An Introduction to q-analysis

The phrase "q-analysis" was used in the first referee's report I ever got. While the subject of this book has a flavor all its own, and has been studied for almost 300 years, there is no term in common use that describes it really well. The closest standard name is "q-series", which is not bad—finite and infinite series occur almost everywhere, as does the letter q—but it is a little too restrictive. I think it needs another appellation, q-analysis is the best one I can think of, and I thank that anonymous referee for it (and an excellent report). Peter Paule used it in [**181**].

I have tried very hard to write a book that can be read by undergraduates. The prerequisites are minimal. One cannot have "the fear of all sums" that plagues many calculus students, but very little specific knowledge of calculus 2 will be required. In particular, you do not need an extensive knowledge of convergence tests, since for q-series the ratio test is nearly always appropriate. (The root test is marginally better in a few cases, and once in a while the n^{th} term test is helpful.) Moreover, we will be much less concerned with when or whether an infinite series converges than with what it converges to.

We will also be seeing zillions of finite and infinite products of a certain kind (this is one reason why I don't want to just say "q-series"), but no prior knowledge of these is assumed. What little we need is developed in Appendix B, and even this can be skipped if one is willing to believe that the infinite products converge. Previous experience with mathematical induction would surely be helpful, but few if any subjects are as well suited to teach induction as q-analysis. The instructor should try to ensure that the students are comfortable with induction (or at least getting more comfortable) early in the term. It is used less often after the first two chapters, but it never goes away.

In the first section of the book it would help to know (or to be told) what $\binom{n}{2}$ means. No further knowledge of binomial coefficients is really necessary since we will make an extensive study of q-binomial coefficients and theorems in the first two chapters that will fill in any gaps. In a few places we will use complex numbers at the level of $e^{i\theta} = \cos\theta + i\sin\theta$. There are some allusions to deeper parts of complex analysis in Chapter 13.

Where to start studying q-analysis is not completely clear. In this book one could start with any of the first three chapters. Chronologically, the story begins with the pathbreaking work on partitions in Chapter 16 of the greatest mathematics book ever written, Leonhard Euler's *Introductio in Analysin Infinitorum*, which is the subject of my Chapter 3, and I have started before with what is now the first part of Chapter 2. The motivation for the current order is that I would rather work with finite sums (Chapter 2) before infinite ones (Chapter 3 and beyond), and I now prefer to discuss the combinatorial properties of q-binomial coefficients (Chapter 1) before q-binomial theorems (Chapter 2).

When I began this project many years ago, I intended that the q-derivative would play a larger role, but it now appears only in the optional sections 2.4 and 3.8, and in the last few sections of Chapter 9. The lovely little book [**154**] takes this point of view, which relieves me of some responsibility, and my own thinking has changed somewhat. While the formal analogies with ordinary calculus are undeniably beautiful, strictly speaking one can't go much beyond Euler that way, and I would rather develop Euler's theorems hand-in-hand with their combinatorial meaning.

There is, I believe, enough material here for two semesters. In a single semester, the instructor has a lot of flexibility. Aside from Chapter 3, the only really essential sections are 1.2, 1.3, two of 2.1–2.3, and 5.1. One can choose to emphasize the history of the subject, or its combinatorial aspects, or the applications to number theory; or one can just pick out the results that one finds the most beautiful. No other subject has as many beautiful formulas (in my possibly biased opinion), so one will get a lot of that no matter what one does. Here are some specific comments on each chapter:

I think one should try to resist the temptation to go too rapidly over the first several sections of Chapter 1, at least with undergraduates. I include Terquem's proof partly for historical reasons and partly to emphasize the cleverness of Rodrigues by comparison, but partly also as a device to slow myself down. Even though the ideas are initially very simple, nothing less than a parallel universe of mathematics is being constructed here, and students need time to become citizens of it. (The notation alone takes some getting used to. I have included a brief summary for reference in Appendix A.) Conjugate permutations and Rothe diagrams can be skipped, although they foreshadow the more important idea of conjugate partitions in Chapter 3. Section 1.6 is not vital, but it shows that sections 1.2 and 1.3 are hinting at something more general. A course emphasizing combinatorics should do sections 1.5 and 1.7.

The centerpiece of Chapter 2 is Rothe's q-binomial theorem in section 2.3. It is equivalent to the Potter–Schützenberger theorem of sections 2.1 and 2.2—perhaps less beautiful, but more useful. As much as anything else, what one wants to get out of the first two chapters is the sense that these two theorems and what I call the Fundamental Property of q-binomial coefficients (section 1.3) are really all saying the same thing.

One can do any two of sections 2.1–2.3. If you like both 2.1 and 2.2, then you can do Rothe's q-binomial theorem by Gruson's method (see the exercises in section 2.2) rather than as in section 2.3. (Instructors should in general be alive to the possibility of doing something in the exercises instead of or in addition to the text.) Gauss's identities in section 2.5 are largely of historical interest, but the second has an important connection with Sylvester's fishhook bijection in section 4.5, the first plays a minor role in section 9.7, and both are used in section 11.4. They are proved again in section 3.5, so one can at least postpone them. Although Jacobi's q-binomial theorem has historically been underrated, in my opinion, it could just be stated here, as there is a natural proof in section 3.6. Euler's theorem in section 2.9 also deserves to be better known, but the rest of the section can be skipped.

Sometimes one proves a q-identity by first establishing a finite form of it and then taking a limit. The limiting process often technically requires a little-known result from analysis called **Tannery's theorem**, which I have included in Appendix C, though the application is usually not made explicit. The parts of Chapter 2 not already mentioned present finite forms of identities yet to come. MacMahon's q-binomial theorem in section 2.7 is of some independent interest, but it is mainly a finite form of the Jacobi triple product, the most important identity appearing after Chapter 3, for which several other proofs are also given. MacMahon's identity might better be presented as in problem 7 of section 2.7 instead of as in the text. The partial fractions identity in section 2.8 has been included mainly to allow an instructor to do the number-theoretic applications in Chapter 7 without having to do Ramanujan's ${}_1\psi_1$ summation in Chapter 6. The Chen–Chu–Gu identity in section 2.10 is a finite form of the quintuple product identity in section 5.3, but that section is also optional, and it contains a second proof.

Chapter 3 introduces partitions, a subject at the intersection of combinatorics and number theory that pervades the rest of the book. The subset of q-identities with natural partition-theoretic interpretations is so large that partitions must be considered an essential part of q-analysis, not just an application of it. Analytically, nearly everything in Chapter 3 is a corollary of the Cauchy/Crelle series (my name; experts will know it as the infinite series version of the q-binomial theorem), which is not difficult to prove directly. However, it is not just the truth of the theorems that I want to establish, but their significance and their inevitability. (This I think is a word that mathematicians should use more often. I have seen it in Hardy's moving obituary of Ramanujan [**131**] and in Rota's equally beautiful essays [**204**] and [**205**]. It is also on the back cover of [**154**]. For Rota, it comes from Immanuel Kant.) I believe it is more illuminating to work from the bottom up here than from the top down, so I recommend that most of the material in the first seven sections of Chapter 3 be done in the order in which it appears. (In a graduate course, one might try to save some time here.) Franklin's "excesses" argument in section 3.4 can be skipped, as can the combinatorial proof of Cauchy/Crelle in section 3.7. The material on ee partitions in section 3.3 is included mostly as background for the Göllnitz–Gordon identities in Chapter 11.

Any course on q-analysis should include Euler's pentagonal number theorem, but one has several options. The historically minded reader might do section 4.1, which is essentially Euler's argument, but others may just note the recurrence in (4.1.4). The combinatorially minded reader should do sections 4.2 and 4.3, which present Franklin's gorgeous partition-theoretic proof; this is one of the greatest achievements of Sylvester's group at Johns Hopkins in the early 1880s and is highly recommended. The other attractive option is to wait for Jacobi's triple product identity in section 5.1, as the pentagonal number theorem is an easy corollary. Euler's theorem on divisor sums in section 4.4 explains why he worked so hard to prove the pentagonal number theorem, so it should be done by readers interested either in number theory or in history. The latter can skip the rest of the section. Another triumph of the Johns Hopkins school is Sylvester's fishhook bijection in section 4.5.

Section 5.1 proves Jacobi's triple product identity. None of the other sections in Chapter 5 is vital, although they are all interesting. The most important are sections 5.5 and 5.7, but section 5.3 is used in Chapter 8 and section 5.4 has a connection with section 4.5.

The next three chapters are shorter than the first five. Chapter 6 is devoted to Ramanujan's ${}_1\psi_1$ summation formula, which I think should be in any course. As evidence of this I give several different proofs. Section 6.2 has four, two in the text that rely on the q-Gauss sum from section 5.5, and two more in the problems that don't. Section 6.3 has a proof of the finite to infinite type, due to Michael Schlosser, that needs the q-Pfaff–Saalschütz identity from section 5.7. The last three sections outline another proof, due in part to Cauchy, that does not require Chapter 5 and develops Jacobi's triple product as a byproduct.

Chapter 7 contains applications of the ${}_1\psi_1$ to Jacobi's theorems on sums of two and four squares. As mentioned above, one can do this material without Chapter 6 if one does section 2.8, which leads to a simple proof of the relevant special case. But I think that most readers will like one or more of the proofs of the ${}_1\psi_1$ at least as well as the argument of section 2.8.

Chapter 8 is also on number theory, specifically congruence properties of partitions. The key theorem in the chapter was stated by Ramanujan, and has often been called his "most beautiful" identity. The approach to it given here requires the quintuple product identity from section 5.3. I might skip this chapter if I planned to cover section 13.2.

Chapter 9 returns to combinatorics. The first five sections are a natural continuation of Chapter 1, although some of the material requires the first few sections of Chapter 3. A highlight of this chapter is Foata's bijective proof in section 9.3 of MacMahon's theorem that the inversion number and the major index are equidistributed, and this requires only Chapter 1—it is not even necessary to do sections 9.1 and 9.2 first, although these simpler arguments may provide motivation. Section 9.4 gives MacMahon's original proof, and section 9.5 a pretty related result. The last three sections in this chapter use the q-derivative, with section 9.8 giving combinatorial properties of q-trigonometric functions.

Chapters 10–12 are on the Rogers–Ramanujan identities and related topics. These can be done anytime after the Jacobi triple product, in a variety of ways. Chapter 10 begins with Schur's combinatorial proof, an extension of Franklin's argument from Chapter 4. Perhaps the simplest proof, due to Robin Chapman, is in section 10.2. It is another finite to infinite type argument and can be viewed as a simplification of Schur's second proof. Ramanujan's proof is in section 11.1, and a version of one of Rogers's proofs is in section 11.3. Another of Rogers's proofs is nearly the same as Selberg's proof, which is in Chapter 12. At a minimum, I suggest doing one of the proofs and section 11.2, which interprets the Rogers–Ramanujan identities in terms of partitions. These three chapters also contain similar q-identities with some further material on partitions, for example the Göllnitz–Gordon identities, which are to the number 8 what the Rogers–Ramanujan identities are to the number 5, in sections 11.6 and 11.7.

Chapter 13 is on Bailey's "very well poised ${}_6\psi_6$ sum", probably the deepest result in the book. The first few sections focus on a special case, with a more elementary proof, that is still strong enough to give Ramanujan's "most beautiful" identity from Chapter 8 and Jacobi's eight square theorem as corollaries. I give Askey's proof of the ${}_6\psi_6$ formula, his similar evaluation of an integral, and finally Watson's transformation, another key fact about very well poised q-hypergeometric series. Following Andrews's fundamental survey paper from the mid 1970s, some of the exercises obtain the quintuple product and the two, four, and eight square theorems as corollaries of the ${}_6\psi_6$.

I should say something about the exercises. There are a lot of them, especially in the first five chapters. They are vital to learning the subject, and I have worked very hard on them. Many are routine, though there are more of these earlier in the book than later; some are even trivial. For example, anything that looks like "check equation so-and-so" is generally not difficult. There are also many longer exercises, including some extremely long ones which if fleshed out could be (and in several cases once were) entire sections. I have often broken the harder problems into several parts, and no doubt some readers will feel that I have overdone this. A student who feels spoon-fed might try guessing what the next step of the problem ought to be. This could lead you to a better proof. If it does, please write to me and tell me about it.

The manuscript has benefitted greatly from a detailed report by an anonymous reviewer. Despite that person's best efforts and mine, there are undoubtedly some remaining errors, obscurities, and other infelicities. Please write to me if you find any.

The three authors of [**24**] have each played an important role in my evolution as a mathematician. My debt to George Andrews will be obvious to the most casual reader of this book. He has profoundly affected my view of many of the subjects presented here, both personally and through his writings, and he has done me many kindnesses.

Ranjan Roy, who passed away while I was making the final edits to the book, was a trusted mentor, friend, colleague, and role model for many years. More than anyone else, he showed me the kind of career that I could have. It is also thanks to him that I first got to teach some of this material to a fondly remembered group of seven students at Beloit College in Fall 1996.

But I must dedicate the book to my late thesis advisor, Richard Askey. Dick rescued my mathematical career in my second year of graduate school, just by being himself, and he was very patient with me afterward throughout a slow process of development. No one else has had as much influence on my adult life.