division concept would become much more learnable because it is in fact the same for all these number systems (thus we also witness the incoherence of TSM). The lack of a definition for division leads inevitably to the impossibility of reasoning about the division of fractions, which then leads to "ours is not to reason why, just invert-and-multiply". We have here a prime example of the convergence of the lack of definitions, the lack of reasoning, and the lack of coherence.

The reason we need precise *definitions* is that they create a level playing field for all learners, in the sense that each person—including the teacher—has *all the needed information* about a given concept from the very beginning and this information is the same for everyone. This eliminates any need to spend time looking for "tricks", "insider knowledge", or hidden agendas. The level playing field makes every concept accessible to all learners, and this fact is what the discussion of *equity* in school mathematics education seems to have overlooked thus far. To put this statement in context, think of TSM's definition of a fraction as a piece of pizza: even elementary students can immediately see that there is more to a fraction than just being a piece of pizza. For example, "$\frac{5}{8}$ miles of dirt road" has nothing to do with pieces of a pizza. The credibility gap between what students are made to learn and what they subconsciously recognize to be false disrupts the learning process, often fatally.

[5]We will provide a definition of this term on page xxv.

[6]In other words, every theorem is completely proved. Of course there are a few theorems that cannot be proved in context, such as the fundamental theorem of algebra.

In mathematics, there can be no valid reasoning without precise definitions. Consider, for example, TSM's proof of $(-2)(-3) = 2 \times 3$. Such a proof requires that we know what -2 is, what -3 is, what properties these negative integers are assumed to possess, and what it means to multiply (-2) by (-3) so that we can use them to justify this claim. Since TSM does not offer any information of this kind, it argues instead as follows: $3 \cdot (-3)$, being 3 copies of -3, is equal to -9, and likewise, $2 \cdot (-3) = -6$, $1 \cdot (-3) = -3$, and of course $0 \cdot (-3) = 0$. Now look at the pattern formed by these consecutive products:

$$3 \cdot (-3) = -9, \quad 2 \cdot (-3) = -6, \quad 1 \cdot (-3) = -3, \quad 0 \cdot (-3) = 0.$$

Clearly when the first factor decreases by 1, the product *increases* by 3. Now, when the 0 in the product $0 \cdot (-3)$ decreases by 1 (so that 0 becomes -1), the product $(-1)(-3)$ ceases to make sense. Nevertheless, TSM urges students to believe that *the pattern must persist no matter what* so that this product will once more increase by 3 and therefore $(-1)(-3) = 3$. By the same token, when the -1 in $(-1)(-3)$ decreases by 1 again (so that -1 becomes -2), the product must again increase by 3 for the same reason and $(-2)(-3) = 6 = 2 \times 3$, as desired. This is what TSM considers to be "reasoning".

TSM goes further. Using a similar argument for $(-2)(-3) = 2 \times 3$, one can show that $(-a)(-b) = ab$ for all *whole numbers* a and b. Now, TSM asks students to take another big leap of faith: if $(-a)(-b) = ab$ is true for whole numbers a and b, then it must also be true when a and b are *arbitrary numbers*. This is how TSM "proves" that *negative times negative is positive.*

Slighting definitions in TSM can also take a different form: the graph of a linear inequality $ax + by \leq c$ is claimed to be a half-plane of the line $ax + by = c$, and the "proof" usually consists of checking a few examples. Thus the points $(0, 0)$, $(-2, 0)$, and $(1, -1)$ are found to lie below the line defined by $x + 3y = 2$ and, since they all satisfy $x + 3y \leq 2$, it is believable that the "lower half-plane" of the line $x + 3y = 2$ is the graph of $x + 3y \leq 2$. Further experimentation with other points below the line defined by $x + 3y = 2$ adds to this conviction. Again, no reasoning is involved and, more importantly, neither "graph of an inequality" nor "half-plane" is defined in such a discussion because these terms sound so familiar that TSM apparently believes no definition is necessary. At other times, reasoning is simply suppressed, such as when the coordinates of the vertex of the graph of $ax^2 + bx + c$ are peremptorily declared to be

$$\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a} \right).$$

End of discussion.

Our emphasis on the importance of definitions in school mathematics compels us to address a misconception about the role of definitions in school mathematics education. To many teachers and educators, the word "definition" connotes something tedious and nonessential that students must memorize for standardized tests. It may also conjure an image of cut-and-dried, top-down instruction that begins with a rigid and unmotivated definition and continues with the definition's formal and equally unmotivated appearance in a chain of logical arguments. Understandably, most educators find this scenario unappetizing. Their response is that, at least in school mathematics, the definition of a concept should emerge at the end—but not at the beginning—of an extended *intuitive discussion* of the hows and whys of the concept.[7] In addition, the so-called conceptual understanding of the concept is believed to lie in the intuitive discussion but not in the *formal definition* itself, the latter being nothing more than an afterthought.

These two opposite conceptions of definition ignore the possibility of a middle ground: one can state the precise definition of a concept at the beginning of a lesson to set the tone of the subsequent mathematical discussion and exploration, which is to show students that *this is all they will ever need to know about the concept as far as doing mathematics is concerned.* Such transparency—demanded by the mathematical culture of the past century (cf. [**Quinn**])—is what is most sorely missing in TSM, which consistently leaves students in doubt about what a *fraction* is or might be, what a *negative number* is, what *congruence* means, etc. In this middle ground, a definition can be explored and explained in intuitive terms in the ensuing discussion on the one hand and, on the other, *put to use in proofs—in its precise formulation—to show how and why the definition is absolutely indispensable to any kind of reasoning concerning the concept.* With the consistent use of precise definitions, the line between what is correct and what is intuitive but maybe incorrect (such as the TSM-proof of *negative times negative is positive*) becomes clearly drawn. It is the frequent blurring of this line in TSM that contributes massively to the general misapprehension in mathematics education about what a proof is (part of this misapprehension is described in, e.g., [**NCTM2009**], [**Ellis-Bieda-Knuth**], and [**Arbaugh et al.**]).

These three volumes (this volume, [**Wu2020a**], and [**Wu2020b**]) will always take a position in the aforementioned middle ground. Consider the definition of a fraction, for example: it is one of a special collection of points on the number line (see Section 1.1 of [**Wu2020a**]). This is the only meaning of a fraction that is needed to drive the fairly intricate *mathematical development* of fractions, and, for this reason, the definition of a fraction as a certain point on the number line is the one that will be unapologetically used all through these three volumes. To help teachers and students feel comfortable with this definition, we give an extensive intuitive discussion of why such a definition for a fraction is necessary at the beginning of Section 1.1 in [**Wu2020a**] before giving the formal definition. This intuitive discussion, naturally, opens the door to whatever pedagogical strategy a teacher wants to invest in it. Unlike in TSM, however, this definition is not given to be forgotten. On the contrary, *all* subsequent discussions about fractions will refer to this *precise definition* (but not to the intuitive discussion that preceded it) and, of course, all the proofs about fractions will also depend on this formal definition because mathematics demands no less. Students need to learn what a proof is and how it works; the exposition here tries to meet this need by (gently) laying bare the fact that reasoning in proofs requires precise definitions. As a second example, we give the definition of the slope of a line only after an extensive intuitive discussion in Section 6.4 of [**Wu2020a**] about *what* slope is supposed to measure and *how* we may hope to measure it. Again, the emphasis is on the fact that this definition of slope is not the conclusion, but the *beginning* of a long logical development that occupies the second half of Chapter 6 in [**Wu2020a**], reappears in trigonometry (relation

[7]Proponents of this approach to definitions often seem to forget that, after the emergence of a precise definition, students are still owed a *systematic exposition of mathematics using the definition* so that they can learn about how the definition fits into the overall logical structure of mathematics.

with the tangent function; see page 39), calculus (definition of the derivative; see pp. 310ff.), and beyond.

(2) Reasoning. Reasoning is the lifeblood of mathematics, and the main reason for learning mathematics is to learn how to reason. In the context of school mathematics, *reasoning* is important to students because it is the tool that empowers them to explore on their own and verify for themselves what is true and what is false without having to take other people's words on faith. *Reasoning* gives them confidence and independence. But when students have to accustom themselves to performing one unexplained rote skill after another, year after year, their ability to reason will naturally atrophy. Many students find it more expedient to stop asking *why* and simply take any order that comes their way sight unseen just to get by.[8] One can only speculate on the cumulative effect this kind of mathematics "learning" has on those students who go on to become teachers and mathematics educators.

(3) Precision. The purpose of *precision* is to eliminate errors and minimize misconceptions, but in TSM students learn at every turn that they should not believe exactly what they are told but must learn to be creative in interpreting it. For example, TSM preaches the virtue of using *the theorem on equivalent fractions* to simplify fractions and does not hesitate to simplify a rational expression in x as follows:

$$\frac{(x-1)(x^2+3)}{x(x-1)} = \frac{x^2+3}{x}.$$

This looks familiar because "canceling the same number from top and bottom" is exactly what the theorem on equivalent fractions is supposed to do. Unfortunately, this theorem only guarantees

$$\frac{ca}{bc} = \frac{a}{b}$$

when a, b, and c are *whole numbers* (b and c understood to be nonzero). In the previous rational expression, however, none of $(x-1)$, (x^2+3), and x is necessarily a whole number because x could be, for example, $\sqrt{5}$. Therefore, according to TSM, students in algebra should look back at equivalent fractions and realize that the theorem on equivalent fractions—*in spite of what it says*—can actually be applied to "fractions" whose "numerators" and "denominators" are not whole numbers. Thus TSM encourages students to believe that "nothing needs to be taken precisely and one must be flexible in interpreting what one learns". This extrapolation-happy mindset is the opposite of what it takes to learn a precise subject like mathematics or any of the exact sciences. For example, we cannot allow students to believe that the domain of definition of $\log x$ is $[0, \infty)$ since $[0, \infty)$ is more or less the same as $(0, \infty)$. Indeed, the presence or absence of the single point "0" is the difference between true and false.

Another example of how a lack of precision leads to misconceptions is the statement that "$\beta^0 = 1$", where β is a nonzero number. Because TSM does not use precise language, it does not—or cannot—draw a sharp distinction between a *heuristic argument*, a *definition*, and a *proof*. Consequently, it has misled numerous students and teachers into believing that the heuristic argument for defining β^0 to be 1 is in fact a "proof" that $\beta^0 = 1$. The same misconception persists for negative exponents (e.g., $\beta^{-n} = 1/\beta^n$). The lack of precision is so pervasive in TSM that there is no end to such examples.

[8]There is consistent anecdotal evidence from teachers in the trenches that such is the case.

(4) Coherence. Another reason why TSM is less than learnable is its incoherence. Skills in TSM are framed as part of a long laundry list, and the lack of definitions for concepts ensures that skills and their underlying concepts remain forever disconnected. Mathematics, on the other hand, unfolds from a few central ideas, and concepts and skills are developed along the way to meet the needs that emerge in the process of unfolding. An acceptable exposition of mathematics therefore tells a coherent story that makes mathematics memorable. For example, consider the fact that TSM makes the four standard algorithms for whole numbers four separate rote-learning skills. Thus TSM hides from students the overriding theme that the Hindu-Arabic numeral system is universally adopted because it makes possible a simple, algorithmic procedure for computations; namely, if we can carry out an operation ($+$, $-$, $\times$, or $\div$) for *single-digit numbers,* then we can carry out this operation for all whole numbers no matter how many digits they have (see Chapter 3 of [**Wu2011**]). The standard algorithms are the vehicles that bridge operations with *single-digit* numbers and operations on *all* whole numbers. Moreover, the standard algorithms can be simply explained by a straightforward application of the associative, commutative, and distributive laws. From this perspective, a teacher can explain to students, convincingly, why the multiplication table is very much worth learning; this would ease one of the main pedagogical bottlenecks in elementary school. Moreover, a teacher can also make sense of the associative, commutative, and distributive laws to elementary students and help them see that these are vital tools for doing mathematics rather than dinosaurs in an outdated school curriculum. If these facts had been widely known during the 1990s, the senseless debate on whether the standard algorithms should be taught might not have arisen and the Math Wars might not have taken place at all.

TSM also treats whole numbers, fractions, (finite) decimals, and rational numbers as four different kinds of numbers. The reality is that, first of all, decimals are a special class of fractions (see Section 1.1 of [**Wu2020a**]), whole numbers are part of fractions, and fractions are part of rational numbers. Moreover, the four arithmetic operations ($+$, $-$, $\times$, and $\div$) in each of these number systems do not essentially change from system to system. There is a smooth conceptual transition at each step of the passage from whole numbers to fractions and from fractions to rational numbers; see Parts 2 and 3 of [**Wu2011**] or Sections 2.2, 2.4, and 2.5 of [**Wu2020a**]. This coherence facilitates learning: instead of having to learn about four different kinds of numbers, students basically only need to learn about one number system (the rational numbers). Yet another example is the conceptual unity between linear functions and quadratic functions: in each case, the leading term—ax for linear functions and ax^2 for quadratic functions—determines the shape of the graph of the function completely, and the studies of the two kinds of functions become similar as each revolves around the shape of the graph (see Section 2.1 in [**Wu2020b**]). Mathematical coherence gives us many such storylines, and a few more will be detailed below.

(5) Purposefulness. In addition to the preceding four shortcomings—a lack of clear definitions, faulty or nonexistent reasoning, pervasive imprecision, and general incoherence—TSM has a fifth fatal flaw: it lacks *purposefulness.* Purposefulness is what gives mathematics its vitality and focus: the fact is that a mathematical investigation, at any level, is always carried out with a specific goal in mind. When a mathematics textbook reflects this goal-oriented character of mathematics, it propels the mathematical narrative forward and facilitates its learning by making students aware of where the discussion is headed, and *why.* Too often, TSM lurches from topic to topic with no apparent purpose, leading students to wonder why they should bother to tag along. One example is the introduction of the absolute value of a number. Many teachers and students are mystified by being saddled with such a "frivolous" skill: "just kill the negative sign", as one teacher put it. Yet TSM never tries to demystify his concept. (For an explanation of the need to introduce *absolute value,* see, e.g., the Pedagogical Comments in Section 2.6 of [**Wu2020a**]). Another is the seemingly inexplicable replacement of the square root and cube root symbols of a positive number b, i.e., $\sqrt{b}$ and $\sqrt[3]{b}$, by rational exponents, $b^{1/2}$ and $b^{1/3}$, respectively (see, e.g., Section 4.2 in [**Wu2020b**]). Because TSM teaches the laws of exponents as merely "number facts", it is inevitable that it would fail to point out the *purpose* of this change of notation, which is to shift focus from the operation of taking roots to the properties of the *exponential function* b^x for a fixed positive b. A final example is the way TSM teaches estimation completely by rote, without ever telling students why and when estimation is important and therefore worth learning. Indeed, we often *have to* make estimates, either because precision is unattainable or unnecessary, or because we purposely use estimation as a tool to help achieve precision (see [**Wu2011**, Section 10.3]).

To summarize, if we want students to be taught mathematics that is learnable, then we must discard TSM and replace it with the kind of mathematics that possesses these five qualities:

Every concept has a clear definition.
Every statement is precise.
Every assertion is supported by reasoning.
Its development is coherent.
Its development is purposeful.

We call these the **Fundamental Principles of Mathematics** (also see Section 2.1 in [**Wu2018**]). We say a mathematical exposition has **mathematical integrity** if it embodies these fundamental principles. As we have just seen, we find in TSM a consistent pattern of violating all five fundamental principles. We believe that the dominance of TSM in school mathematics in the past five decades is a principal cause of the ongoing crisis in school mathematics education.

One consequence of the dominance of TSM is that most students come out of K–12 knowing only TSM, not mathematics that respects these fundamental principles. To them, learning mathematics is not about learning how to reason or distinguish true from false but about memorizing facts and tricks to get correct answers. Faced with this crisis, what should be the responsibility of institutions of higher learning? Should it be to create courses for future teachers and educators to help them *systematically* replace their knowledge of TSM with mathematics that is consistent with the five fundamental principles? Or should it be, rather, to leave TSM alone but make it more palatable by helping teachers infuse their classrooms with activities that *suggest* visions of reasoning, problem solving, and sense making? As of this writing, an overwhelming majority of the institutions of higher learning are choosing the latter alternative.

At this point, we return to the earlier question about some of the ways both university mathematicians and educators might misunderstand and misuse these three volumes.

Potential misuse by mathematicians

First, consider the case of mathematicians. They are likely to scoff at what they perceive to be the triviality of the content in these volumes: no groups, no homomorphisms, no compact sets, no holomorphic functions, and no Gaussian curvature. They may therefore be tempted to elevate the level of the presentation, for example, by introducing the concept of a *field* and show that, when two fractions symbols m/n and k/ℓ (with whole numbers m, n, k, ℓ, and $n \neq 0$, $\ell \neq 0$) satisfying $m\ell = nk$ are identified, and when $+$ and $\times$ are defined by the usual formulas, the fraction symbols form a field. In this elegant manner, they can efficiently cover all the standard facts in the arithmetic of fractions in the school curriculum.[9] This is certainly a better way than *defining fractions as points on the number line* to teach teachers and educators about fractions, is it not? Likewise, mathematicians may find finite geometry to be a more exciting introduction to axiomatic systems than any proposed improvements on the high school geometry course in TSM. The list goes on. Consequently, pre-service teachers and educators may end up learning from mathematicians some interesting mathematics, but not mathematics that would help them overcome the handicap of knowing only TSM.

Mathematicians may also engage in another popular approach to the professional development of teachers and educators: teaching the solution of hard problems. Because mathematicians tend to take their own mastery of fundamental skills and concepts for granted, many do not realize that it is nearly impossible for teachers who have been immersed in thirteen years or more of TSM to acquire, *on their own*, a mastery of a *mathematically correct* version of the basic skills and concepts. Mathematicians are therefore likely to consider their major goal in the professional development of teachers and educators to be teaching them how to solve *hard problems*. Surely, so the belief goes, if teachers can handle the "hard stuff", they will be able to handle the "easy stuff" in K–12. Since this belief is entirely in line with one of the current slogans in school mathematics education about the critical importance of *problem solving*, many teachers may be all too eager to teach their students the extracurricular skills of solving challenging problems *in addition to* teaching them TSM day in and day out. In any case, the relatively unglamorous content of these three volumes (this volume, [**Wu2020a**], and [**Wu2020b**])—designed to replace TSM—will get shunted aside into supplementary reading assignments.

At the risk of belaboring the point, the focus of these three volumes is on showing how to *replace teachers' and educators' knowledge of TSM in grades 9–12* with mathematics that respects the fundamental principles of mathematics. Therefore, reformulating the mathematics of grades 9–12 from an advanced mathematical standpoint to obtain a more elegant presentation is not the point. Introducing novel elementary topics (such as Pick's theorem or the 4-point affine plane) into the mathematics education of teachers and educators is also not the point. Rather, the point in year 2020 is to do the essential spadework of revisiting the standard 9–12 curriculum—topic by topic, along the lines laid out in these three volumes—showing teachers and educators how the TSM in each case can be supplanted by mathematics that makes sense to them and to their students. For example, since most pre-service teachers and educators have not been exposed to the use of precise

[9] This is my paraphrase of a mathematician's account of his professional development institute around year 2000.

definitions in mathematics, they are unlikely to know that definitions are supposed to be used, *exactly as written, no more and no less*, in logical arguments. One of the most formidable tasks confronting mathematicians is, in fact, how to change educators' and teachers' perception of the role of definitions in reasoning.

As illustration, consider how TSM handles slope. There are two ways, but we will mention only one of them.[10] TSM pretends that, by defining the slope of a line L using the difference quotient with respect to two *pre-chosen* points P and Q on L,[11] such a difference quotient is a property *of the line itself* (rather than a property of the two points P and Q). In addition, TSM pretends that it can use "reasoning" based on this defective definition to derive the equation of a line when (for example) its slope and a given point on it are prescribed. Here is the inherent danger of thirteen years of continuous exposure to this kind of pseudo-reasoning: teachers cease to recognize that (a) such a definition of slope is defective and (b) such a defective definition of slope cannot possibly support the purported derivation (= proof) of the equation of a line. It therefore comes to pass that—as a result of the flaws in our education system—many teachers and educators end up being confused about even the meaning of the simplest kind of reasoning: "A implies B". They need—and deserve—all the help we can give so that they can finally experience *genuine mathematics*, i.e., mathematics that is based on the fundamental principles of mathematics.

Of course, the ultimate goal is for teachers to *use* this new knowledge to teach their own students so that those students can achieve a true understanding of what "A implies B" means and what real *reasoning* is all about. With this in mind, we introduce in Section 6.4 of [**Wu2020a**] the concept of slope by discussing what slope is supposed to measure (an example of *purposefulness*) and how to measure it, which then leads to the formulation of a precise definition. With the availability of the AA-criterion for triangle similarity (Theorem G22 in Section 5.3 of [**Wu2020a**]), we then show how this definition leads to the formula for the slope of a line as the difference quotient of the coordinates of *any* two points on the line (the "rise-over-run"). Having this critical flexibility to compute the slope—plus an earlier elucidation of what an *equation* is (see Section 6.2 in [**Wu2020a**])—we easily obtain the equation of a line passing through a given point with a given slope, *with correct reasoning* this time around (see Section 6.5 in [**Wu2020a**]). Of course the same kind of reasoning can be applied to similar problems when other reasonable geometric data are prescribed for the line.

By guiding teachers and educators systematically through the *correction of TSM errors on a case-by-case basis,* we believe they will gain a new and deeper understanding of school mathematics. Ultimately, we hope that if institutions of higher learning and the education establishment can persevere in committing themselves to this painstaking work, the students of these teachers and educators will be spared the ravages of TSM. If there is an easier way to undo thirteen years and more of mis-education in mathematics, we are not aware of it.

A main emphasis in using these three volumes should therefore be on providing patient guidance to teachers and educators to help them overcome the many

[10] A second way is to *define* a line to be the graph of a linear equation $y = mx + b$ and then *define* the slope of this line to be m. This is the definition of a line in advanced mathematics, but it is *so profoundly inappropriate* for use in K–12 that we will just ignore it.

[11] This is the "rise-over-run".

handicaps inflicted on them by TSM. In this light, we can say with confidence that, for now, the best way for mathematicians to help educate teachers and educators is to firm up the mathematical foundations of the latter. Let us repair the damage TSM has done to their mathematics content knowledge by helping them to acquire a knowledge of school mathematics that is consistent with the fundamental principles of mathematics.

Potential misuse by educators

Next, we address the issue of how educators may misuse these three volumes. Educators may very well frown on the volumes' insistence on precise definitions and precise reasoning and their unremitting emphasis on proofs while, apparently, neglecting *problem solving*, *conceptual understanding*, and *sense making*. To them, good professional development concentrates on all of these issues plus contextual learning, student thinking, and communication with students. Because these three volumes never explicitly mention problem solving, conceptual understanding, or sense making per se (or, for that matter, contextual learning or student thinking), their content may be dismissed by educators as merely skills-oriented or *technical knowledge for its own sake* and, as such, get relegated to reading assignments outside of class. They may believe that precious class time can be put to better use by calling on students to share their solutions to difficult problems or by holding small group discussions about problem-solving strategies.

We believe this attitude is also misguided because the critical missing piece in the contemporary mathematical education of teachers and educators is an exposure to a systematic exposition of the standard topics of the school curriculum that respects the fundamental principles of mathematics. Teachers' lack of access to such a mathematical exposition is what lies at the heart of much of the current education crisis. Let us explain.

Consider *problem solving.* At the moment, the goal of getting all students to be proficient in solving problems is being pursued with missionary zeal, but what seems to be missing in this single-minded pursuit is the recognition that the body of knowledge we call mathematics consists of nothing more than a sequence of problems posed, and then solved, by making logical deductions on the basis of precise definitions, clearly stated hypotheses, and known results.[12] This is after all the whole point of the classic two-volume work [**Pólya-Szegö**], which introduces students to mathematical research through the solutions to a long list of problems. For example, the Pythagorean theorem and its many proofs are nothing more than solutions to the problem posed by people from diverse cultures long ago: "Is there any relationship among the three sides of a right triangle?" *There is no essential difference between problem solving and theorem proving in mathematics.* Each time we solve a problem, we in effect prove a theorem (trivial as that theorem may sometimes be).

[12] It is in this light that the previous remark about the purposefulness of mathematics can be better understood: before solving a problem, one should know *why* the problem was posed in the first place. Note that, for beginners (i.e., school students), the overwhelming emphasis has to be on *solving* problems rather than the more elusive issue of posing problems.

The main point of this observation is that if we want students to be proficient in problem solving, then we must give them plenty of examples of grade-appropriate proofs all through (at least) grades 4–12 and engage them regularly in grade-appropriate theorem-proving activities. If we can get students to see, day in and day out, that problem solving is a way of life in mathematics and if we also *routinely get them involved in problem solving* (i.e., *theorem proving*), students will learn problem solving naturally through such a long-term immersion. In the process, they will get to experience that, to solve problems, they need to have precise definitions and precise hypotheses as a starting point, know the direction they are headed before they make a move (sense making), and be able to make deductions from precise definitions and known facts. Definitions, sense making, and reasoning will therefore come together *naturally* for students if they learn mathematics that is consistent with the five fundamental principles.

We make the effort to put problem solving in the context of the fundamental principles of mathematics because there is a danger in pursuing problem solving per se in the midst of the TSM-induced corruption of school mathematics. In a generic situation, teachers teach TSM and only pay lip service to "problem solving", while in the best case scenario, teachers *keep TSM intact* while teaching students how to solve problems on a separate, parallel track *outside of TSM*. Lest we forget, TSM considers "out of a hundred" to be a correct definition of *percent*, expands the product of two linear polynomials by "FOILing", and assumes that in any problem about *rate*, one can automatically assume that the rate is constant ("Lynnette can wash 95 cars in 5 days. How many cars can Lynnette wash in 11 days?"), etc. In this environment, it is futile to talk about (*correct*) problem solving. Until we can rid school classrooms of TSM, the most we can hope for is having teachers teach, on the one hand, definition-free concepts with a bag of tricks-sans-reasoning to get correct answers and, on the other hand, reasoning skills for solving a separate collection of problems for special occasions. In other words, two parallel universes will co-exist in school mathematics classrooms. So long as TSM continues to reign in school classrooms, most students will only be comfortable doing one-step problems and any problem-solving ability they possess will only be something that is artificially grafted onto the TSM they know.

If we want to avert this kind of **bipolar mathematics education** in schools, we must begin by providing teachers with a better mathematical education. Then we can hope that teachers will teach mathematics consistent with the fundamental principles of mathematics[13] so that students' problem-solving abilities can evolve naturally from the mathematics they learn. It is partly for this reason that the six volumes under discussion[14] choose to present the mathematics of K–12 with explanations (= proofs) for all the skills. In particular, these three volumes on the mathematics of grades 9–12 provide proofs for every theorem. (At the same time, they also caution against certain proofs that are simply too long or too tedious to be presented in a high school classroom.) The hope is that when teachers and educators get to experience firsthand that every part of school mathematics is

[13]And, of course, to also get school textbooks that are unsullied by TSM. However, it seems likely as of 2020 that major publishers will hold onto TSM until there are sufficiently large numbers of knowledgeable teachers who demand better textbooks. See the end of [**Wu2015**].

[14]These three volumes, together with [**Wu2011**], [**Wu2016a**], and [**Wu2016b**].

suffused with reasoning, they will not fail to teach reasoning to their own students as a matter of routine. Only then will it make sense to consider problem solving to be an integral part of school mathematics.

The importance of correct content knowledge

In general, the idea is that if we give teachers and educators an exposition of mathematics that *makes sense* and has built-in *conceptual understanding* and *reasoning,* then we can hope to create classrooms with an intellectual climate that enables students to absorb these qualities as if by osmosis. Perhaps an analogy can further clarify this issue: if we want to teach writing, it would be more effective to let students read good writing and learn from it directly rather than to let them read bad writing and simultaneously attend special sessions on the fine points of effective written communication.

If we want school mathematics to be suffused with reasoning, conceptual understanding, and sense making, then we must recognize that these are not qualities that can stand apart from mathematical details. Rather, they are firmly anchored to hard-and-fast mathematical facts. Take proofs (= reasoning), for example. If we only talk about proofs in the context of TSM, then our conception of what a proof is will be extremely flawed because there are essentially no correct proofs in TSM. For starters, since TSM has no precise definitions, there can be no hope of finding a completely correct proof in TSM. Therefore, when teaching from these three volumes,[15] it is imperative to first concentrate on getting across to teachers and educators the *details* of the mathematical reformulation of the school curriculum. Specifically, we stress the importance of offering educators a *valid* alternative to TSM for their future research. Only then can we hope to witness a reconceptualization—*in mathematics education*—of reasoning, conceptual understanding, problem solving, etc., on the basis of a solid mathematical foundation.

Reasoning, conceptual understanding, and *sense making* are qualities intrinsic to school mathematics that respects the fundamental principles of mathematics. We see in these three volumes a continuous narrative from topic to topic and from chapter to chapter to guide the reader through this long journey. The sense making will be self-evident to the reader. Moreover, when every assertion is backed up by an explanation (= proof), reasoning will rise to the surface for all to see. In their presentation of the natural unfolding of mathematical ideas, these volumes also routinely point out connections between definitions, concepts, theorems, and proofs. Some connections may not be immediately apparent. For example, in Section 6.1 of [**Wu2020a**], we explicitly point out the connection between Mersenne primes and the summation of finite geometric series. Other connections span several grades: there is a striking similarity between the *proofs* of the area formula for rectangles whose sides are fractions (Theorem 1.7 in Section 1.4 of [**Wu2020a**]), the ASA congruence criterion (Theorem G9 in Section 4.6 of [**Wu2020a**]), the SSS congruence criterion (Theorem G28 in Section 6.2 of [**Wu2020b**]), the fundamental theorem of similarity (Theorem G10 in Section 6.4 of [**Wu2020b**]), and the theorem about the equality of angles on a circle subtending the same arc (Theorem G52 in Section 6.8 of [**Wu2020b**]). All these proofs are achieved by breaking up a complicated argument into two or more clear-cut steps, each involving simpler arguments. In

[15] As well as from the other three volumes, [**Wu2011**], [**Wu2016a**], and [**Wu2016b**]).

other words, they demonstrate how to reduce the complex to the simple so that prospective teachers and educators can learn from such instructive examples about the fine art of problem solving.

The foregoing unrelenting emphasis on mathematical content should not lead readers to believe that these three volumes deal with mathematics at the expense of pedagogy. To the extent that these volumes are designed to promote better teaching in the schools, they do not sidestep pedagogical issues. Extensive pedagogical comments are offered whenever they are called for, and they are clearly displayed as such; see, for example, pp. 29, 40, 46, 65, 80, 91, 162, 179, 235, 359, etc., in the present volume. Nevertheless, our most urgent task—*the* fundamental task—in the mathematical education of teachers and educators as of 2020 has to be the *reconstruction of their mathematical knowledge base*. This is not about judiciously *tinkering* with what teachers and educators already know or *tweaking* their existing knowledge here and there. Rather, it is about the hard work of *replacing* their knowledge of TSM with mathematics that is consistent with the fundamental principles of mathematics *from the ground up.* The primary goal of these three volumes is to give a detailed exposition of school mathematics in grades 9–12 to help educators and teachers achieve this reconstruction.

To the Pre-Service Teacher

In one sense, these three volumes are just textbooks, and you may feel you have gone through too many textbooks in your life to need any fresh advice. Nevertheless, we are going to suggest that you approach these volumes with a different mindset than what you may have used with other textbooks, because you will soon be using the knowledge you gain from these volumes to teach your students. Reading other textbooks, you would likely congratulate yourself if you could achieve mastery over 90% of the material. That would normally guarantee an A. More is at stake with these volumes, however, because they directly address what you will need to know in order to write your lessons. Ask yourself whether a *mathematics* teacher whose lessons are correct only 90% of the time should be considered a good teacher. To be blunt, such a teacher would be a near disaster. So your mission in reading these volumes should be to achieve nothing short of total mastery. **You are expected to know this material 100%**. To the extent that the content of these three volumes is just K–12 mathematics, this is an achievable goal. This is the standard you have to set for yourself. Having said that, we also note explicitly that many *Mathematical Asides* are sprinkled all through the text, sometimes in the form of footnotes. These are comments—usually from an advanced mathematical perspective—that try to shed light on the mathematics under discussion. *The above reference to "total mastery" does not include these comments.*

You should approach these volumes differently in yet another respect. Students' typical attitude towards a math course is that if they can do all the homework problems, then most of their work is done. Think back on your calculus courses or any of the math courses when you were in school, and you will understand how true this is. But since these volumes are designed specifically for teachers, your emphasis cannot be limited to merely doing the homework assignments because your job will be more than just helping students to do homework problems. When you stand in front of a class, what you will be talking about, *most of the time*, will not be the exercises at the end of each section but the concepts and skills in the exposition proper.[1] For example, very likely you will soon have to convince a class on geometry why the Pythagorean theorem is correct. There are two proofs of this theorem in these volumes, one in Section 5.3 of [**Wu2020a**] and the other on pp. 233ff. Yet on neither occasion is it possible to assign a problem that asks for a proof of this theorem. There are problems that can assess whether you know enough about the Pythagorean theorem to apply it, but how do you assess whether

[1] I will be realistic and acknowledge that there *are* teachers who use class time only to drill students on how to get the right answers to exercises, often without reasoning. But one of the missions of these three volumes is to steer you away from that kind of teaching. See **To the Instructor** on page xix.

you know *how to prove the theorem* when the proofs have already been given in the text? It is therefore entirely up to you to achieve mastery of everything in the text itself. One way to check is to pick a theorem at random and ask yourself: Can I prove it without looking at the book? Can I explain its significance? Can I convince someone else why it is worth knowing? Can I give an intuitive summary of the proof? These are questions that you will have to answer as a teacher. To the extent possible, these volumes try to provide information that will help you answer questions of this kind. I may add that the most taxing part of writing these volumes was in fact to do it in a way that would allow you, as much as possible, to adapt them for use in a school classroom with minimal changes. (Compare, for example, **To the Instructor** on pp. xix ff.)

There is another special feature of these volumes that I would like to bring to your attention: these volumes are essentially *school textbooks written for teachers*, and as such, you should read them with the eyes of a school student. When you read Chapter 1 of [**Wu2020a**] on fractions, for instance, picture yourself in a sixth-grade classroom and therefore, no matter how much abstract algebra you may know or how well you can explain the construction of the quotient field of an integral domain, you have to be able to give explanations in the language of sixth-grade mathematics (i.e., to sixth graders). Similarly, when you come to Chapter 6 of [**Wu2020a**], you are developing algebra from the beginning, so even the use of symbols will be an issue (it is in fact the *key* issue; see Section 6.1 of [**Wu2020a**]). Therefore, be very deliberate and explicit when you introduce a symbol, at least for a while.

The major conclusions in these volumes, as in all mathematics books, are summarized into **theorems**. Depending on the author's (and other mathematicians') whims, theorems are sometimes called **propositions**, **lemmas**, or **corollaries** as a way of indicating which theorems are deemed more important than others. Roughly speaking, a proposition is not regarded to be as important as a theorem, a lemma is conceptually less important than a proposition, and a corollary is supposed to follow immediately from the theorem or proposition to which it is attached. (Incidentally, a formula or an algorithm is just a theorem.) This idiosyncratic classification of theorems started with Euclid around 300 BC, and it is too late to do anything about it now. The main concepts of mathematics are codified into **definitions**. Definitions are set in **boldface** in these volumes when they appear for the first time; a few truly basic ones are even individually displayed in a separate paragraph, but most of the definitions are embedded in the text itself, so you should watch out for them.

The statements of the theorems, and especially their proofs, depend on the definitions, and proofs are the guts of mathematics.

Please note that when I said above that I expect you to know everything in these volumes, I was using the word "*know*" in the way mathematicians normally use the word. They do not use it to mean simply "know the statement by heart". Rather, *to know* a theorem, for instance, means *know the statement by heart, know its proof, know why it is worth knowing, know what its potential implications are, and finally, know how to apply it in new situations.* If you know anything short of this, how can you expect to be able to answer your students' questions? At the very least, you should know by heart all the theorems and definitions as well as the main ideas of each proof because, if you do not, it will be futile to talk about the other aspects of knowing. Therefore, a preliminary suggestion to help you master the content of these volumes is for you to

> copy out the statements of every definition, theorem, proposition, lemma, and corollary, along with page references so that they can be examined in detail when necessary,

and also to

> form the habit of summarizing the main idea(s) of each proof.

These are good study habits. When it is your turn to teach your students, be sure to pass along these suggestions to them.

You should also be aware that reading a mathematics book is not the same as reading a gossip magazine. You can probably flip through one of the latter in an hour or less. But in these volumes, there will be many passages that require slow reading and re-reading, perhaps many times. I cannot single out those passages for you because they will be different for different people. We do not all learn the same way. What you can take for granted, however, is that mathematics books make for exceedingly slow reading. (Nothing good comes easy.) Therefore if you get stuck, time and time again, on a sentence or two in these volumes, take heart, because this is the norm in mathematics learning.

Prerequisites

In terms of the *mathematical development* of this volume, only a knowledge of **whole numbers**, 0, 1, 2, 3, ..., is assumed. Thus along with place value, you are assumed to know the four arithmetic operations, their standard algorithms, and the concept of **division-with-remainder** and how it is related to the long division algorithm.[1] Division-with-remainder assigns to each pair of whole numbers b (the dividend) and d (the divisor), where $d \neq 0$, another *pair* of whole numbers q (the quotient) and r (the remainder), so that

$$b \ = qd + r \quad \text{where } 0 \leq r < d.$$

Some subtle points about the concept of *division* among whole numbers will be briefly recalled at the beginning of Section 1.5 of [**Wu2020a**]. A detailed exposition of the concept of "division" among whole numbers is given in Chapter 7 of [**Wu2011**].

Note that 0 is included among the whole numbers.

A knowledge of negative numbers, particularly integers, is not assumed. Negative numbers will be developed ab initio in Chapter 2 of [**Wu2020a**].

Because every assertion in these three volumes (this volume, together with [**Wu2020a**] and [**Wu2020b**]) will be proved, students should be comfortable with mathematical reasoning. It is hoped that as they progress through the volumes, all students will become increasingly at ease with proofs. In terms of the *undergraduate curriculum*, readers of this volume—as a rule of thumb—should have already taken the usual two years of college calculus or their equivalents.

[1] Unfortunately, a correct exposition of this topic is difficult to come by. Try Chapter 7 of [**Wu2011**].