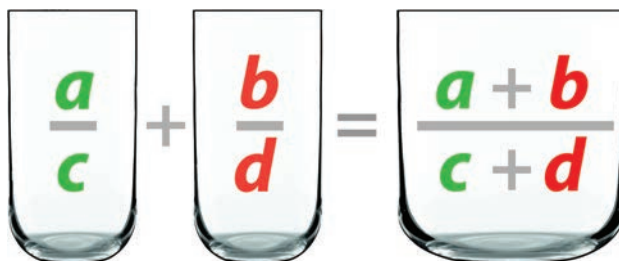


Part 1

Putting Two and Two Together

CHAPTER 1

Cordial math



One of the trickiest topics in school mathematics is fractions. Why can't they just behave like familiar, friendly whole numbers? But your Maths Masters are here to help, and we've discovered a wonderful new way to add fractions. Here's an example:

$$\frac{4}{7} + \frac{5}{6} = \frac{9}{13}.$$

Much easier. And for those who would like to apply the method more generally, the formula is

$$\frac{a}{c} + \frac{b}{d} = \frac{a+b}{c+d}.$$

At this stage you may be suspicious. So, ok, we'll confess: this excellent method for addition is not really our invention. This method of "adding" fractions has been discovered and rediscovered by schoolchildren for centuries. Of course, there's at least one problem with the method: it usually gives the wrong answer.

So, we'll all still have to add fractions in the traditional manner, with those annoying common denominators. However, the weird addition above does turn out to have some very remarkable properties.

Here's an interesting experiment, perfect for a sunny spring day. (So, you may have to leave Victoria.) Buy some concentrated raspberry cordial and mix yourself a glass of cordial and water. Of course the more cordial you add, the redder the liquid. We'll now consider two different mixes, in equal-sized glasses:

First Glass: 4 parts cordial and 7 parts water.

Second Glass: 5 parts cordial and 6 parts water.

Now create a third mix by combining the contents of the two glasses:

Third Glass: $4 + 5$ parts cordial and $7 + 6$ parts of water.

So, the proportions of cordial and water in the third glass are exactly given by the result our weird fraction sum. Intriguing.



Now, since $5/6$ is greater than $4/7$, the second glass will be redder than the first. What about the third glass? Since we've just combined the contents of the first two glasses, the third glass will be in between, redder than the first but not as red as the second. That means we have a cordial-powered proof that

$$\frac{4}{7} < \frac{9}{13} < \frac{5}{6}.$$

Very neat.

The strange sum we've been considering is called a *mediant*, the name reflecting the in between property that we've just observed. Of course it's misleading to use a plus sign, and so our weird operation should be represented in some other way. Most commonly the symbol \oplus is used. Then, for example,

$$\frac{4}{7} \oplus \frac{5}{6} = \frac{9}{13}.$$

However, there is something very strange about the mediant. To illustrate, first note that $5/6$ obviously equals $10/12$. However, if we calculate the mediant with $10/12$ in place of $5/6$ we find that

$$\frac{4}{7} \oplus \frac{10}{12} = \frac{14}{19}.$$

The two fractions $9/13$ and $14/19$ are definitely not equal. So, the mediant cannot be operating on the actual fractions, the numbers. Rather, the mediant is an operation on particular *representations* of fractions.

The mediant is definitely a peculiar creature, but it is still of genuine use. Our cordial mixing above is one illustration, but there are much more impressive applications.

Start with any positive whole number: we'll choose 6 to illustrate. Now write down, in order from smallest to largest, all the fractions with denominators at most 6; the fractions should be in lowest form, and we'll include $0/1$ and $1/1$ at the beginning and end. So, starting with 6, our list of fractions is

$$\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}.$$

Lists constructed in this way are called *Farey sequences*. These sequences have an amazing property: any fraction in a Farey sequence is the mediant of the fractions on either side. So for example, in our list above $3/5 = 1/2 \oplus 2/3$. That is very strange, and very, very cool.

It turns out that Farey sequences are much more than just weird fun. The *Riemann hypothesis* is perhaps the most famous and most important unsolved problem in mathematics (and is worth \$1,000,000).¹ And, the Riemann hypothesis can be expressed as a question about Farey sequences.²

Amazing. And all that from a glass or two of cordial.

Puzzles to ponder

Can you find an example where the mediant of two “fractions”, a/b and c/d , is equal to the sum of the two fractions? That is, your task is to find an example where

$$\frac{a}{c} + \frac{b}{d} = \frac{a+b}{c+d}.$$

Suppose now that a, b, c and d are all positive. Can you prove our cordial theorem? That is, assuming $a/b < c/d$, your job is to prove algebraically that

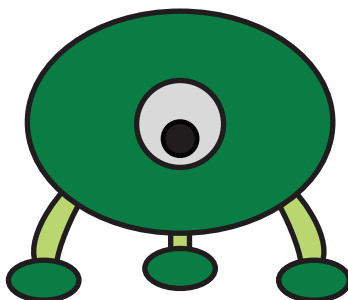
$$\frac{a}{c} < \frac{a+b}{c+d} < \frac{b}{d}.$$

¹The Riemann Hypothesis is a “Millennium Prize Problem”, with the Clay Mathematics Institute offering \$1,000,000 for its solution. Burkard has a Mathologer video that explains some of the underlying mathematics.

²S. Kanemitsu and M. Yoshimoto, *Farey sequences and the Riemann hypothesis*, Acta Arithmetica, **75**, 363–378, 1996.

CHAPTER 2

Uncovering base motives



10.01021101222...

Once upon a time, many years ago, there was a review of the school mathematics curriculum. In those days it was believed that mathematicians could contribute some insight. The good mathematicians were pleased to assist, and the result was a brilliant curriculum. The teachers and students were delighted, and everybody learned mathematics happily ever after.

Well, not quite. Our tale is of the “New Math” movement, from the 1960s. The New Math did include a significant involvement of mathematicians, and the proposals were indeed mathematically sophisticated, but overall the results were farcical. It is perhaps the reason why mathematicians to this day reside in pedagogical purgatory.

A brilliant encapsulation of what went wrong is provided by Tom Lehrer’s famous and funny song, *New Math*. Lehrer invites his audience to subtract 173 from 342. Having barely coped with that, Lehrer then declares that the subtraction should actually be done in base eight, chirpily singing that the 4 is in the “eights place”, the 1 is in the “sixty-fours place”, and so on. Lehrer’s lesson is hilarious, full of New Math jargon, and all of it self-evidently pointless.

Arithmetic in anything but base ten has now disappeared entirely from the curriculum, and we bid good riddance to the nonsense lampooned by Lehrer. But why were these bases ever taught at all? Have we lost anything by their removal? Indeed, we have: the bases have been thrown out with the bathwater.

One purpose of using different bases is simply to have fun, to play with numbers. True, writing π in other bases may not appeal to everyone. We, however, enjoyed the exercise when trying to determine our ideal number plate.¹ But, apart from the games, an important message has been lost.

¹In 2010, we wrote a column about wanting a customer car license plate, reading π in some suitable base.

Numbers are abstract. They are ideal, mental objects, difficult to discern and difficult to discuss. Because of this it is very easy, for example, to confuse the *numeral* “8”, the symbol, with the *number* for which the symbol stands. Sadly, many current textbooks are riddled with such confusion.

Writing a number in different bases can be an attempt to distinguish the number from the symbols representing that number. Admittedly, the attempt may be so abstruse that the message is lost, and then we sing along with Tom Lehrer. However, given the current infestation of calculators, it is now common to view decimal representations as the be-all and end-all, to regard these decimals themselves as the numbers. They are *not* numbers, and they are usually not even insightful representations of numbers: this message is more important than ever.

That’s all very general, so what about particular bases? The ancient inventor of our number system chose base ten simply because humans have ten fingers. In certain contexts, however, there are other natural bases. The clear example is base two, where all numbers are built up from 0 and 1. The on-or-off nature of base two arithmetic makes it perfect for the logic underlying computers, and for many related areas of mathematics. Indeed, given the technology fetishism of our curriculum masters, the absence of base two in the Australian curriculum is particularly puzzling.

Other bases have uses as well, and we’ll end with a truly beautiful illustration. We have written before about irrational numbers,² and we remarked then that when numbers such as $\sqrt{2}$ are declared irrational it is usually with no hint of how we *know* them to be irrational. Well, now we will ponder that.

Recall that $\sqrt{2}$ being irrational means that it could not be written as a fraction, $\sqrt{2} = A/B$ with A and B whole numbers. If we square both sides of this equation, and multiply to get rid of the denominator, we then have the equation

$$A^2 = 2B^2.$$

So, what we are claiming is that this equation is impossible, that no positive whole numbers A and B will solve it. But how can we possibly rule out all of the infinitely many choices for A and B ? Here comes the magic: we shall imagine that the numbers A and B are written in base three.

A number written in base three will have a ones place, a threes place, a nines place and so on. So, for example, we would normally write the number fifteen as 15, but in base three it would be written as 120: this amounts to $(1 \times 9) + (2 \times 3) + (0 \times 1)$.

²See Chapters 33 and 53 of our first book of math columns, *A Dingo Ate My Math Book*, American Mathematical Society, 2017.

We can perform arithmetic in base three just as in base ten. For example, the base ten equation $2 \times 2 = 4$ would be written $2 \times 2 = 11$ in base three. And, here is the working for a harder one, the number 120 multiplied by itself:

$$\begin{array}{r}
 120 \\
 \times 120 \\
 \hline
 10100 \\
 12000 \\
 \hline
 22100
 \end{array}$$

Of course you could convert 120 to base ten form (i.e. 15), and then calculate as usual, but the point is you don't need to: all the familiar methods of arithmetic apply just as well in base three, and in any base.

Now, for what follows, you only need to know one special fact about base three arithmetic:

Ignoring final zeroes, any squared number written in base three always ends in a 1.

For instance, reviewing our examples above, $1 \times 1 = 1$, and $2 \times 2 = 11$ and $120 \times 120 = 22100$. Why this is always so, why there is always a 1 at the end, may not be so obvious, although it is not that difficult to show. But hopefully the claim we're making is clear.

Now, with that fact in mind, look again at our A, B equation for $\sqrt{2}$. Written in base three, we now know that the A^2 will end in a 1 (ignoring zeroes). However, B^2 will also end in a 1, and that means $2B^2$ will end in a 2. But if A^2 and $2B^2$ end in different digits then they cannot possibly be equal. That is, the equation $A^2 = 2B^2$ is impossible to solve with positive whole numbers. And that means $\sqrt{2}$ cannot be written as a fraction. We have *proved* that $\sqrt{2}$ is irrational.

This amazingly simple proof is due to mathematician Robert Gauntt, who was a freshman at Purdue at the time.³ Tom Lehrer is himself a mathematician, and we are sure that even the sceptical Lehrer would be captivated by this beautiful application of base arithmetic.

Puzzles to ponder

What is our alien up above doing?

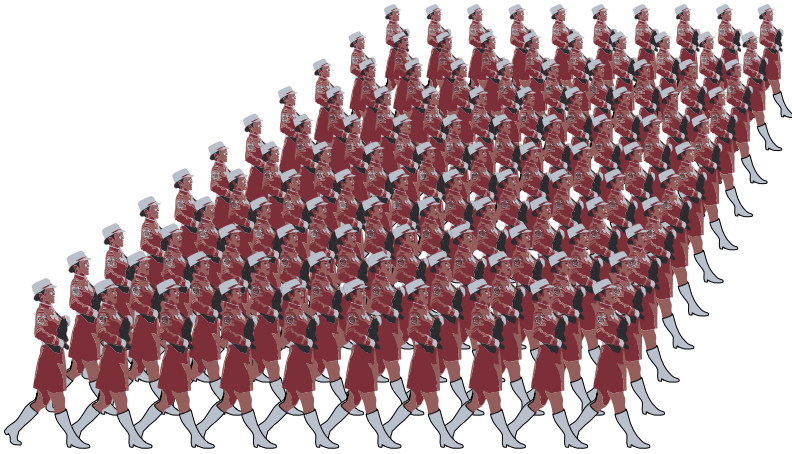
Why does any squared number end in a 1 when written in base three?

Can you find another base, which proves that $\sqrt{3}$ is irrational?

³The irrationality of $\sqrt{2}$, American Mathematical Monthly, **63**, 247, 1956.

CHAPTER 3

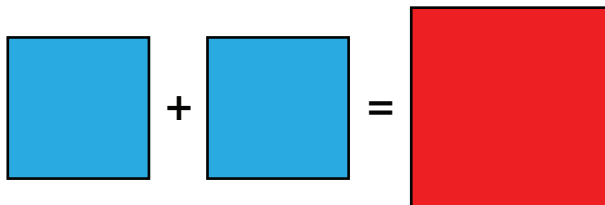
Sneaky square dance



It is pleasing to know that there are parts of the world where everyone loves squares. What a paradise it must be, a country where people are forever marching in perfect square formation.

Well, maybe not. Geometric marching is probably not as much fun when its main purpose is to please a Glorious Leader. Still, those marching squares are impressive. And, given the unfortunate folk will be marching anyway, we have a great idea for a very mathematical flourish.

Our plan is to have two identical squares of marchers, each square performing the usual stunning steps. Then, the grand finale will consist of the squares being merged into one big (Red) square.



It'd definitely be a showstopper but first there are details to sort out. The interlacing of the squares will be tricky, requiring planning, practice and fancy footwork.

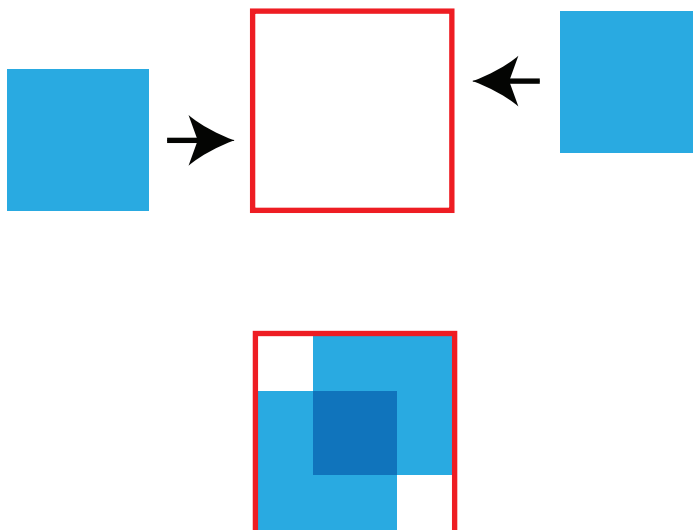
We also have to decide the size of squares to use, which would need to consist of a suitable number of marchers. For example, beginning with two tiny 2×2

squares wouldn't work: that would give us eight marchers in total, one short of the number required to rearrange into a 3×3 square. Similarly, beginning with two 3×3 squares of marchers would mean we have eighteen marchers, too many to form a 4×4 square and insufficient for a 5×5 square.

Hmmm. This will take some time to figure out, but we should be able to do it. We'll try 4×4 , then 5×5 and so on, and eventually we should have in hand the smallest squares that work.

For now, let's leave that calculation and just assume we've determined the smallest squares that can be merged. Then, imagining we have sufficient marchers to occupy those squares, we can plan the marching steps.

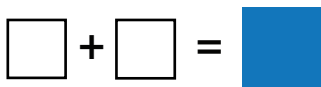
Let's begin with an empty red quadrangle of just the right size to accommodate all our marchers. Then, a stylish approach would be to have the two identical squares of marchers enter the quadrangle from opposite sides.



At this stage the two little white squares are unoccupied and the blue squares are overlapping. It would be crowded in the middle dark blue square, but that's not a problem: we can simply arrange for half of the marchers to stand on the shoulders of the others.

Finally, we'll have the marchers leap off their comrades' shoulders and into the two empty white squares, a spectacular finish to our merging of the two blue squares into the Red square. Ta da!

But wait a minute. If, as planned, we have precisely the correct number of leaping marchers to fill the two little white squares then this can be represented by the following picture.



Uh oh. We assumed that we began with the very smallest squares that could be merged to make a larger square. Yet, somehow we created even smaller squares that would work. How can that be?

Simply, it cannot be. Sure, there is nothing wrong with our marching plan. However, for our *smallest* possible squares to result in even *smaller* squares is a plain logical impossibility. The unavoidable conclusion is that *no* squares can be merged in the way we had originally contemplated.

Well, bum. So much for our plans to impress our Glorious Leader with a great marching finale. However, perhaps he'll be impressed by some intriguing mathematics that emerges from our failed attempt.

What we have outlined above is a *proof by contradiction*. We began by *assuming* that certain squares were possible, and that assumption led to a *contradiction*, a logical impossibility. This contradiction *proves* that our original assumption was wrong, and that no such squares can exist.

We can now reconsider this conclusion in terms of numbers rather than squares. Consider again the two hypothetical blue squares. Say each square contains $B \times B$ marchers and that the larger red square into which they can supposedly merge has $R \times R$ marchers. We have proved this is impossible, and that means that there are no positive whole numbers B and R that solve the equation

$$2B^2 = R^2.$$

Rearranging, it follows that there is no *fraction* R/B that solves the equation

$$\left(\frac{R}{B}\right)^2 = 2.$$

So, there is no fraction whose square is 2, which is exactly the same as saying $\sqrt{2}$ is not a fraction. That is, our marching ponderings have *proved* that $\sqrt{2}$ is irrational.

Now that is pretty cool. It is one thing to punch $\sqrt{2}$ into a calculator and to stare at a few unilluminating decimals; it is another to *know* that $\sqrt{2}$ can never be written as a fraction.

We've written about $\sqrt{2}$ before, when discussing the clever principle behind the dimensions of A-sized paper.¹ On another occasion we used base arithmetic to give a different proof that $\sqrt{2}$ is irrational.²

There are a number of proofs that $\sqrt{2}$ is irrational, but the marching band proof above is possibly our favorite.³ It has been popularized by the great John Conway, who attributed it to mathematician Stanley Tennenbaum.⁴

And will the Glorious Leader be impressed with Tennenbaum's proof? We don't know. But, if he isn't, and perhaps anyway, it may be time to start looking for a new Glorious Leader.

Puzzle to ponder

Can you come up with a similar marching proof that $\sqrt{3}$ is irrational?

¹See Chapter 33 of *A Dingo Ate My Math Book*.

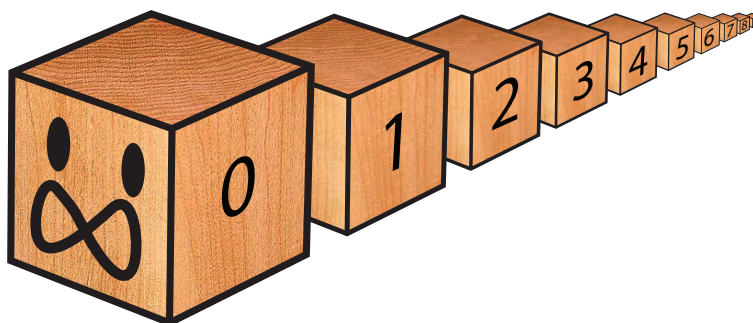
²See the previous Chapter.

³One of your Maths Masters leans towards the proof in the previous Chapter.

⁴J. Conway and J. Shipman, *Extreme Proofs I: The Irrationality of $\sqrt{2}$* , Mathematical Intelligencer, **35**, 2–7, 2013. We also generalized Tennenbaum's argument in *Marching in Squares*, College Mathematics Journal, **49**, 181–186, 2018.

CHAPTER 4

A very strange set of blocks



Your Maths Masters have been lecturing at universities for about three hundred years, and for the last two hundred years we have been engaged fulltime in popularizing mathematics. Something like that. Anyway, we have worked long and hard on many projects, but all our efforts have been guided by one fundamental goal: to convince as many people as possible that $0.999\ldots = 1$.

We look forward to the day when we can visit a school, ask the students what the infinite decimal $0.999\ldots$ is, and have them all shout back “One!” On that day we can happily retire. It is not likely to be soon.

Infinity is a very tricky concept, which has been bamboozling mathematicians for millennia. In fact it was only in the 19th century that infinite constructions, including $0.999\ldots$, were completely understood.

We have another infinity puzzler in store, so won’t revisit $0.999\ldots = 1$ today.¹ We will pause, however, to note that an infinite decimal is in fact an infinite sum. Our friend $0.999\ldots$, for example, is the infinite sum

$$9/10 + 9/100 + 9/1000 + \cdots .$$

A similar and possibly more familiar infinite sum is

$$1/2 + 1/4 + 1/8 + \cdots .$$

This sum also totals exactly to 1.



¹See Chapter 48 of *A Dingo Ate Our Math Book*, and the puzzle for this Chapter.

A sum where each number is a fixed multiple of the previous number is known as a *geometric series*. That is true for both the sums above: at each stage we're multiplying by $1/10$ to create our old friend $0.999\ldots$, and by $1/2$ in the second sum.

The nice thing about a geometric series is that, just as for the above examples, the total can be worked out exactly. However, it will come as no surprise that infinite sums other than geometric series can be much trickier.

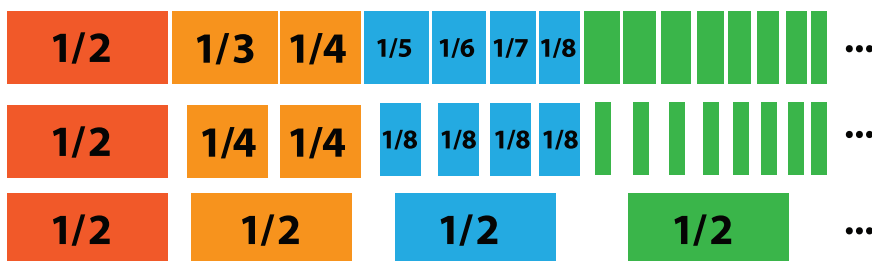
Consider the following sum, known as the *harmonic series*:

$$1 + 1/2 + 1/3 + 1/4 + \cdots$$

What can we say about it? Well, even though we're adding progressively smaller numbers, the total is very large. Notice that $1/3 > 1/4$, and so

$$1/2 + (1/3 + 1/4) > 1/2 + (1/4 + 1/4) = 1/2 + 1/2.$$

Not so large yet. But now have a look at this.



As pictured, the next four (blue) fractions also sum to at least $1/2$, as do the next eight (green) fractions, and so on, forever. So, since we're adding $1/2$ over and over forever, the only possibility is that the harmonic series totals to infinity. We told you it was large.

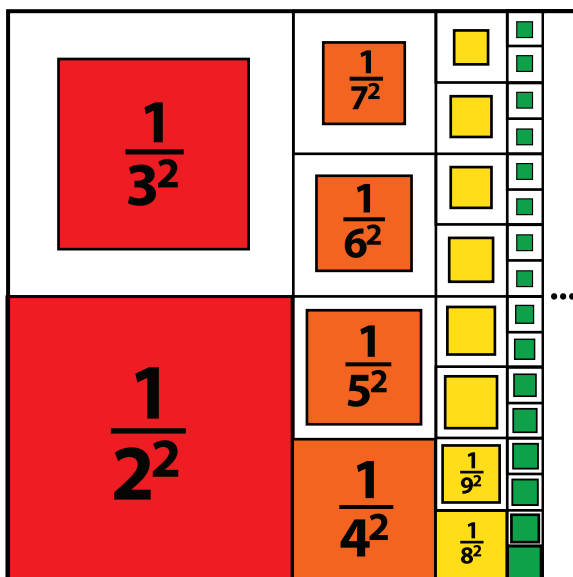
OK, now onto another sum, the *quadratic series*:

$$1/1 + 1/4 + 1/9 + 1/16 + \cdots$$

This time the denominators are squares, and so the sum is smaller. But how small?

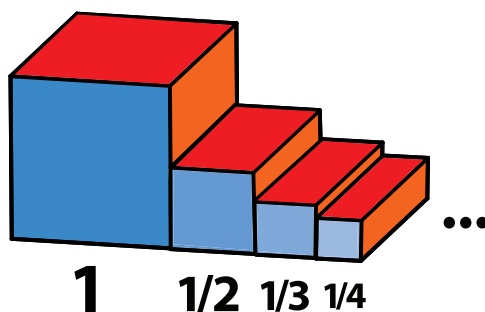
Imagine infinitely many squares, the first with side length 1, the second with side length $1/2$, then $1/3$, and so on. The areas of these squares are $1 \times 1, 1/2 \times 1/2, 1/3 \times 1/3 \cdots$. It follows that the sum of the areas of all the squares is exactly the quadratic series.

However, as the picture below illustrates, the initial 1×1 square is large enough to accommodate all the subsequent squares.



That means that the total sum of the areas is at most twice that of the first square. So, the quadratic series totals to less than 2: definitely finite.

We now want to consider a very puzzling toy, suitable for a budding baby maths master. The toy we have in mind is an infinite set of blocks: the first block has dimensions $1 \times 1 \times 1$, the second is $1 \times 1/2 \times 1/2$, the third $1 \times 1/3 \times 1/3$, and so on. Let's consider how we could make these blocks.



From the familiar “length times width times height”, we can calculate the volume of each block. It is then straightforward to calculate that the sum of the volumes of all the blocks is exactly our quadratic series. That means the total volume of the blocks is less than two: it'd take a lot of work to make them, but we'd only require a finite amount of wood.

But, what if we wanted to paint the blocks? Just considering the red sides on top, the first has area 1, the second area $1/2$, the third $1/3$ and so on. That means the sum of the red areas is exactly the harmonic series, and so must infinite.

Hmm. These are truly strange blocks: it would take a finite amount of wood to make them, but the infinite surface area means it would require an infinite amount of paint to paint them.

But what if we just hollowed out the blocks? It would only require a finite amount of paint to fill them: wouldn't that effectively paint that same infinite area? It's all very strange!

One final matter: we have yet to say what the quadratic series actually totals to. Believe it or not, we have the exact, astonishing answer:

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}.$$

How on Earth did π get in there? In fact, the quadratic series is the starting point for many fascinating stories, on π , and prime numbers, and the famed *Riemann Hypothesis*.

We'd love to write on that sum one day.² But, first, there is some homework to be done: a little matter of $0.999\cdots = 1$.

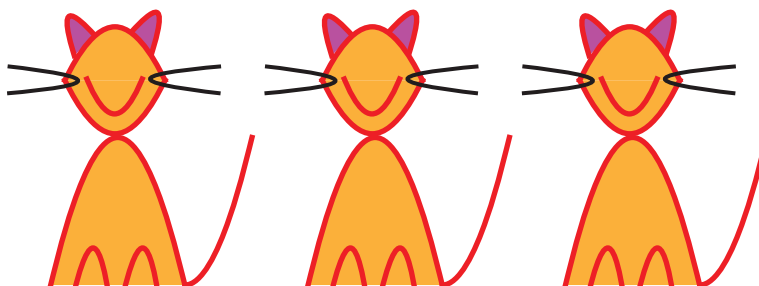
Puzzle to ponder

Can you *prove* that $1/2 + 1/4 + 1/8 + \cdots = 1$? (Chapter 24 provides a hint.)

²Burkard has some *Mathologer* videos devoted to the sum and to the Riemann Hypothesis.

CHAPTER 5

Parabolic production line



One of your Maths Masters is blessed to have a very young maths mistress to assist him with his work. Regular readers of this column will not be surprised that little Eva's favorite word is "cat". She loves cats. Amusingly, though, it seems that everything is a cat. Eva greets all animals (including her parents) with a loud and confident cry of "Cat!"

Eva's omni-catting is oddly familiar. It is reminiscent of many high school math classes, where any curvy graph is as likely as not to be greeted with confident cries of "Parabola!"

The parabola-spotting is significantly less amusing, but of course it is not the students' fault. The textbooks overflow with parabolas and quadratic equations, including the most absurdly contrived applications: what to make of a shop displaying $x^2 + 2x - 48$ types of cheese?¹ Moreover, students are introduced to few other curves, which are seldom distinguished in a meaningful or memorable manner.

It is all sad and silly. And needless. There are genuine, beautiful applications of parabolas which are rarely if ever discussed. Yes, eventually a few lucky students briefly study projectile motion (and seemingly more briefly when the senior Australian Curriculum kicks in), but that's about it.

Physics students still see such applications, in the form of parabolic mirrors and lenses. These are welcome mainstays of science museums such as Scienceworks,² particularly as very impressive parabolic whispering walls.

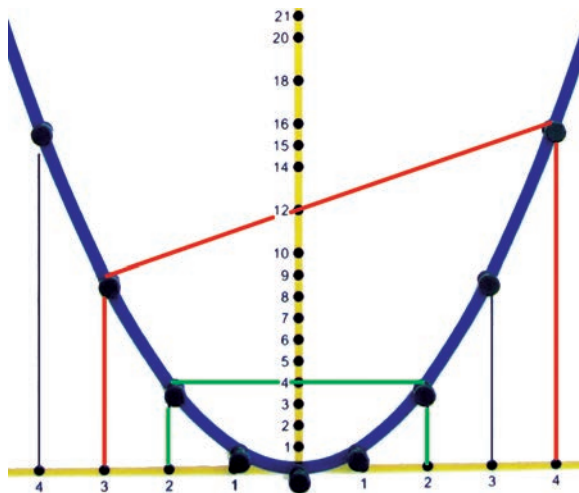
Wouldn't it be refreshing for students to know how these displays work, beyond parroting "focal point" as if reciting jargon explains anything? Wouldn't it be worthwhile for students to know what a focal point is, and the mathematics to explain it? We live in (not overly much) hope.

¹We're not making this up.

²Melbourne's not very good hands-on science museum, which we've had cause to mention a few times. See Chapter 40, and Chapters 63 and 64 of *A Dingo Ate My Math Book*.

One day we'll give focal points a go.³ Today, however, we'll demonstrate a simpler and very cool feature of parabolas.

Last year, one of your Maths Masters visited the *Mathematikum* in Giessen, Germany. This is a seriously amazing mathematics museum, featuring many impressive exhibits, including a parabolic whispering wall. One exhibit was based upon a parabola drawn on a white wall:



What appear as black dots on the parabola are cylinders protruding from the wall. The numbers on the coordinate axes indicate that the parabola is the archetypal $y = x^2$, with the cylinders placed at points with whole number coordinates, $(1, 1)$, $(2, 4)$ and so on.

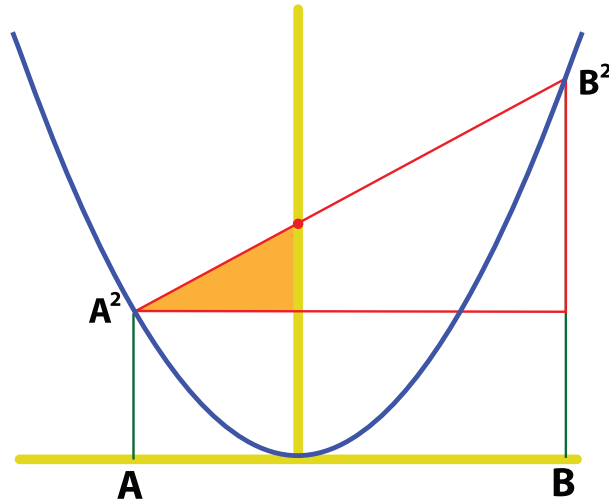
The purpose of the exhibit is to demonstrate how to graphically multiply two numbers. To find 3×4 , for example, the visitor is instructed to tie a rope between the cylinders at distances 3 and 4 to the left and right of the vertical axis. This rope is represented by the sloping red line.

Then, the spot where the rope crosses the vertical (y) axis indicates the product of the two numbers. So, in the above example we conclude that $3 \times 4 = 12$. Very nice.

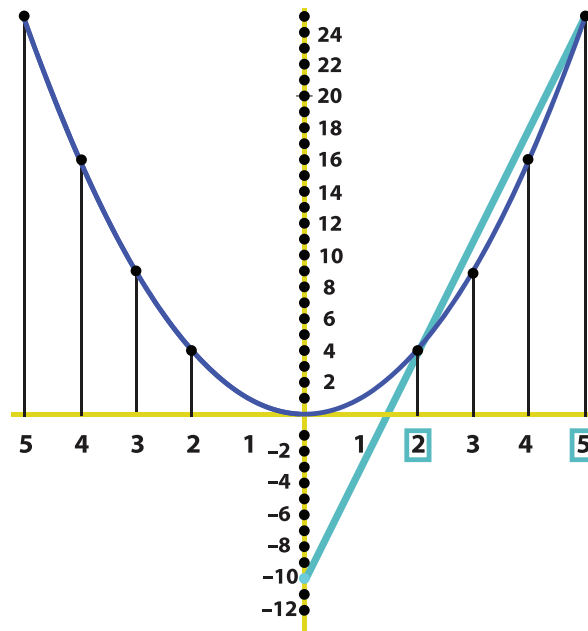
It is a surprising and clever demonstration, but why does it work? It is not hard to understand when the two numbers are the same, amounting to squaring a single number: in this case the rope will be horizontal and at just the correct height. This is illustrated by the green line, indicating the product $2 \times 2 = 4$.

³See Chapter 64 of *A Dingo Ate My Math Book*.

If the numbers A and B to be multiplied are unequal there is more work to be done. A natural approach is to begin by obtaining the equation of the straight red line. However, it is simpler to just draw in horizontal and vertical lines, creating a big red triangle:



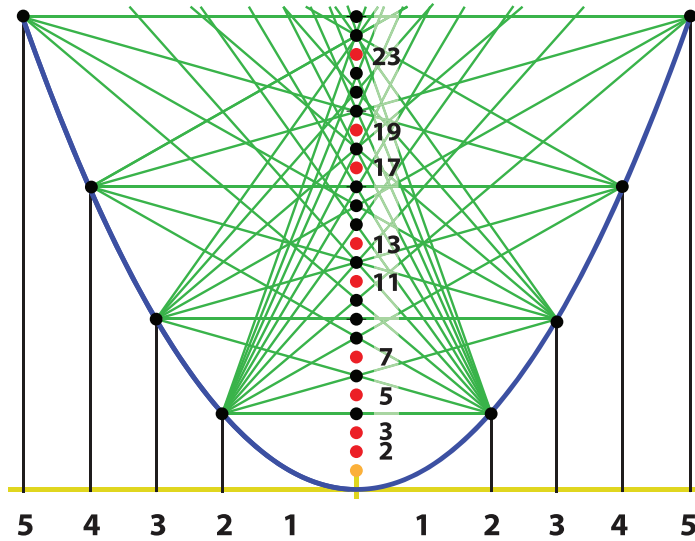
Then, we notice that the smaller triangle inside is *similar*, that is, it's the same shape. It is then straightforward to use this similarity to obtain an equation, indicating where the red line crosses the vertical axis. We'll leave the details to the triangle aficionados.



What about negative numbers? The Mathematikum exhibit only demonstrated the multiplication of positive numbers, but the same construction works just as well in general. Above, for example, we have calculated $2 \times (-5) = -10$.

This graphical multiplying is very pretty, and it also has a very pretty consequence: it can locate for us all the prime numbers. Or, which amounts to the same thing, it can locate all the numbers that are *not* prime.

Consider all possible multiplications of two whole numbers greater than one, and draw the straight lines for each of these multiplications. Then, every possible product will be indicated by a line running through the vertical axis. The only numbers left untouched will be 1 and the prime numbers. It is a striking, pictorial variation of the famous *sieve of Eratosthenes*.



It's all very nice stuff and we could write plenty more, but that's probably enough parabolic fun for one day. After all, there's serious schoolwork to get done: somewhere out there, there is a cheese shop that desperately needs factorizing.

Puzzle to ponder

Use the pictured similar triangles to prove that the parabolic trick for multiplying really works.

CHAPTER 6

The magic of the imaginary



One of our favorite mathematical writers is John Stillwell, formerly at our home of Monash University, and perhaps our favorite among John’s many excellent books is *Yearning for the Impossible*.¹ In this book John describes the manner in which mathematicians fantasize about the “impossible”, and how they make these fantasies real. More bluntly put, John tells the story of mathematics as an extended history of cheating.

Probably the most notorious example of mathematical cheating is the introduction of *imaginary numbers*. The name hardly inspires confidence, and the alternative designation of “complex numbers” is not much better.

What are imaginary numbers? The common and unsatisfying answer is that they are, for example, square roots of negative numbers, $\sqrt{-1}$ and $\sqrt{-2}$ and so forth. The eminently reasonable complaint is that a number multiplied by itself cannot be negative, and so these weird roots simply don’t make sense. The cheater’s rejoinder is that he doesn’t care whether or not such numbers exist; he’s just going to cheat and pretend that they do.

The cheater’s approach is to avoid asking what imaginary numbers *are*, or where we might find them, and instead focus upon what they *do*. That may seem intellectually dishonest but the approach is very common.

Let’s step back and consider an everyday square root, an apparently harmless fellow such as $\sqrt{5}$. Even here it is very difficult to say what the number actually is. Sure, we can write $\sqrt{5} = 2.36067\dots$, or whatever. However, writing out a few decimals followed by some dots is doing no more than giving the illusion of precision and understanding. It is extraordinarily difficult, however, to make sense of infinite decimals. Indeed, even apparently simple repeating decimals, such as $0.99999\dots$, can be very tricky.²

¹A K Peters, 2006.

²See Chapter 4.

Nonetheless, if we cannot say precisely what $\sqrt{5}$ is, we still know exactly how it works. The whole point of “root” is that it’s the reverse process of “squaring”. So, for example $\sqrt{4} = 2$ because $2 \times 2 = 4$. Similarly, for our troublesome $\sqrt{5}$ fellow, the one thing we can be sure of is

$$\sqrt{5} \times \sqrt{5} = 5.$$

Well, in for a penny, in for a pound. We may well not believe that $\sqrt{-2}$ exists. If however, we shut out our protesting brain and just pretend that it does, that there is something meaningful there, then the one equation we will accept is

$$\sqrt{-2} \times \sqrt{-2} = -2.$$

That’s all good fun but it also appears to be tautological. Is there anything substantial to be gained? Amazingly, there is.

To illustrate, consider the following sequence of numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, \dots$$

These are the megafamous *Fibonacci numbers*. They’re not actually due to Fibonacci although, as we’ve discussed elsewhere, there are other good reasons to honor him.³ So, we begin with 1 and 1, then $2 = 1 + 1$, $3 = 1 + 2$, and so on, each Fibonacci number being the sum of the two previous ones.

That’s all well and good, but what if we want the 1000th Fibonacci number? Sure, we can churn the numbers out one by one: the 9th Fibonacci number is 34, the 10th is 55, and on and on. We’ll eventually get there. But what we want is a formula that will simply provide us with the answer immediately. That’s where the magic begins.

Fibonacci wrote down his famous sequence in 1202. The magic formula, now known as *Binet’s formula*, came over 500 years later. Discovered by French mathematician Abraham de Moivre,⁴ Binet’s formula gives the 1000th Fibonacci number as

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{1000} - \left(\frac{1-\sqrt{5}}{2}\right)^{1000}}{\sqrt{5}}.$$

It is hard to exaggerate the amazingness of Binet’s formula. Clearly the Fibonacci numbers will all be whole numbers. Nonetheless, Binet’s formula enters the world of irrational numbers and then exits again. All we need use, and all we can use, is that $\sqrt{5} \times \sqrt{5} = 5$, and all the irrational roots magically cancel out.

(As an interesting side point, notice the golden ratio $(1 + \sqrt{5})/2$ appearing in Binet’s formula. It is pleasing to see the golden ratio taking time away from the tawdry business of selling cars to perform an honest day’s mathematical work.⁵)

We can perform exactly the same magic with imaginary numbers. To illustrate, let’s consider another sequence:

$$1, -1, -5, -7, 1, 23, 43, \dots$$

³We wrote a column based upon Keith Devlin’s excellent book, *The Man of Numbers*. Devlin focusses on the importance of Fibonacci’s work to the rebirth of mathematics in Europe.

⁴Binet’s formula is one of the *many* examples of a formula or theorem being named after someone other than the discoverer. See Chapter 58.

⁵See Chapter 59 of *A Dingo Ate My Math Book*. In this chapter we hammered the use of the golden ratio as a sales gimmick.

Known for historical reasons as the *Marty numbers*,⁶ they are deservedly much less famous than Fibonacci's sequence. They are constructed, however, in a very similar manner.

For the Marty numbers, we begin with 1 and -1 . Then, any subsequent Marty number is twice the previous number minus three times the one before that. So, the third Marty number is $(2 \times -1) - (3 \times 1) = -5$. The 8th Marty number would be $(2 \times 43) - (3 \times 23) = 17$, and so on.

And now the question: what if we want the 1000th Marty number? Here it is:

$$\frac{(1 + \sqrt{-2})^{1000} + (1 - \sqrt{-2})^{1000}}{2}.$$

And, see where we've ended up. Whereas the Fibonacci numbers required us to consider the irrational, the Marty numbers have led us all the way into the imaginary world. Once again, all the roots magically cancel out, giving an ordinary everyday whole number, and to accomplish this all we need is to apply the equation $\sqrt{-2} \times \sqrt{-2} = -2$.

Not that the Marty numbers are particularly interesting; it is only your Maths Masters and a few of their mates who have ever bothered with them. But the Marty numbers enable a simple and historically faithful demonstration of how to cheat with imaginary numbers, how to extract a real, workable answer from these weird, semi-real creatures.

The practice of employing imaginary numbers to solve problems about everyday numbers has a proud and puzzling history. It began in the 1500s, when Scipione del Ferro and other Italian mathematicians discovered a formula for the solutions to cubic equations, the higher degree counterpart to quadratic equations. Their formula always worked, but sometimes in a perplexing manner: even if the solutions of a polynomial were everyday numbers, the formula might express these solutions in terms of imaginary numbers; just as happened with the Marty numbers. Del Ferro and his colleagues had no idea what these imaginary numbers were or why they were necessary. They just knew that they worked.

Cheating with imaginary numbers continued for centuries. The finest mathematicians, including the great Leonhard Euler, became masters at manipulating imaginary numbers without ever knowing what these numbers were or whether they existed. There is a sense in which *Euler's formula*, the gem of imaginary numbers, was never properly understood by Euler.⁷

Finally, in 1799 the Norwegian mathematician Caspar Wessell explained it all. Wessell provided a convincing explanation of what imaginary numbers actually are, making them as tangible, as "real", as everyday numbers. (They are arguably even more real: it was another 50 years before "harmless" irrationals such as $\sqrt{5}$ were satisfactorily explained.)

So, did the cheating end? Yes and no. By the end of 19th century mathematicians were much clearer on the rules of the game. It was no longer permitted to blithely concoct new numbers without a solid sense of what these new numbers were. Nonetheless, new numbers and whole new mathematical worlds were, and are, still being concocted.

⁶Patent pending.

⁷Euler's formula states that $e^{\pi i} + 1 = 0$, where i stands for $\sqrt{-1}$ and e is *not* "Euler's number": see the next Chapter.

Mathematicians continue to just make things up, just as they always have. There is a famous quote by mathematician Leopold Kronecker:

God made the integers; all else is the work of Man.

Indeed, Kronecker doesn't go far enough. Nobel prize winning physicist Percy William Bridgman responded to Kronecker's quotation:

Nature does not count nor do integers occur in nature. Man made them all, integers and all the rest.

Bridgman was correct. All mathematics is cheating. It's all a fiction.

Puzzle to ponder

Write $\phi = \frac{1+\sqrt{5}}{2}$ for the *golden ratio*, and write $\mu = 1 + \sqrt{-2}$ for the (just now coined) *Marty ratio*. Prove that

$$\phi^2 = \phi + 1$$

and

$$\mu^2 = 2\mu - 3.$$

(These equations imply that the sequences $1, \phi, \phi^2, \phi^3, \dots$ and $1, \mu, \mu^2, \mu^3, \dots$ follow, respectively, the same pattern as the Fibonacci sequence and the Marty sequence. That, in turn, is the key to proving Binet's formula and (the just now coined) Marty's formula.)

CHAPTER 7

There's no e in Euler



Recently, while admiring Melbourne's gorgeous display of overhead wires,¹ we had cause to mention the number e . We also remarked that common references to e as "Euler's number" were inaccurate. (And, no, that's *not* the great Swiss mathematician pictured above). Some of our readers queried that claim. We'll now reply, taking the opportunity to tell a small part of the story of e .

Though extremely important, e is not the most inviting of numbers. Unlike π , the number cannot be illustrated or motivated or explained by very simple geometry: we cannot simply point to a circle or similar and exclaim "Look, there's e !"

This difficulty of e is exemplified in the Victorian curriculum. The curriculum exhibits no concern for what e is, or why it is what it is. It also appears that the forthcoming Australian curriculum is inclined to do little more. Although e requires some effort, however, it is not nearly as difficult a number as is suggested by this dereliction of duty.

The historical origins of e are clouded by the mists of time, but it seems likely that the number first arose as the result of financial considerations. It is still the easiest way to get a grasp of the number.

To begin, imagine we come across a very generous bank, Bank Simple, offering 100% annual interest. (Yes, this is a fantasy.) So, if we invested \$1 then after a year our dollar would have doubled to \$2.

¹See Chapter 34.

That is a very good offer, but then we find a second bank, Bank Compound, with a different scheme: they will give us 50% interest every six months. Would we prefer to invest with Bank Compound? Definitely.

After six months at Bank Compound, we will have 50% on top of our original \$1, amounting to \$1.50; in effect, we've multiplied by 1.5. Then, after the next six months, we will have earned an additional 50% of that \$1.50: we again multiply by 1.5, to arrive at the year's total of \$2.25.

This is illustrating the familiar and important notion of *compound interest*. The point is that calculating smaller interest at correspondingly smaller time intervals means that we are obtaining interest on our interest, resulting in a greater overall return on our investment. What does this have to do with e ? We're getting there.

Imagine we've found a third bank, Bank Super Compound, which returns 25% interest every three months. We would then obtain a 25% increase in our investment, compounded four times over the year. So, starting again with our faithful \$1, at the end of the year we would have $(1 + 1/4)^4 = \$2.44$.

We can keep going. We locate Bank Super Duper Compound, which calculates the interest every month, contributing an extra $1/12$ on top of our investment on each occasion. The result is, at the end of the year our \$1 would have grown to $(1 + 1/12)^{12} = \$2.61$.

Finally, we come across Bank Infinity, which goes the whole hog. This last bank divides the year into a zillion nanoseconds and then calculates the appropriately tiny amount of interest at each nanosecond. The result is, at the end of the year our \$1 will have returned $\$(1 + 1/\text{zillion})^{\text{zillion}}$. (To be precise, Bank Infinity calculates the *limit* of this quantity as the number of time intervals goes beyond a zillion and tends to infinity).

What is the amount returned by Bank infinity, what is this final number? It is the number we now denote by e . It is the result of compounding interest to the theoretical limit, what is known as *continuously compounded interest*. For those who love decimals, or calculating their interest *really* precisely, the expansion of this special number begins

$$2.7182818284590452353602874713526624977572470936999 \dots$$

Do those final 9s indicate that the number is actually a terminating decimal? No: we've simply been cheeky in choosing where to stop. As is π , our new special number is irrational.

But what does any of this have to do with Leonhard Euler? Nothing.

The earliest known appearance of the number is an indirect reference in a 1618 work, probably by the English mathematician William Oughtred: it is Oughtred's portrait that we have included above. Our new number was then used throughout the 17th Century. Around 1690, the great German mathematician Gottfried Leibniz explicitly denoted this number by the letter b . That was seventeen years before Euler's birth, in 1707.

There is a second part to the story, however, which does involve Euler. Though we have indicated how Oughtred's number naturally arises as a financial concept, this only begins to explain the central, critical role that the number plays in calculus. This was demonstrated by Sir Isaac Newton and the other great 17th Century mathematicians.

This work on calculus was then carried to brilliant extremes by, among others, Leonhard Euler. And, along the way, Euler chose to denote Oughtred's number by the letter e , a labeling that has endured. There is absolutely no evidence, however, that Euler chose the letter e to refer to himself, or for any particular reason other than that the letter was relatively unencumbered by uses in other contexts.

Now, exactly how does e claim such a central role in calculus? And how, if at all, does e capture the notion of exponential growth? This is yet another issue ducked by our curricula. The question is simply handballed to the universities, which then typically drop the ball. Someday we'll get to that, too.²

Puzzle to ponder

Show that any compounding bank like the ones above will return at least \$2 by the end of the first year. That is, show that the quantity $(1 + 1/N)^N$ is always at least 2. (Since Bank Infinity is the "limit" of what these banks offer, this also proves $e \geq 2$. See the puzzle solution for a quick discussion on how to similarly go about proving $e \leq 3$.)

²Burkard's *Mathologer* YouTube channel has a video devoted to everything about e .

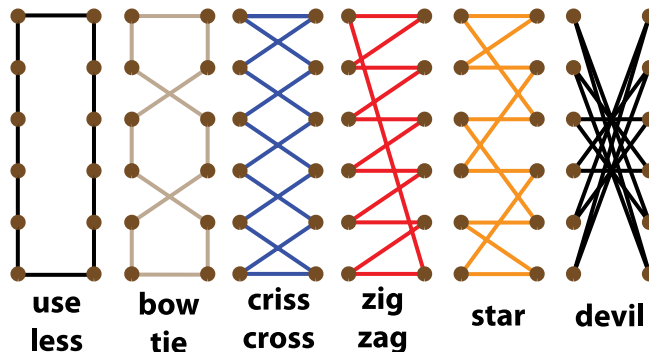
CHAPTER 8

What’s the best way to lace your shoes?



Recently, we (possibly) figured out a shortest tour of Victoria.¹ Today we’ll take you on some different tours, much closer to home.

When you lace and tie one of your shoes, the shoelace takes a tour of the eyelets. Two of the most popular such tours are the crisscross and zigzag lacings. However, there are many other possible lacings.



Which among these tours give the best lacing? To answer this we must first decide what we mean by “best”. The simplest notion to capture mathematically is “best = shortest”. So we’ll begin by considering the lengths of lacings.

It is quite obvious that the lacing on the left is the shortest, while also being close to useless. We can preclude such short but silly lacings by requiring that

¹See Chapter 44 of *A Dingo Ate My Math Book* where we used the idea of a trip around Victoria to discuss the traveling salesman problem.

each eyelet actually contributes to pulling the two sides of the shoe together. This amounts to having one or both segments that end in the eyelet connecting to an eyelet on the other side of the shoe. This is the case for all but the first lacing pictured above.

It turns out that, no matter the dimensions of the shoe and no matter the number of eyelets, the shortest *useful lacing* will always be the bowtie. Moreover, the lacings shown above are always in that order, from shortest to longest.

These are actually very surprising statements. For example, for a shoe with six pairs of eyelets there are 3,758,400 different useful lacings to compare. For God's Shoes, with 100 pairs of eyelets, the number of different useful lacings has grown astronomically, to:

```
18609042680740081966219159618202966147674648634776613609422844999556517
44439776524407967069878507881476622415958837227535961285224560640376326
00625910253777564367173849849651318613992974488058381689718183963494277
31395000391555199195008915051014033339502366067550230464677872323986936
376533403322840432771072000000000000000000000000000000000000000000000000
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In most familiar lacings there are no vertical segments at all, with every segment connecting opposite sides of the shoe, which makes these *very useful lacings*. Among all such lacings, the crisscross is always the shortest possible, and the devil lacing shown on the right is the longest possible.

There is an alternative notion of “best lacing”, which is perhaps more natural. We can view a lacing as a pulley system, pulling the two sides of a shoe together. We can then compare the strength of different lacings. Here there is no clear winner: depending upon the dimensions of the shoe, either the crisscross or the zigzag is the strongest. For dimensions close to those of real shoes – although who cares about those? – these two lacings are about equally strong.

So it turns out that the very popular crisscross scores well in terms of both length and strength. In addition, it is easy to remember, symmetric and pretty. It is fair to conclude that crisscross is indeed the best way to lace your shoes.

Finally, we should mention that the world's two leading shoelace experts reside in Melbourne: your writer here, Maths Master Burkard, who did the math for an article in the journal *Nature* in 2002;² and Ian Fieggen, who knows absolutely everything there is to know about shoelaces. You *must* check out Ian's amazing website.³

Puzzle to ponder

Count the number of useful and very useful lacings with 2×3 eyelets.

²*Nature*, **420**, 476. And, the real shoe fanatics can check Burkard's *The Shoelace Book*, American Mathematical Society, 2006.

³At the time of writing, Ian's website is still active.

CHAPTER 9

Ringling the changes

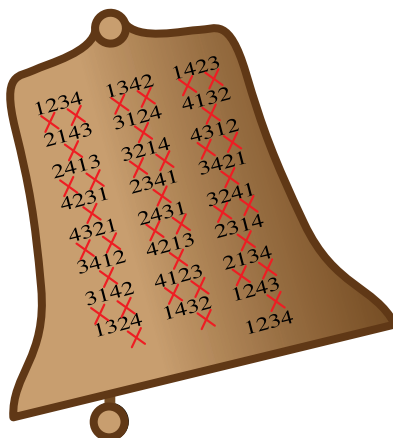
Do you like your math in exotic locations? Then why not join the band of bell ringers at Melbourne's St. Paul's Cathedral for the ringing of the changes. What does bell ringing have to do with math? A lot! We'll explain.

St. Paul's – the church opposite Federation Square – has twelve main bells, tuned in the key of C# major. The smallest bell, no. 1, rings the highest note, with the largest bell, no. 12, ringing the lowest. Changes can be rung using any number of bells. So, let's simplify things, and just use bells 1, 2, 3, and 4.

A *change* is what mathematicians call a *permutation*, the ringing of each of the four bells exactly once. For example, 3214 refers to the change of ringing bell 3, then bell 2, then bell 1, and finally bell 4. Then, *ringing the changes* means to ring a sequence of changes, whilst obeying three mathematical rules:

- First, the sequence starts and ends with the change 1234;
- Second, except for the start and end, no change is repeated;
- Third. from one change to the next, any bell can move by at most one position in its order of ringing. For example, this third rule says that the change 3214 cannot be followed by 2134, since bell 3 would have shifted by two spots.

One possible sequence of changes, known as *Plain Bob*, goes like this:



Here we first move down the first column, then the second and finally the third. Have a close look and you can see the third rule in action – from one change to the next a bell will either stay in the same position or swap its place with a neighboring bell. In the diagram, these swaps are indicated by crosses between the two changes.

Here are a few facts to set things into perspective. To ring one change takes between 1.5 to 2.5 seconds, the time it takes a bell to complete a natural swing.

When bell ringers go wild, they will ring sequences of changes consisting of more than 5000 changes, which translates into several hours of amusement for the neighbors. When ringing the changes, tradition dictates that bell ringers are not allowed any memory aids such as sheet music, nor can they be relieved by another bell ringer (for example, to relieve themselves). So, this means that a bell ringer has to effectively recite a sequence of several thousand numbers, one every two seconds, and to translate this sequence into perfect bell ringing. It takes a bell ringer several months to master ringing a bell by themselves, and years before they can dream of performing this kind of marathon bell ringing as a member of a team.

When ringing bells, one of the grand aims is to ring a sequence that includes every possible change. In the case of four bells such a sequence must be $(1 \times 2 \times 3 \times 4) + 1 = 25$ changes long, and Plain Bob is such an example. The general formula for n bells tells us that $(1 \times 2 \times 3 \times \cdots \times n) + 1$ changes are required.

Mathematicians have only recently proved that, no matter how many bells we want to ring, it is always possible to compose a complete ringing sequence.¹ Here is a table that shows you the numbers of changes we're talking about, and how long it would take if you rang at a rate of two seconds per change.

3	Singles	7	13 seconds
4	Minimus	25	49 seconds
5	Doubles	121	4 minutes
6	Minor	721	24 minutes
7	Triples	5,041	2 hours 48 minutes
8	Major	40,321	22 hours 24 minutes
9	Caters	362,881	8 days 10 hours
10	Royal	3,628,801	84 days
11	Cinques	39,916,801	2 years 194 days
12	Maximus	479,001,601	30 years 138 days

To ring a complete sequence on eight bells is the most that seems humanly possible. In recorded history, such a sequence has been rung on church tower bells only once. This took place at the Loughborough Bell Foundry in the U.K., beginning at 6.52 a.m. on 27 July 1963 and ending at 12.50 a.m. 28th July after 17 hours 58 minutes of continuous ringing. (Did they really not go to the toilet for 18 hours?!) Of course, to actually do this is ridiculously hard, both physically and mentally. So how do they do it? One mental trick is to make up sequences that are as easy as possible to remember. If you have a close look at Plain Bob, you can see that each column is generated from the change at the top by a simple knitting pattern of swaps. Then, at the end of the first and second columns, you swap the

¹Arthur T. White, *Ringing the changes II*, *Ars Combin*, **20 A**, 65–75, 1985.

last two bells and in this way link the three columns together. Based on this simple algorithm, it is very easy to reconstruct the whole sequence from scratch. Try it!

Permutations and collections of permutations play very important roles in many branches of mathematics. In particular, *group theory*, the branch of mathematics concerned with symmetries, is full of permutations. The earliest examples of group theory structures and techniques in action are the highly structured bell-ringing sequences that were developed in the early 17th century. For example, the first column of our Plain Bob sequence captures the eight symmetries of the square. The subdivision of Plain Bob into three columns is something that mathematicians also get excited about.

But why on Earth did anybody dream up this convoluted mathematical way of ringing the bells, instead of just playing tunes? And, how did they get away with doing so for centuries? After all, lots of people do complain about the noise, and together with the bell tower itself the bells do form a musical instrument that you could in theory play tunes on by striking the bells with hammers.

The problem is that if you want to ring the bells by swinging them, which sounds a lot more impressive, and carries a lot further, ringing tunes is not an option. Why? The reason is that we are talking about very large bells, up to 1.5 tons in the case of St Paul's. Once set in motion it is very hard for a bell ringer to vary the interval at which such a monster bell will ring. This is the mechanical constraint that explains the third rule of bell ringing, and motivated bell ringers to invent mathematically perfect bell ringing.

Originally, change ringing was a competitive sport, with bands of ringers of footy player physique and mindset, competing against each other on a regular basis. Often at odds with the church itself, the *exercise*, as it is traditionally called, had as much do with shouting each other drinks according to very intricate penalty system for mess-ups, as it did with ringing the bells.² Bell ringing has come a long way and mellowed a bit since its bloodsport origins.³ However, if you are interested in math, beer and serious mind games, you must visit one of six bell towers in Melbourne where bell ringing is still practised.

Puzzle to ponder

List all complete ringing sequences for three bells.

²In Australia, buying a friend an alcoholic drink is referred to as shouting.

³For more information, look up the Australian and New Zealand Association of Bellringers.