Preface

Mathematics is often seen as a collection of subjects, each learned one at a time. Although those fields rarely overlap, they are all subsumed under the heading of mathematics as they all rely on rigorous logical deductions from a set of axioms.

But in fact, there are numerous overlaps and interconnections between fields. And often the most interesting mathematics comes about when connections are made across the subdisciplines.

Tiling theory happens to be an area where many of the subfields of mathematics overlap. Tools can be applied from linear algebra, algebra, analysis, geometry, topology, and combinatorics. In this book, all of these fields make an appearance. Tiling theory provides an opportunity to demonstrate the interconnectedness of mathematics.

Tiling theory also has the ingredients to get across the beauty of mathematics:

i. There are beautiful pictures.
ii. Open problems can be stated without having to spend inordinate amounts of time explaining background.
iii. There is deep mathematics that applies to the field.

This book assumes an introductory background in linear algebra, analysis and/or topology and algebra, but readers missing one or more of these topics can also get a lot from this book by either avoiding certain sections or filling in the missing background as they go along. For instance, linear algebra is only used for affinities in Section 1.5. Analysis in the Euclidean plane is primarily used in Section 1.4 and 2.6. Topology appears primarily in the definition of a tiling and in Sections 4.4 and 4.5. Algebra mostly appears in Section 1.5. Even in these cases, the amount of material used is minimal and appears in a preliminary chapter (Chapter 0), which is devoted to a quick review of the relevant background. The reader can either start with that chapter or just read on and return to that chapter when the need arises.

I have used much of the material in this book to teach a semester-long course in tiling theory. I have taught it both as a lecture course and as a tutorial, where students in small groups present the material. There is substantially more material here than I could fit in a single semester. Any of the following sections can be omitted while still hitting the most important ideas of tiling theory:

2.3, 2.5, 2.6, 2.7, 2.9, 3.5, 3.6, 4.1, 4.2, 4.3, 4.4, 4.5

There is plenty of fascinating material in these sections, so look them over before deciding which to cut. For students with less background, proofs of any of the Extension Theorem (Theorem 2.9), the Periodicity Theorem (Theorem 2.14) and Theorem 2.18 can be omitted. The proofs of these three are more substantial than the other results proved in the book. Also, Section

2.9 on balanced tilings is more technical than most other sections and can easily be skipped.

There are a few caveats. This is not a book that proves everything from the axiomatic point of view. If there is a geometric fact that is obvious to everyone concerned and for which the reader could find an axiomatic proof, we do not include the proof. As an example, we do not prove that reflection in a line in the plane preserves all distances between points. This is evident enough.

Most of the time, our arguments are geometric. Although some results also have more algebraic proofs, tilings are about pictures. So, we primarily stick with that approach. But algebra also plays an important role in tiling theory and will also have its chance.

This book would not exist if not for the wonderful book “Tilings and Patterns", by Branko Grünbaum and G. C. Shephard. This is where I first learned an appreciation for tilings. Although the original edition is out of print, Dover has produced a paperback reprint. That book is the encyclopedia for the current state of affairs in tiling theory as of 1987. The new edition from 2016 is an unabridged copy of the original with the addition of an appendix that updates some of the topics covered. The reader should look to that book as a much more complete compendium of results in tiling theory, and certainly anyone interested in pursuing research in tiling theory should find a copy of that book, and read it front to back.

Many of the topics covered in this book are a subset of the topics covered in “Tilings and Patterns". That subset was chosen to be what I have found to be the most appealing to students, and what was already the most appealing to me. But there are some additional topics in this book that do not appear in “Tilings and Patterns”. Here we will go into substantially more detail on the isometries of the Euclidean plane. We will also cover random tilings, projections generating aperiodic tilings, the Taylor-Socolar tile and all of the topics covered in Chapter 4, including spherical tilings, hyperbolic tilings, 3-dimensional Euclidean tilings, knotted tilings, and tilings and 3-manifolds.

There are a few sections of this book that overlap more substantially with the corresponding sections of “Tilings and Patterns". In the notes at the end of the book, I have tried to be as explicit as possible about that.

The first appendix to the book provides methods for creating interesting tilings. This can be covered at the very beginning as it allows you to make your own tilings relevant to further topics covered in the book. It’s also a lot of fun.

Every section ends with at least ten exercises. These are a good way to solidify your understanding of the material. Starred exercises are more difficult.

Throughout the book, you will find projects. Any of these could be the basis for either a paper or a presentation. A list of all of them appears in the appendix. You will also find there a list of resources, including books, web sites and kits.

You will also find open questions throughout the book. It is important to realize how many simply stated questions still remain to consider. I have done my best to verify that the particular questions are still unanswered at the time of writing. But it is possible that some of these questions have been answered, either at the time I am writing this or since then.

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A special thanks to the authors, artists, editors and publishers who have given me permission to include figures from other sources. I am forever in their debt. The appendix includes a list of all credits for figures appearing in this book.

And a huge thanks to the reviewers of the earlier draft. The feedback I received was immensely helpful. A draft of the book was also used in senior-level tiling courses taught by David Futer at Temple University, Keiko Kawamura at the University of Iowa and May Mei at Denison University. A very special thanks to all three of them and their students, who gave me extremely useful feedback.

I also want to thank everyone at the American Mathematical Society who helped with the production of this book. In particular, Marcia Almeida helped to keep track of all of the necessary figure credits, a daunting task, Erin Donahue did a great job with the copyediting, and John Brady worked with me to create the cover design. My editor at AMS, Ina Mette, has been supportive of me and my various mathematical projects for years. I owe her a tremendous debt.

Thanks are also owed to all of the students to whom I have taught tiling theory. This book could not exist without their input. I learned from all of them. A special thanks to Cameron Edgar, Peter Hollander and Liza Jacoby, with whom I worked for nine months cataloging the possible non-edge-to-edge tilings of the sphere by regular spherical polygons. And thanks to Eli Miller and John Petrucci, who produced many versions of the figures. They were a huge help!

I hope you enjoy reading the book as much as I enjoyed writing it.

-Colin Adams