INTRODUCTION

It is said that an interpreter – a person who translates from one language to another – does not exchange words for words, but rather exchanges meaning for meaning. As mathematicians, a large part of our job is to explain things to others. We may use words and symbols to do this, but our job is not primarily to convey words and symbols; like an interpreter's, it is to convey *meaning*.

There are many ways to communicate meaning. As the great mathematician William Thurston said, "Mathematics is an art of human understanding. Mathematical concepts are abstract, so it ends up that there are many different ways that they can sit in our brains. A given mathematical concept might be primarily a symbolic equation, a picture, a rhythmic pattern, a short movie – or best of all, an integrated combination of several different representations."

The purpose of this book, then, is to help you find a good representation of the mathematical concept you wish to illustrate. This book does three things:

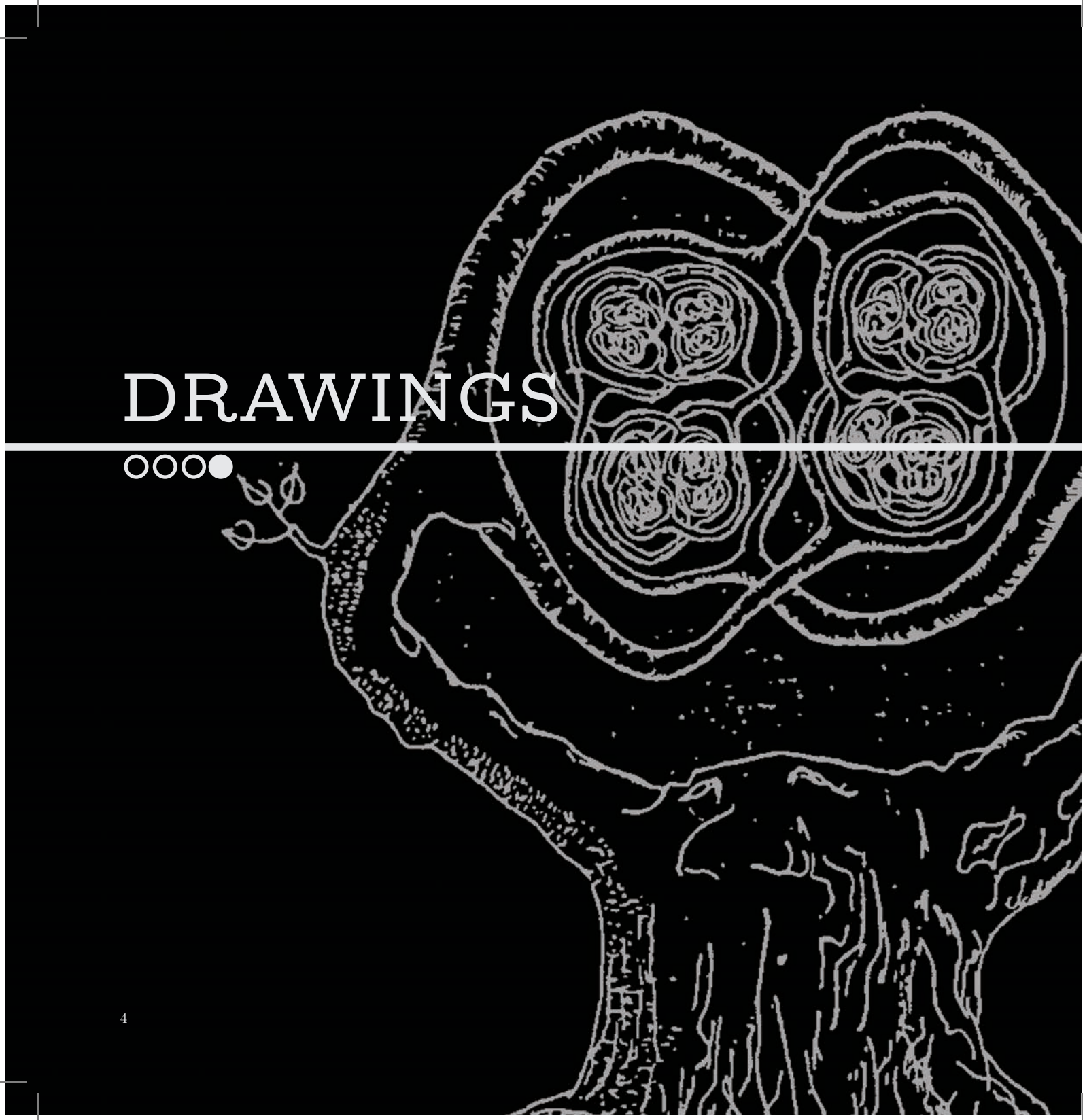
- Showcases the great variety of materials for illustrating mathematics,
- Gives voice to people's stories about illustrating their mathematics, so that we can learn from their experience, and
- Shows the variety of ways that different people use the same materials in very different ways.

In addition, it will introduce you to many of the amazing people who spent time at the Institute for Computational and Experimental Research Mathematics (ICERM) in fall 2019 for the Illustrating Mathematics program, in an attempt to capture some of the creative and generous spirit that flowed through our days there.

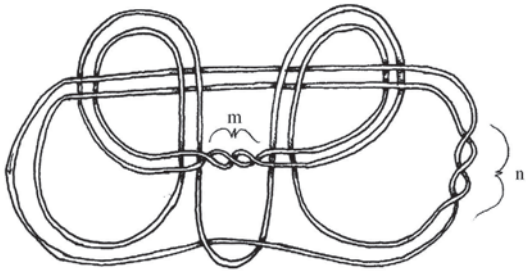
Diana Davis
May 1, 2020
Bures-sur-Yvette, France

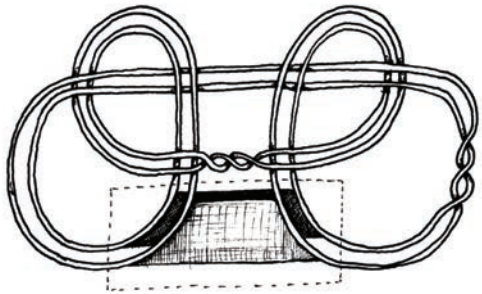
Illustrating Mathematics semester program participants, ICERM, Fall 2019.

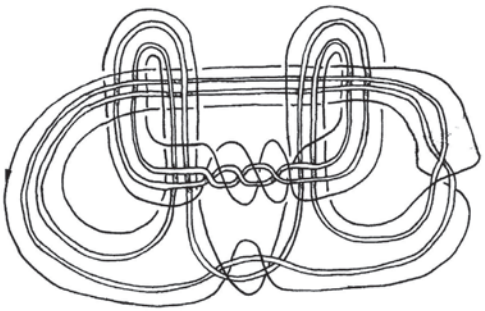
DRAWINGS

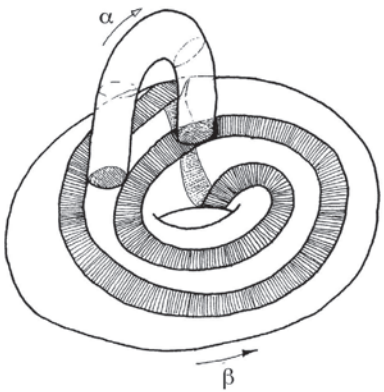


The adage that a picture is worth a thousand words is certainly true in mathematics, in which one carefully chosen figure can eliminate the need for many lines of exposition. For most of us, our most frequent way of illustrating mathematics is by hand – on chalkboards, scrap paper, paper napkins or, increasingly, whiteboards and electronic tablets.









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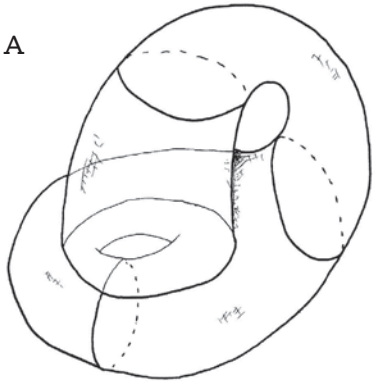

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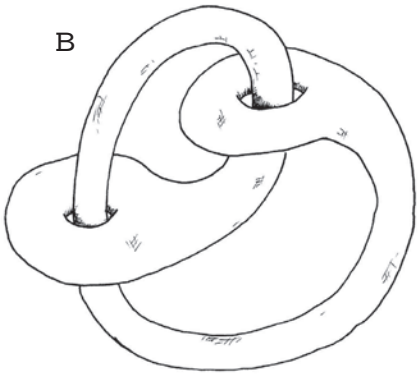

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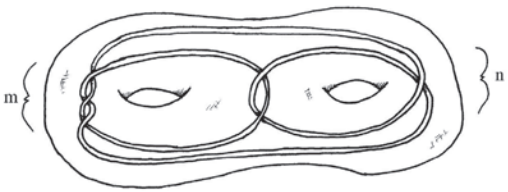

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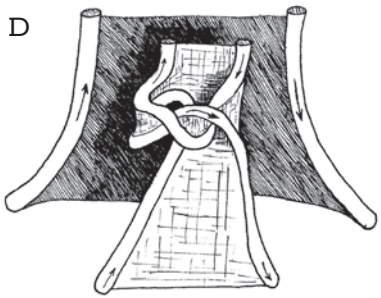

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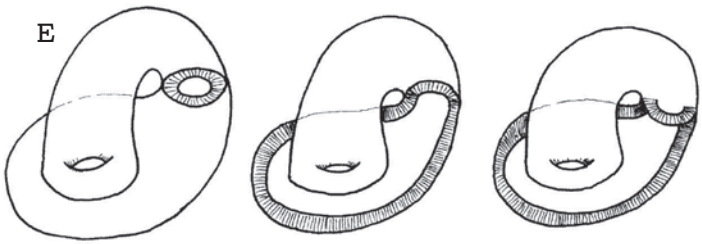


These drawings are from my first solo paper, "Bundles, handcuffs, and local freedom." They illustrate the existence of a knot in the 3-sphere whose complement is hyperbolic, that admits a fibration over the circle, and whose group contains a subgroup that is locally free (finitely generated subgroups are free) and not free. This answered a question from James Anderson: He had observed that if such a thing could not exist, then there would be counterexamples to Thurston's Virtual Fibration Conjecture (which we now know to be true due to the remarkable work of Ian Agol and Dani Wise).

The fundamental group of the 2-complex X (A) contains a subgroup that is locally free and not free. The way to construct the knots is to find a fibered knot in the complement of X so that X's group injects into the knot's. The complement of X is the handlebody (B). The knot is inside the handlebody (C). The knot is built by taking a square knot (which is fibered), cabling (which is again fibered), and then "plumbing." Plumbing is the process of gluing a twisted band along a square in a fiber (D). The rest of the figures are part of the proof that the knot is hyperbolic, and that you can perform surgeries to get closed hyperbolic fibered manifolds with groups having locally free non-free subgroups.

One year while I was in graduate school, Bob Gompf was teaching a course in 4-manifolds that I had lost the thread of. I didn't want to be rude, so I kept attending. So there I was in class, doodling and thinking about Anderson's question. I knew that connect sums, cabling, and plumbing preserved fibering. I sketched the complement of X and saw the square knot there. I cabled it for fun. I wanted the result to be hyperbolic, so I needed to do something else. So I plumbed a little band on to get rid of the essential torus that was ruining hyperbolicity, and I left class with a theorem. It took a while to prove that the examples were hyperbolic, but I had found them. Getting lost in a lecture isn't always bad!

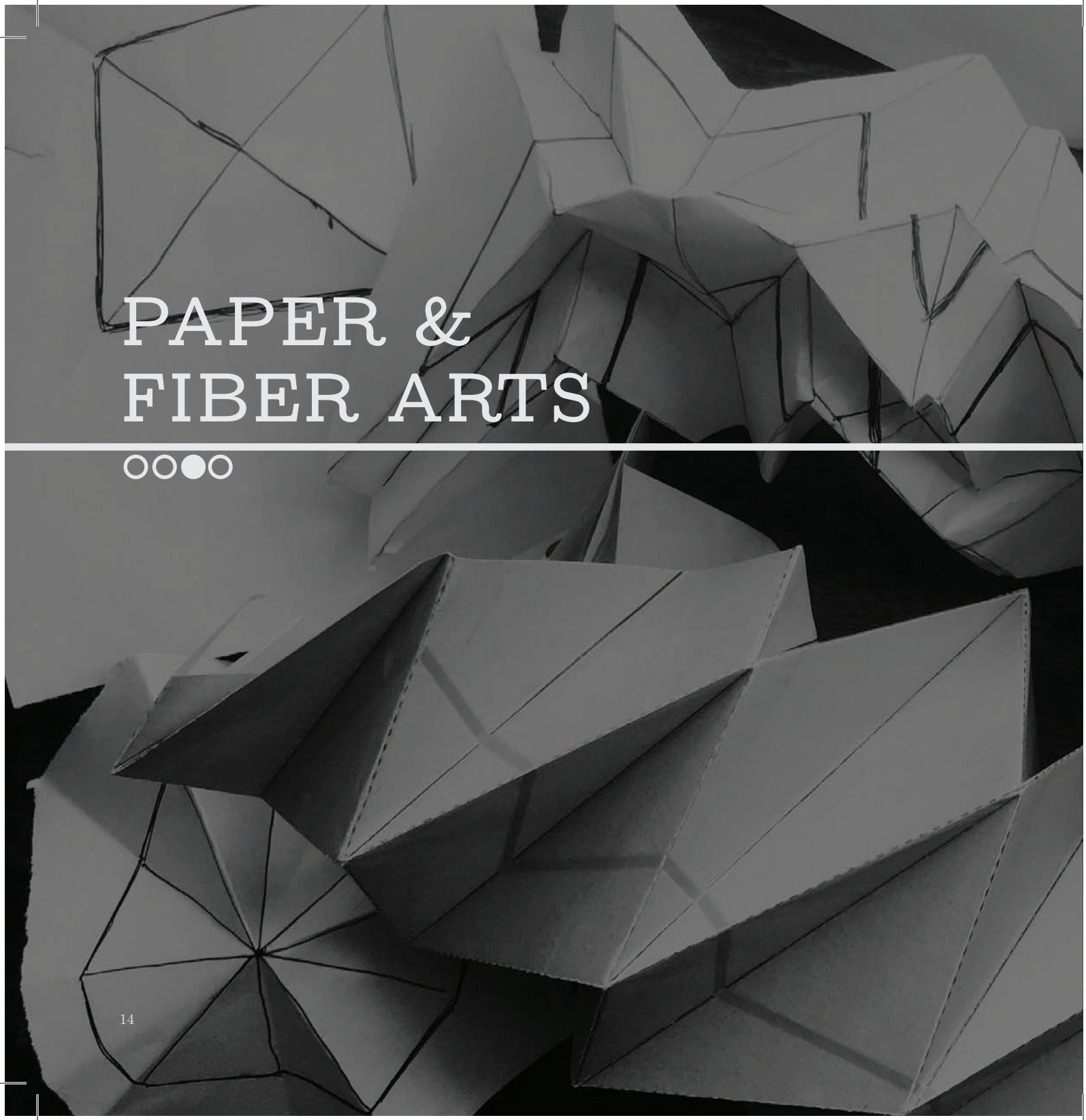
I drew the pictures by hand since I didn't know how to do it any other way!

AUTUMN KENT

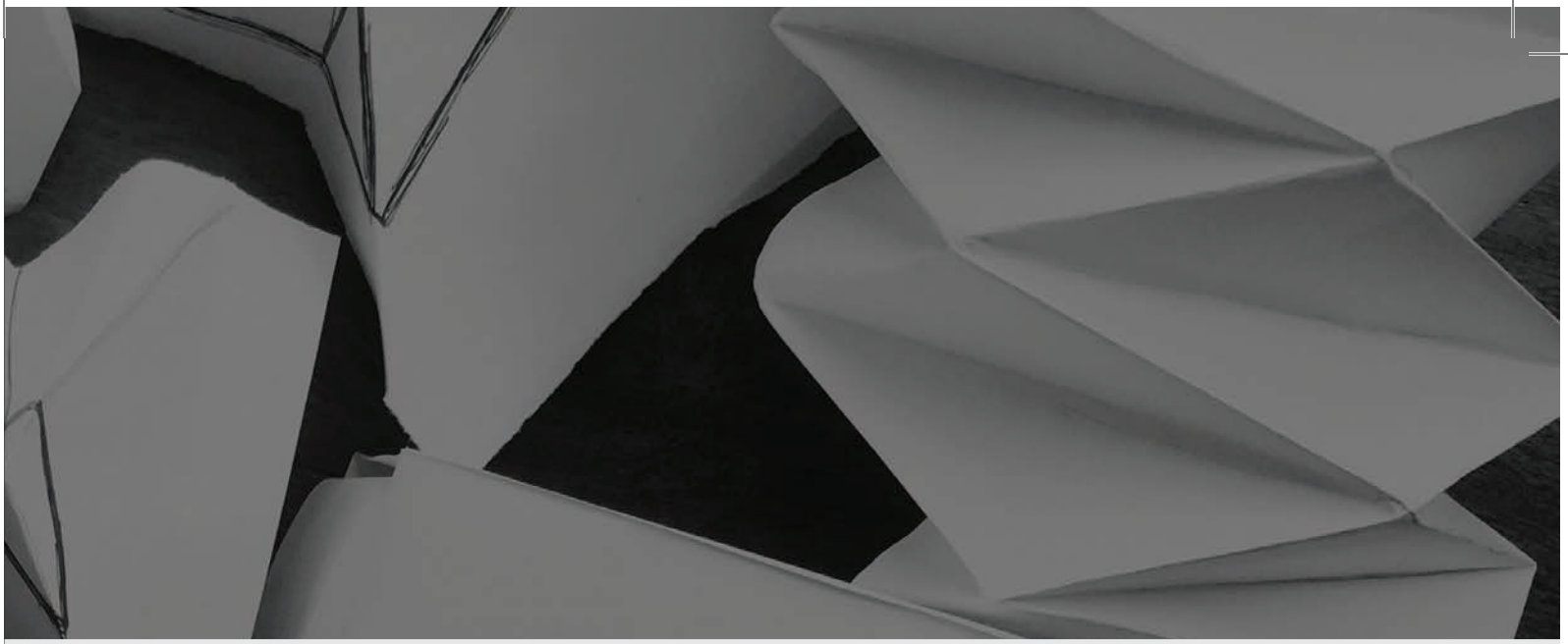
University of Wisconsin

pen drawings

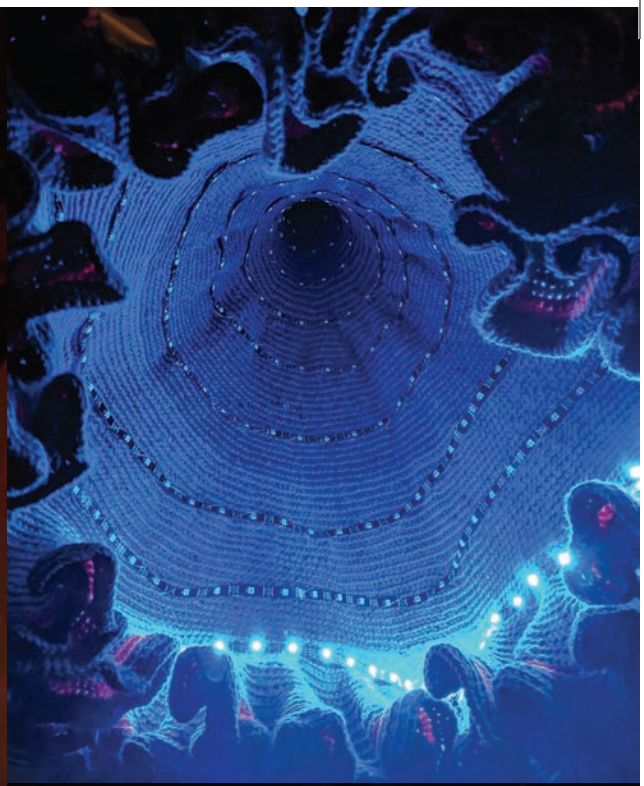
Images reprinted by permission from Springer Nature: Geometriae Dedicata, 106, *Bundles, handcuffs, and local freedom*, Autumn Kent, 2004, https://www.springer.com/journal/10711/.



PAPER & FIBER ARTS



Paper is by far the most abundant resource in most mathematicians' offices, and it is our most readily available tool for creating mathematical illustrations. Most of us created the Platonic solids out of paper at some point in our early education, and have created numerous other models throughout our mathematical careers. For creating objects that people will handle and manipulate, and especially for those that require some stretch, sewn or knitted fabrics may work better than paper.

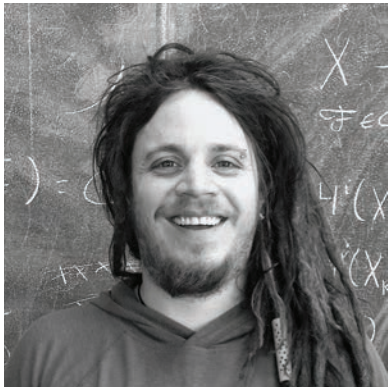


The Fabric of Spacetime is an installation serving as an interactive model of a young universe, combining crochet and electronics to create a dynamic and luminescent experience. The main physical component is a large, hand-crocheted hyperbolic manifold, where the number of stitches in each row increased at an exponential rate. In this way, the circumference grows exponentially faster than the length, introducing negative curvature and resulting in the many folds. This technique was pioneered in 1997 by Daina Taimina, a mathematician at Cornell. The curvature is analogous to the geometry of a very young universe (much less than one second old), where the spatial dimensions grew exponentially with respect to the time dimension, introducing curvature in the geometry of spacetime itself.

Sewed into the fabric model are 264 individually programmable neopixel LEDs, forming a spiral pattern around the inside, and mounted throughout the room are six servo motors, each connected by fishing wire to a fold in the crocheted model. There is also a PIR motion sensor directed toward the underbelly of the piece. All of these are wired to an Arduino MEGA microcontroller.

While undisturbed, the servos pull the model open and closed in a regular breathing motion, shifting to red while opening and to blue while closing. This is in homage to the cosmic red shift and blue shift of the universe: since red has the longest wavelength of the visible spectrum, things moving away from us at relativistic velocities gain a slight red tint in color, and, since blue has the shortest wavelenth, things moving towards us gain a slight blue tint. In this way, the color shift of the piece represents the actual shift that would be visible in an expanding and contracting universe. When the viewer walks under the piece, they trigger chaotic motion and lights.

I (Gabriel) find crochet to be a very flexible medium, both literally and figuratively, and I am fascinated at the way it can create various geometries and symmetry patterns, and also how easy it is to combine with electronics. Using crochet, I really tangibly felt the intensity of exponential growth. To create hyperbolic crochet, you add stitches in such a way that the circumference of the crocheted piece grows exponentially. In practice, this means the workflow really starts to slow down. This piece took hundreds of hours of crochet and weighs 10 pounds. The circumference of the piece at the widest part is over 80 feet (all hidden in the folds).

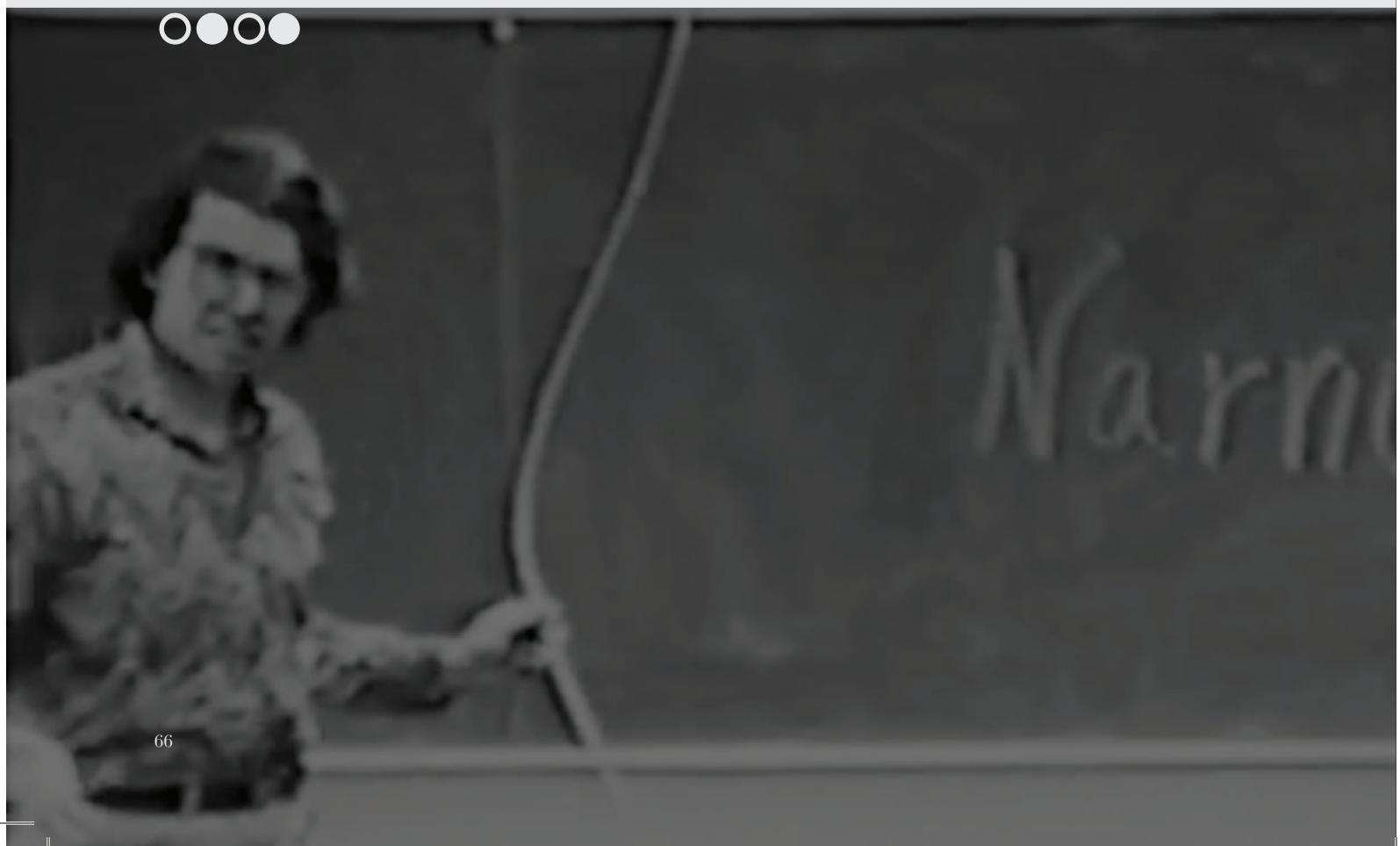


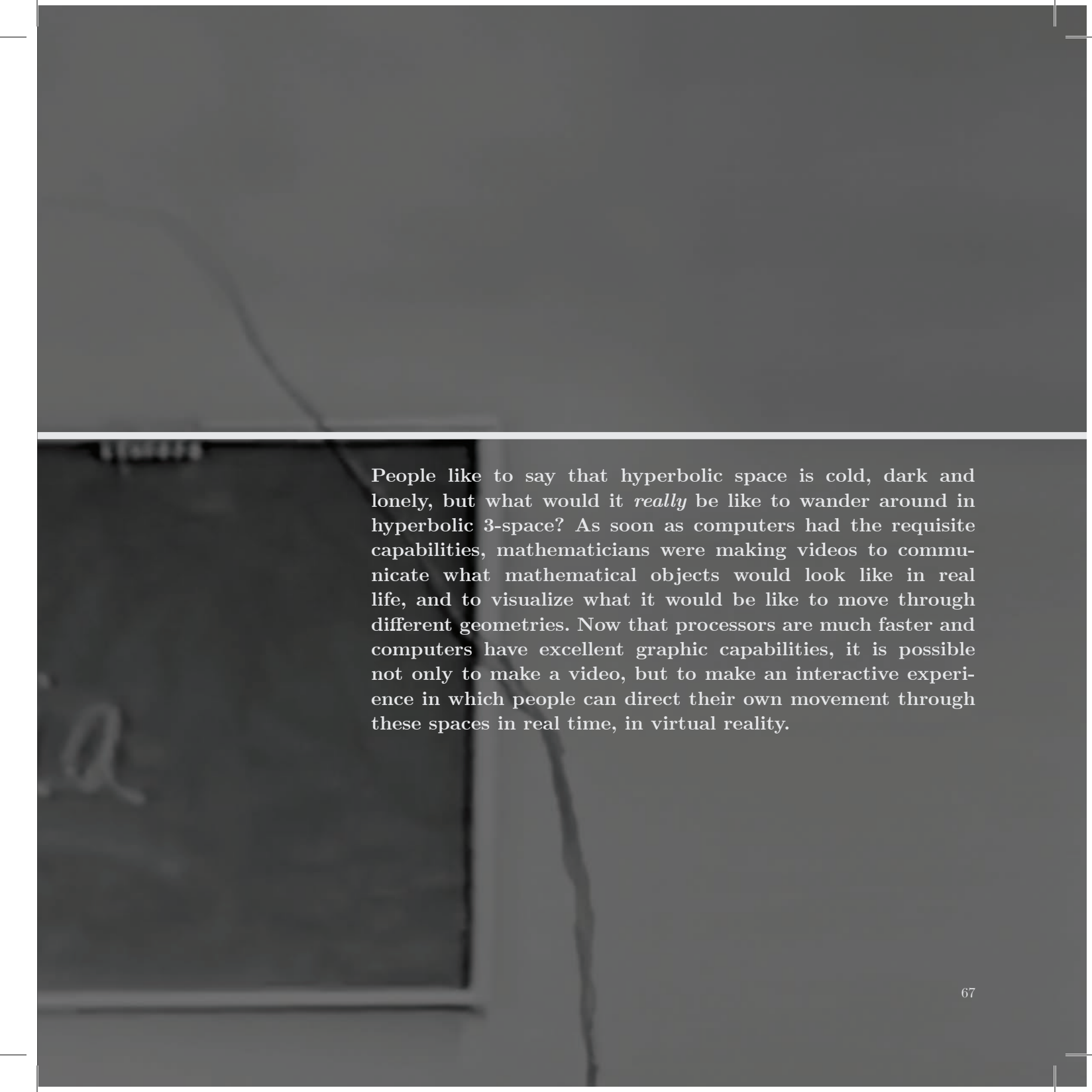
GABRIEL DORFSMAN-HOPKINS & MEGHAN MAYNARD

ICERM; Independent artist

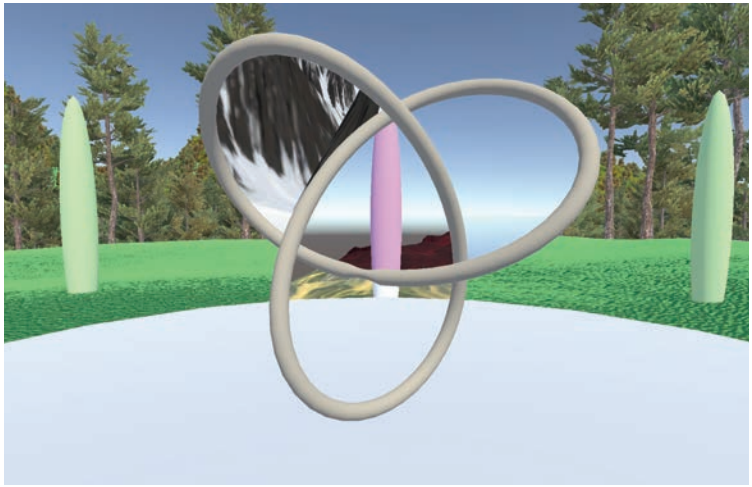
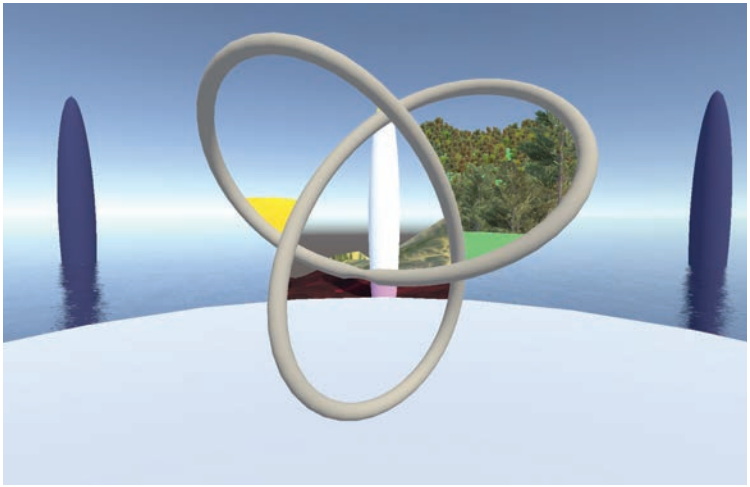
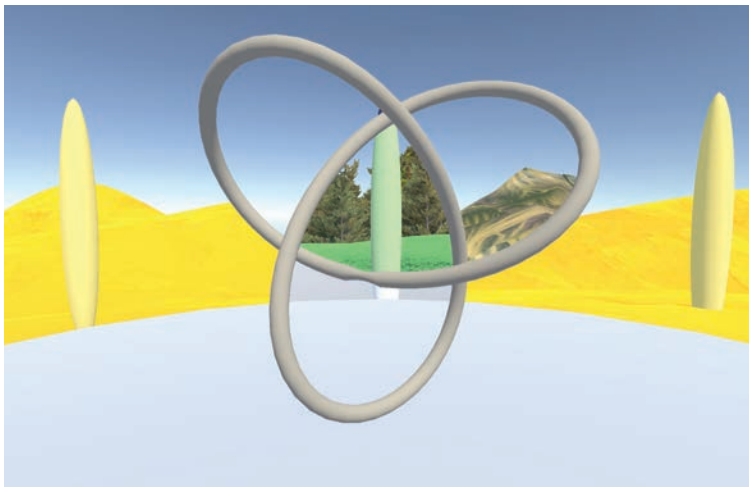
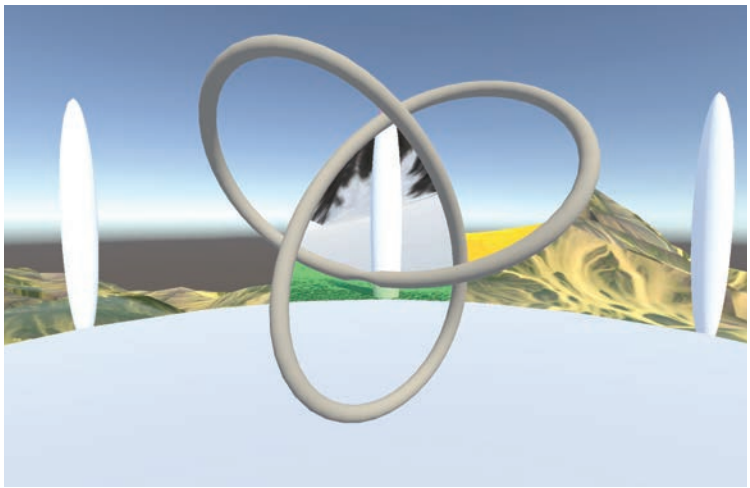
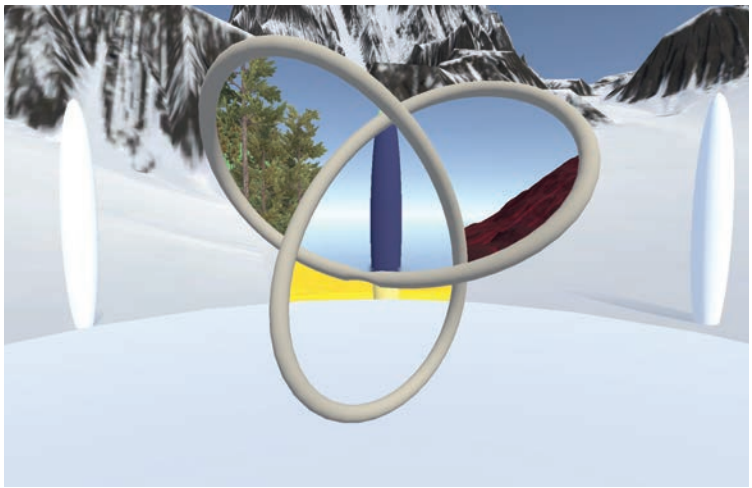
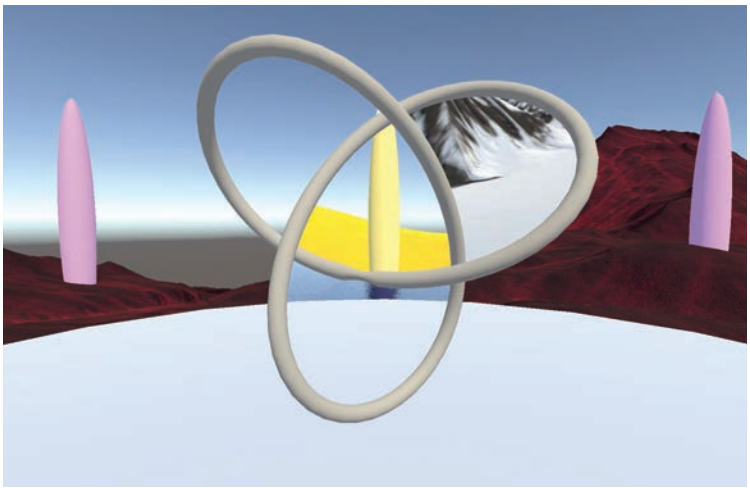
crocheted yarn and LEDs

VIDEO AND VIRTUAL REALITY





People like to say that hyperbolic space is cold, dark and lonely, but what would it *really* be like to wander around in hyperbolic 3-space? As soon as computers had the requisite capabilities, mathematicians were making videos to communicate what mathematical objects would look like in real life, and to visualize what it would be like to move through different geometries. Now that processors are much faster and computers have excellent graphic capabilities, it is possible not only to make a video, but to make an interactive experience in which people can direct their own movement through these spaces in real time, in virtual reality.

In his video "Knots to Narnia," William Thurston uses large wire knots to demonstrate his concept of knots as portals, where he actually steps through the knot to move back and forth between, in his conception, Earth and Narnia (see previous page). Virtual reality gives us the ability to bring to life this experience of actually stepping into other worlds and *seeing*, not just imagining, what it looks like on the other side.

These pictures show the six-fold branched covering of order two of the trefoil knot, generating the dihedral group of symmetries of the triangle. The three outer loops of the knot correspond to reflections, and the inner region corresponds to the rotation by 120°.

To create this virtual reality experience, I had to learn a lot about cyclic branched covers of knots, as I had to construct them in the software implementation. This led me to discover a construction from Poul Heegaard's dissertation from 1898 that could be implemented to simulate these portals in virtual reality. It consists of gluing a cone to the knot, which serves as the branch cut along which the different worlds are glued together.

At first, I thought about implementing portals as surfaces somehow spanned by the knot. The most obvious choice was to try Seifert surfaces, but this turned out to be a dead end. The next attempt was not to construct the branched covering, but only to simulate it through the knot projection on the screen. This approach used a variant of Reidemeister moves to keep track of the worlds the regions led to, but it turned out to be quite complicated and unstable at the crossings. Luckily, I then found the reference to Heegaard's construction in John Stillwell's *Classical Topology and Combinatorial Group Theory*, and I was able to create the virtual reality world.

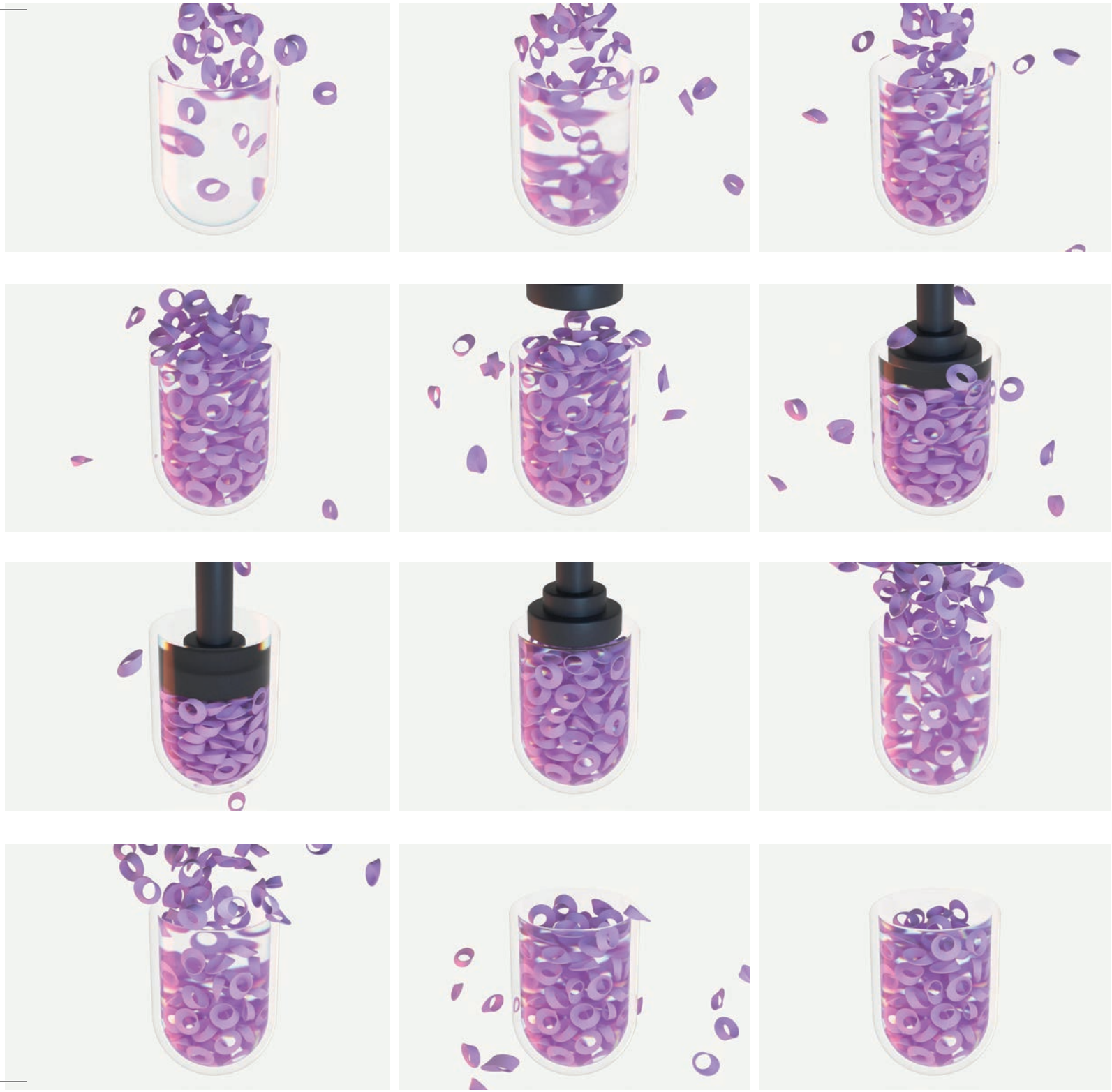
It was a challenge to reduce the computational load of the software, which has to compute which world to show for each pixel of both screens in the head-mounted display. I was able to achieve this by using shaders to offload much of the calculations onto the graphics processing unit, thanks to some tips from Roice Nelson. The main computational load is now the rendering of the worlds. I wanted to create worlds that are interesting to look at, but not so interesting that they take away the focus from the knot itself.

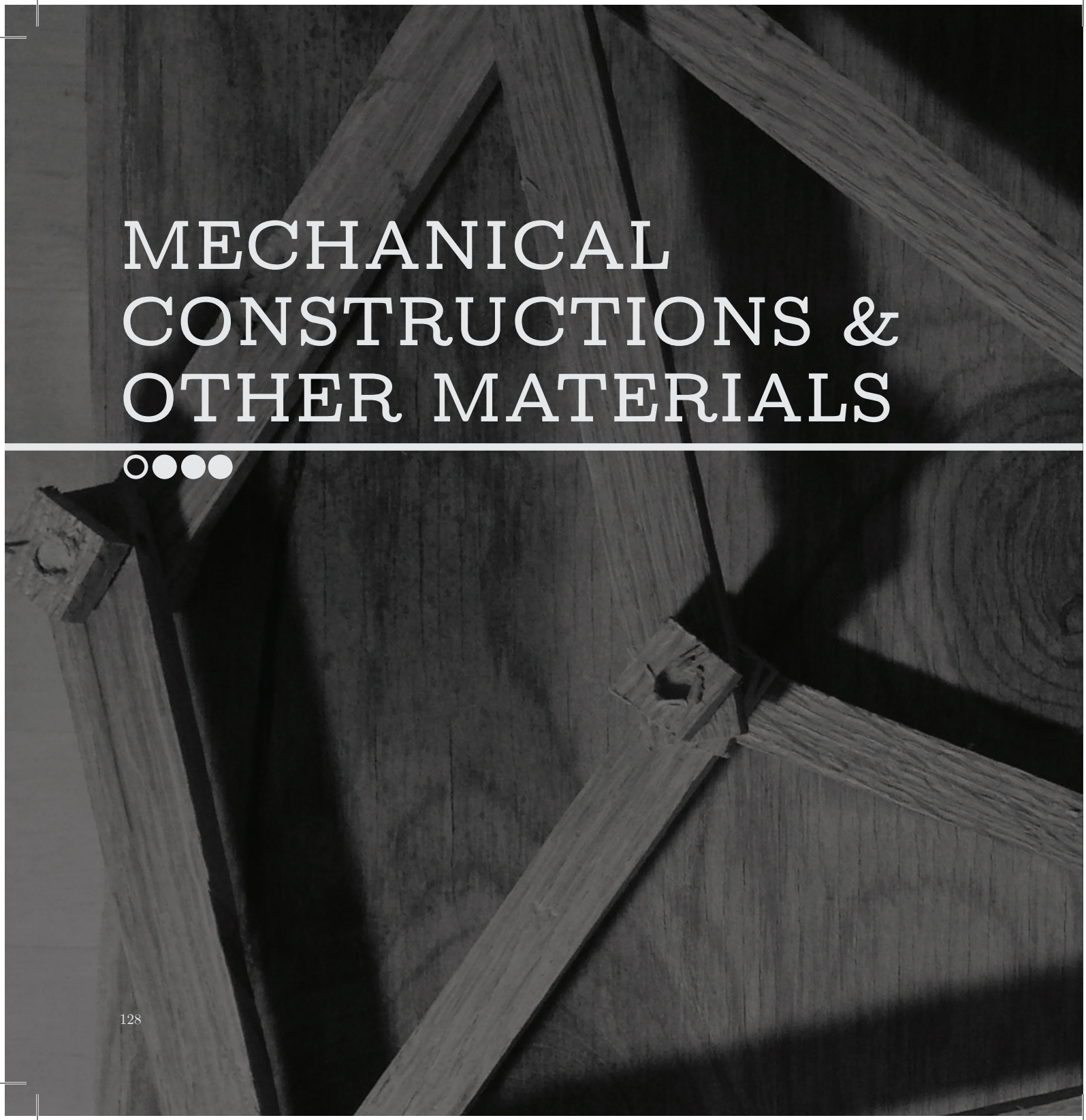
Software download and information: https://imaginary.org/program/knotportal
Thurston's original video, "Knots to Narnia":
https://www.youtube.com/watch?v=IKSrBt2kFD4

MORITZ L. SÜMMERMANN

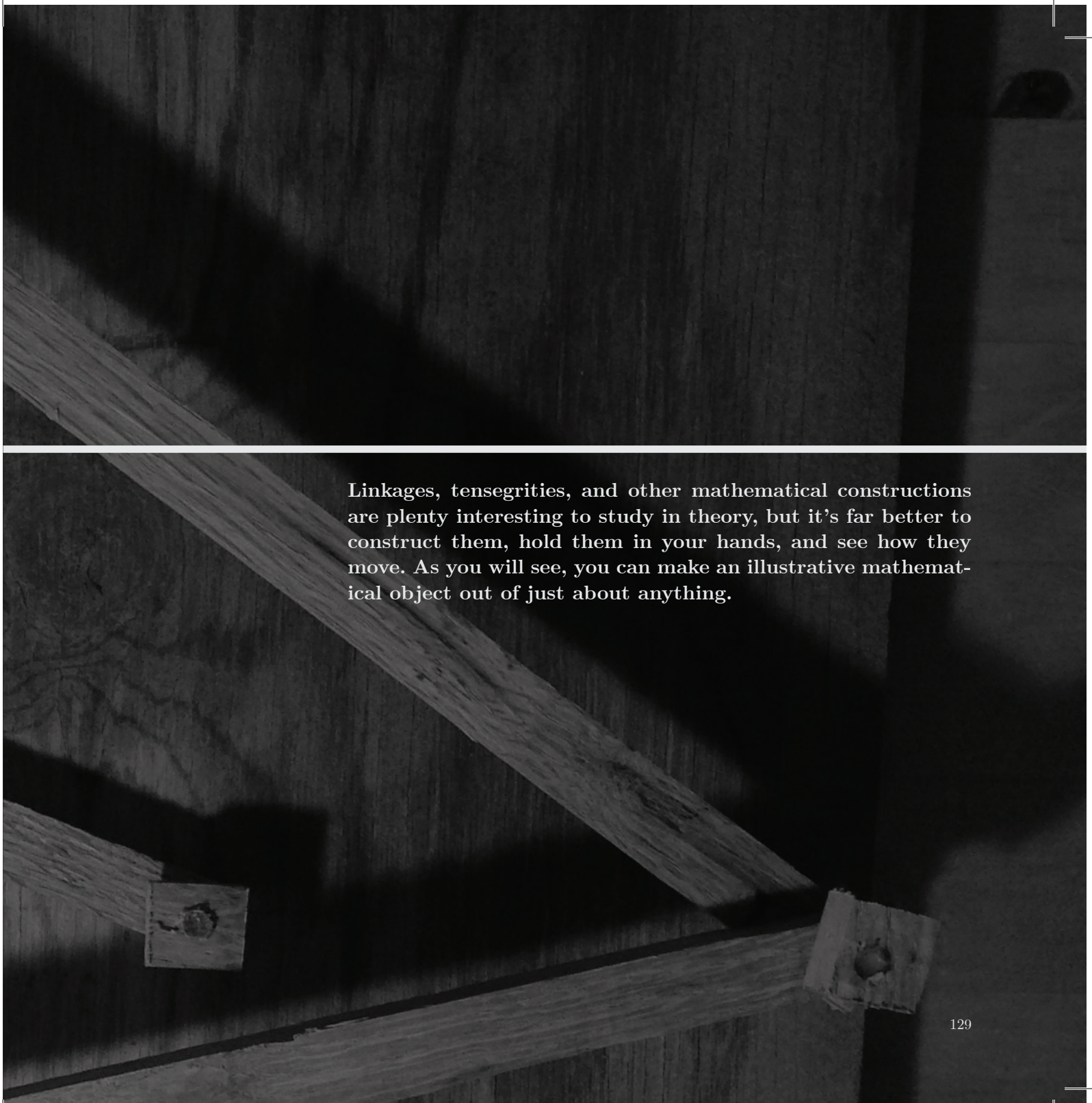
University of Cologne

virtual reality





MECHANICAL CONSTRUCTIONS & OTHER MATERIALS



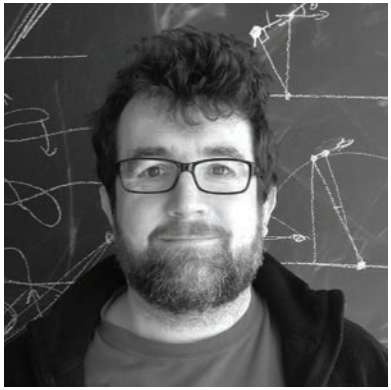
Linkages, tensegrities, and other mathematical constructions are plenty interesting to study in theory, but it's far better to construct them, hold them in your hands, and see how they move. As you will see, you can make an illustrative mathematical object out of just about anything.

This is a hyperboloid, cut from a brick.

This item was the result of a project with Nick Bruscia and Dan Vrana from University at Buffalo and OMAX corporation, who make the water jet. The challenge was simply to explore what was possible with a five-axis water jet. As the cutting "tool" is a straight (well, spiraling) jet of water, *ruled surfaces* are the natural thing to start experimenting with.

A surface is *ruled* if, for every point on the surface, there is a line through that point that is completely contained in the surface. A hyperboloid is a ruled surface, so to cut it out we need the jet of water to follow the ruling lines. This object therefore starts as an illustration of an application of geometry: how to control and think about the forms that can be made by controlling the form of a general tool.

It also presents a compelling version of a classic mathematical object. The surprising material (brick) draws you in, with the question of how it could be made. Even knowing the tool used might not help, as the curved surface seems impossible to make with such a straight tool. This leads directly to an understanding of the behavior of ruled surfaces and the hyperboloid.



EDMUND HARRISS

University of Arkansas

Waterjet-cut brick