Introduction

Tilings appear early in the history of civilization. As soon as dwellings were constructed, the choice of stones for the walls and for the floor determined tilings. As building technology improved, those stones became tiles, and those tiles became decorative, coming in various colors and patterns.

Numerous examples can be found in the wonderful tradition of sophisticated tiling work that appears in Islamic architecture. In Figure A, we see a ceramic wall pattern from Marrakech, Morocco with a variety of tilings appearing in the horizontal strips.

Figure A: A Marrakech design.

Tilings have continued to play an important role in structures up to the present day. In Figure B, we see the floor tiling in the Archaeological Museum in Seville, Spain, constructed from regular triangles, squares and hexagons.

Figure B: The floor of the Archaeological Museum in Seville, Spain.

Figure C shows a tiling by tiles with curved edges, from the Water Grill restaurant in Southern California.

Figure C: A tiling from the Water Grill restaurant in Southern California.

In Figure D, we see three styles of tiling found on sidewalks.

Figure D: Some sidewalk tilings.

Figure E shows a tiling from the floor of a Banana Republic store and a tiling on wallpaper.

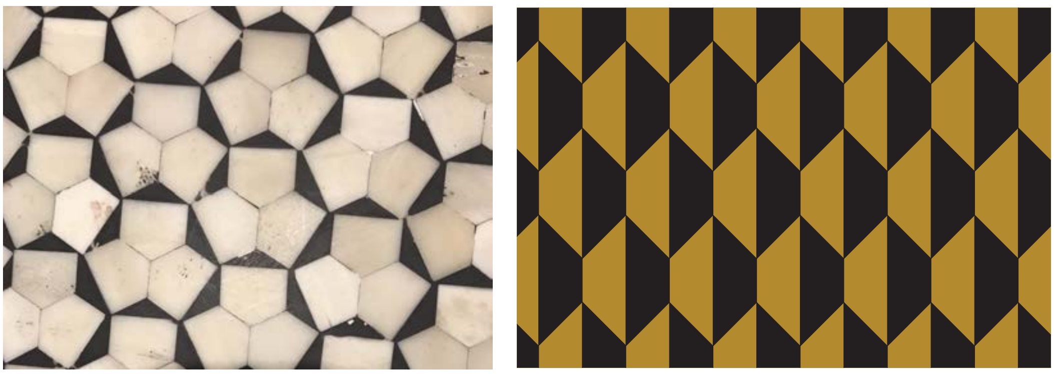


Figure E: A floor tiling and a wallpaper tiling.

In Figure F, we see several models of possible floor tilings created by the artist Anton Bakker, using the Conway Criterion, which we discuss in Section 1.58.

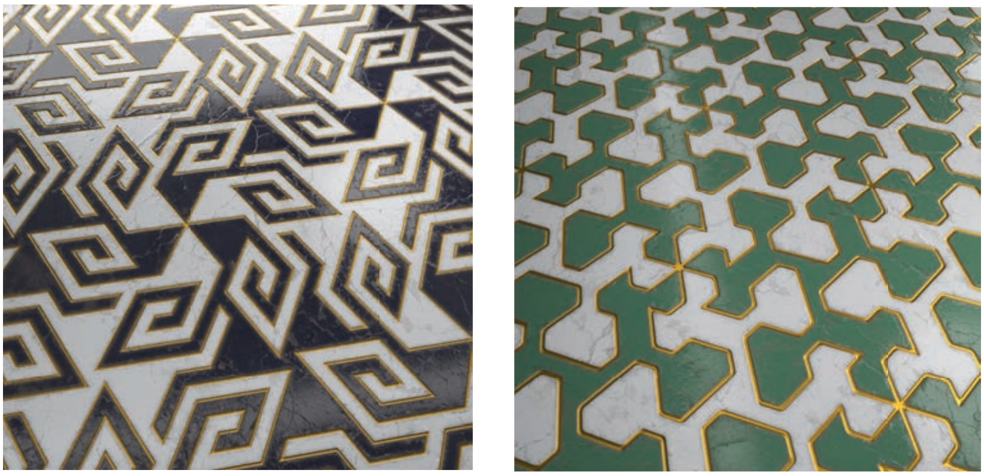


Figure F: Two tilings by Anton Bakker.

Figure G is a view of the tiling on the sides of Ravensbourne College of Design and Communication in London.

Figure G: Tiling on the Ravensbourne College of Design and Communication.

In Figures H, I, and J we see various tilings. Although all of these tilings have a certain translational symmetry to their patterns, we can also find tilings (as in Figure K) that form a spiral-like pattern and have no such symmetry.

Tilings play a key role in design, in art, in architecture, and also in many areas of science. Moreover, tiling theory has an important place in mathematics.

This book is meant as an introduction to the mathematics of tiling. In the classes I have taught on the subject, I have been lucky to have students who took the course because of their interest in any of the topics in which tilings appear. It is a wonderful subject for generating excitement in that it combines beautiful pictures with fascinating mathematics. And it has applications in the real world.

Chapter 1 begins with a careful definition of what constitutes a tiling. Then we change

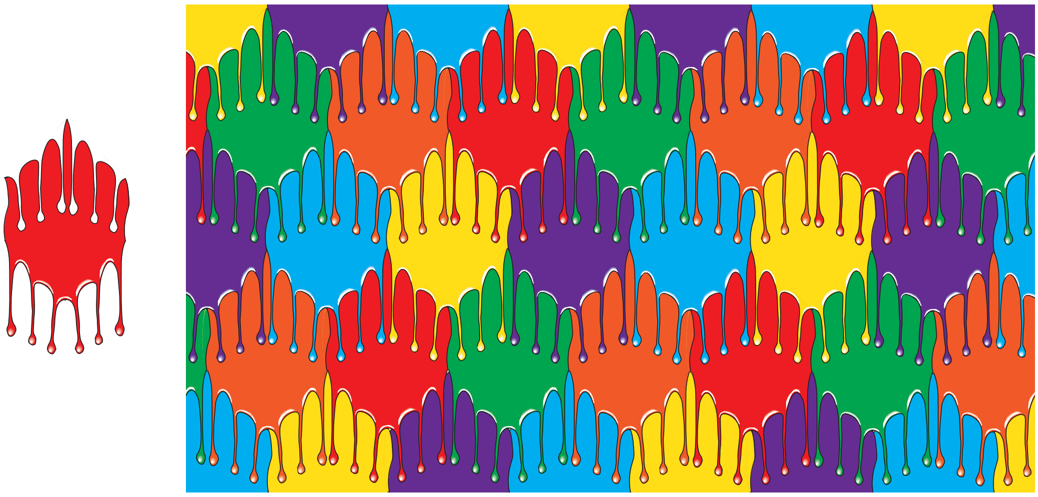


Figure H: A tile and the unique tiling that results from it.

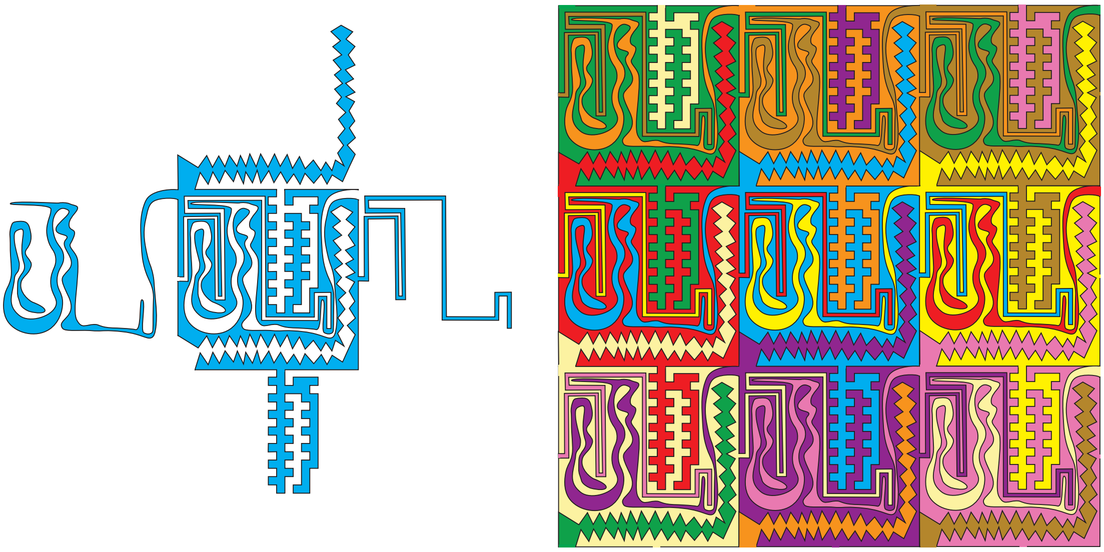


Figure I: Another tile and the unique tiling it generates.

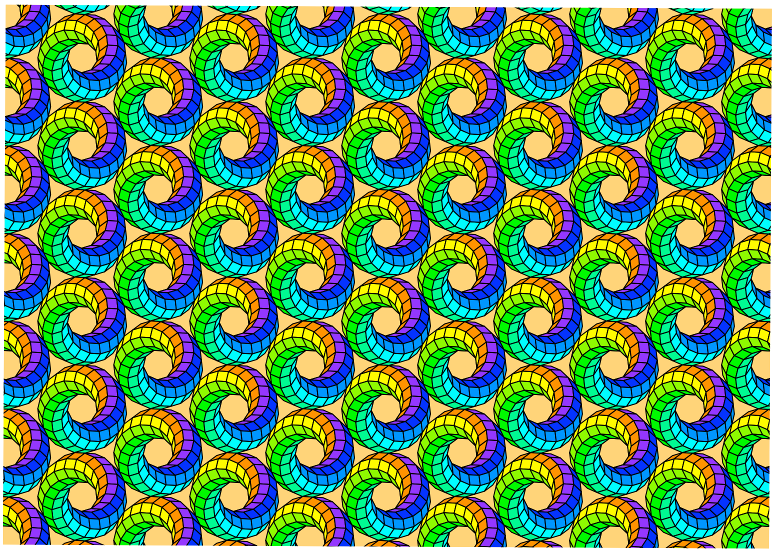


Figure J: A "toroidal" tiling.

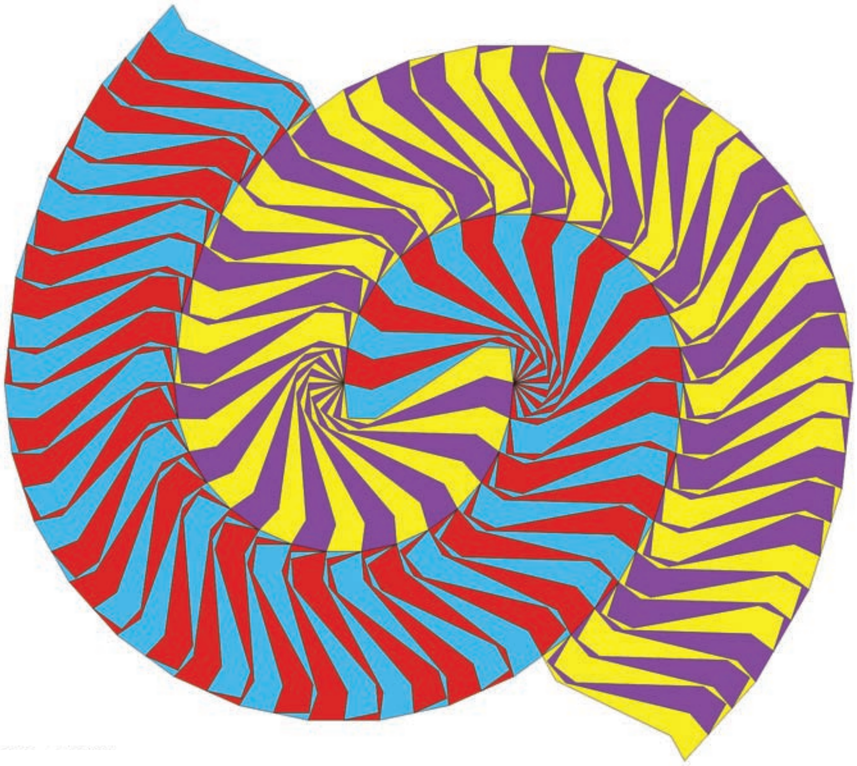


Figure K: A spiral tiling.

direction and spend some time discussing isometries of the Euclidean plane and the isometries that are symmetries, first of tiles and then of tilings. This allows us to define the symmetry groups of tilings and discuss their classification. The chapter ends with a discussion of how many different tilings a given set of tiles can generate.

In Chapter 2, we talk about particular types of tilings, both in terms of particularly nice and particularly interesting classes. But we also consider how to generate tilings and the question of which choices of tiles can generate tilings.

Chapter 3 introduces the idea of aperiodic protosets, which are finite sets of tiles that generate only tilings with no translational symmetries. When they were discovered, they upended many previous assumptions about tilings. These results have had a dramatic impact on materials science; helping, as they have, to explain quasicrystals.

In Chapter 4, we consider tilings of spaces other than the Euclidean plane. We look at tilings of the sphere, the hyperbolic plane and Euclidean 3-space. Once in Euclidean 3-space, we liberalize the definition of a tiling to consider "knotted tilings" as well. And then we look at how tilings are related to surfaces and 3-manifolds and questions about the possibilities for the spatial universe in which we live.

I hope you enjoy having the opportunity to explore this fascinating area of mathematics. There are pretty pictures and beautiful mathematics. What more could you ask?

Chapter 1 Introduction to Tiling

1.1 What is a Tiling?

Although we will consider other spaces in Chapter 4, we are initially interested in tilings of the Euclidean plane $\mathbb{E}^2$. Everyone has an intuition for what a tiling should be. It is a collection of regions in the plane that together cover the entire plane and that do not overlap other than on their boundaries. But in fact that leaves a lot of leeway. Do we allow our tiles to be disconnected? Do we allow holes in our tiles? Do we allow our tiles to have whiskers? (See Figure 1.1.)

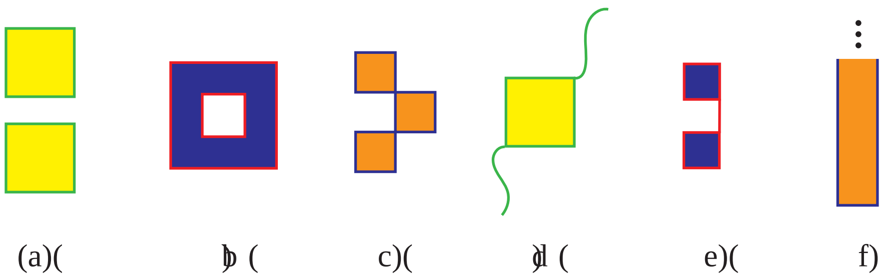


Figure 1.1: Sets we do not allow as tiles.

Although there has been research into these other possibilities, for our purposes, we would like to avoid them. Hence, we have the following definitions.

Definition 1.1

Let $\mathcal{C}$ be a collection of subsets of the plane. We say that $\mathcal{C}$ is a **covering** *of $\mathbb{E}^2$ if $\mathbb{E}^2 = \bigcup_{C \in \mathcal{C}} C$. We say that $\mathcal{C}$ is a* **packing** *of $\mathbb{E}^2$ if any pair of the subsets in the collection $\mathcal{C}$ do not intersect in their interiors.*

In Figure 1.2, we see both a covering and a packing.

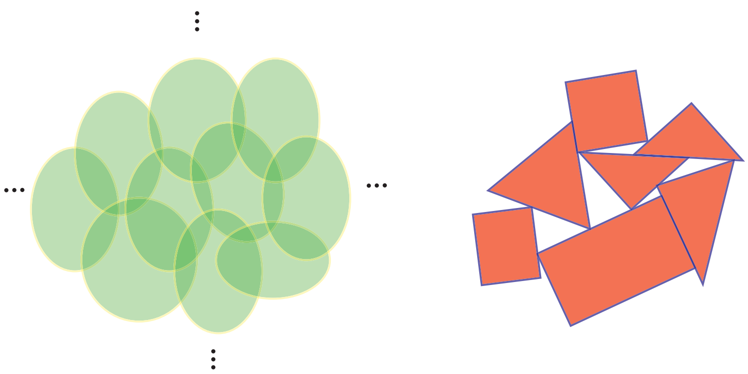


Figure 1.2: A covering and a packing.

Definition 1.2

*A **tiling** of the plane $\mathbb{E}^2$ with* **protoset** *$\mathcal{P}$ is a collection $\mathscr{T}$ of closed sets in the plane, called* **tiles** *and a finite collection $\mathcal{P}$ of closed sets in the plane, each called a* **prototile**, *such that the following hold:*

1. *Each prototile is topologically equivalent to a disk.*
2. *Each tile is congruent to a unique prototile.*
3. $\mathbb{E}^2 = \bigcup_{T \in \mathscr{T}} T$ *($\mathscr{T}$ is a covering.)*
4. $\mathring{T}_i \bigcap \mathring{T}_j = \phi$ *for all $i \neq j$ ($\mathscr{T}$ is a packing.)*

We say that a protoset **admits a tiling** *$\mathscr{T}$ if there is a tiling $\mathscr{T}$ of the plane corresponding to that protoset.*

For example, in Figure 1.3, we see a tiling with protoset consisting of an equilateral triangle, a square and a star polygon. Sometimes, you will see tilings called **tessellations** but we will call them tilings throughout this book.

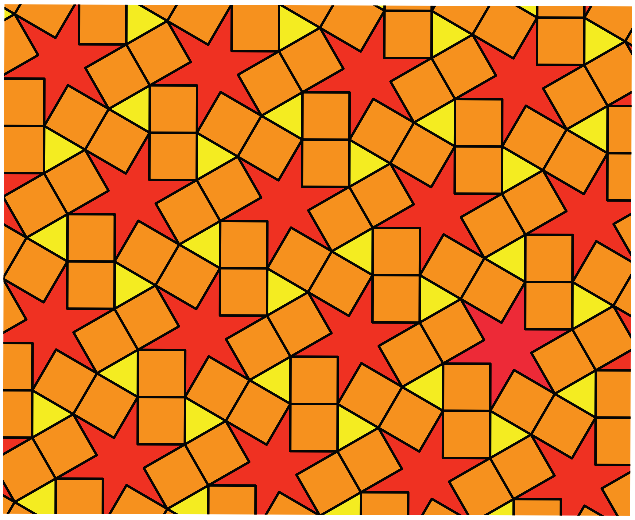


Figure 1.3: A tiling with three prototiles.

As we have already mentioned, Condition 1 of the definition is there to avoid certain pathological conditions. We do not want tiles to be disconnected. We do not want tiles with whiskers. We do not want tiles with holes. And we do not want tiles such that the removal of a finite number of points would disconnect the tile. In addition, we do not want tiles with zero area.

Condition 2 limits the number of shapes of tiles to the finite collection of prototiles, no two of which are congruent. Keep in mind that the congruence between a tile in a tiling and the corresponding prototile could include reflection.

One can also consider infinite sets of prototiles, but we will not go there. Conditions 3 and 4 ensure that the tiles cover the plane and that they only overlap in their boundaries.

We see that these conditions are satisfied for some of the standard tilings with which we are familiar. For instance, the tilings of the plane by squares, equilateral triangles and regular hexagons in Figure 1.4 certainly cover the plane such that the interiors of the tiles do not intersect. Any two tiles only intersect at most on their boundaries. And each tile is itself topologically a disk. We call these tilings the **regular tilings** of the plane, since for each, the one prototile is a regular polygon.

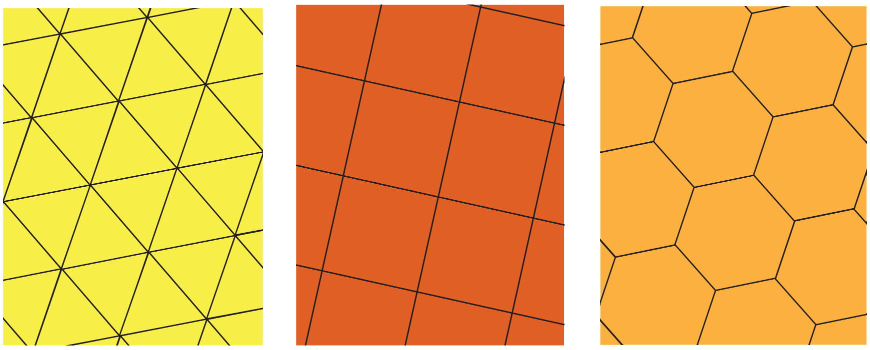


Figure 1.4: The regular tilings of the plane.

Definition 1.3

A tiling $\mathscr{T}$ is **monohedral** *if every tile in $\mathscr{T}$ is congruent to a single tile. Said another way, there is only one prototile in the protoset $\mathcal{P}$ generating $\mathscr{T}$. A tiling $\mathscr{T}$ is* **dihedral** *if there are two prototiles in the protoset $\mathcal{P}$ generating $\mathscr{T}$. More generally, a tiling $\mathscr{T}$ is* **n-hedral** *if there are a total of n prototiles in the protoset $\mathcal{P}$ generating $\mathscr{T}$.*

So for example, all three of the regular tilings are monohedral, whereas the tiling in Figure 1.3 is 3-hedral or trihedral. One of the fundamental problems in tiling theory is to determine, given a single tile, whether there is a monohedral tiling with that tile as protoset.

Definition 1.4

Given a tiling, a **vertex** *of the tiling is a point where three or more tiles intersect. An* **edge** *of the tiling is a connected subset of the boundary of a tile that contains no vertices in its interior and that is bounded by two vertices.*

Since a tile T is topologically equivalent to a disk, its boundary is topologically a circle. Hence, as in Figure 1.5, once we show that there are only a finite number of vertices on the boundary (see Lemma 1.5 below), the collection of those n vertices cuts the boundary into exactly n edges.

In order for the tiling to cover the plane, every point in the boundary of a tile must be in at least two tiles. Since those points that are in three or more tiles are vertices, the non-endpoint points in the edges must lie in exactly two tiles. So each edge is on the boundary of two tiles.

Note that if a tile in a tiling is a polygon, it need not be the case that every vertex of the polygon is a vertex of the tiling, as for example occurs for the tiling in Figure 1.6. So when referring to a polygonal tile, we refer to its **corners** and **sides** so as not to confuse them with the

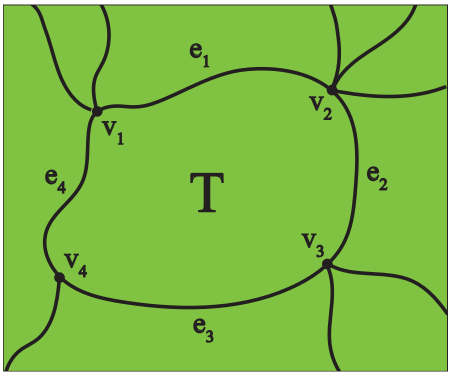


Figure 1.5: The n vertices on the boundary of a tile cut the boundary into exactly n edges.

vertices and edges coming from the tiling.

Definition 1.5

In the case of a tiling by polygons such that the corners and sides of the polygons coincide with the vertices and edges of the tiling, we say the tiling is **edge-to-edge**.

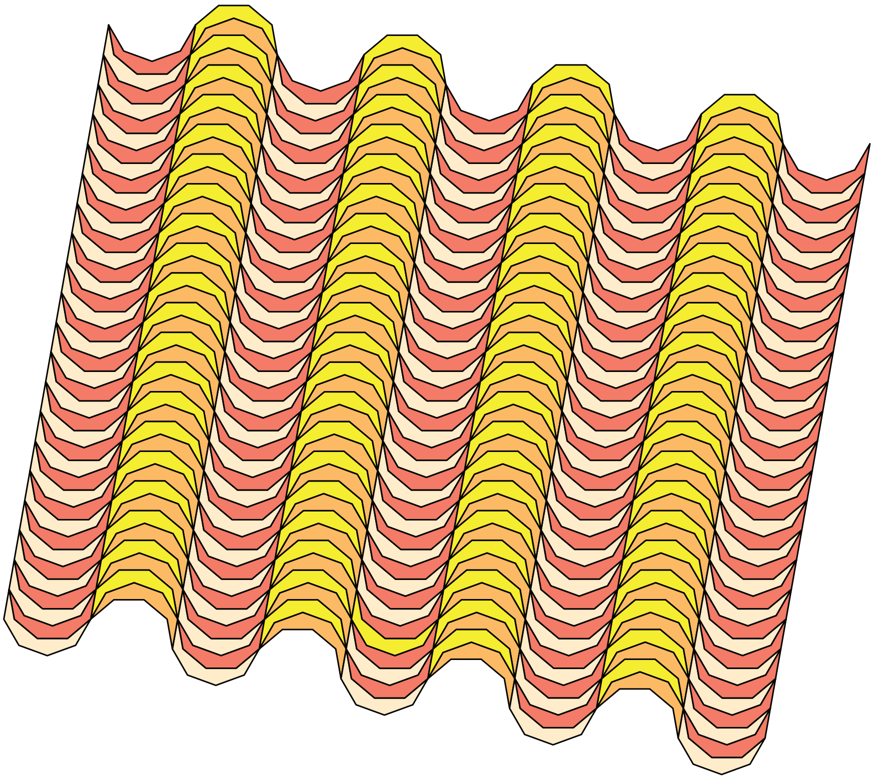


Figure 1.6: Corners need not coincide with vertices.

If we do not restrict to edge-to-edge tilings, even in as simple a case as monohedral tilings by squares, we can construct an uncountable infinity of distinct tilings. To do so, we slide the bottom half of the standard tiling left relative to the top half a distance x where x is any real number in the uncountable interval $[0, \frac{1}{2}]$, as in Figure 1.7. All of these tilings will be distinct.

However, in the exercises, you will prove that when we do restrict to monhedral edge-to-edge tilings by regular polygons, the three regular tilings are the only possibilities.

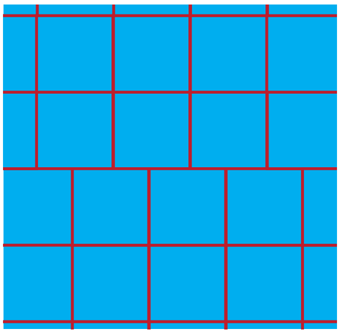


Figure 1.7: An uncountable infinity of monohedal tilings by squares.

Definition 1.6

The **valency (or degree)** *of a vertex is the number of edges that share it. A tiling is said to be j*-**valent** *if every vertex has valency j.*

So, the regular tilings by squares, triangles and hexagons are 4-valent, 6-valent, and 3-valent respectively.

Definition 1.7

Two tiles are **adjacent** *if they share an edge. Two tiles are* **neighbors** *if they intersect. The* **neighborhood** *of a tile T, denoted $\mathcal{N}(T)$, is the union of T with all of its neighbors.*

Definition 1.8

An edge is **incident with a vertex** *if that vertex is one of its endpoints. A tile is* **incident with an edge** *if that edge lies on the boundary of the tile.*

Note that if all the prototiles in the protoset are convex polygons, two tiles can only intersect in a single edge or single vertex. However, if the tiles are not all convex, this no longer holds, as for example occurs for the dihedral tiling in Figure 1.8.

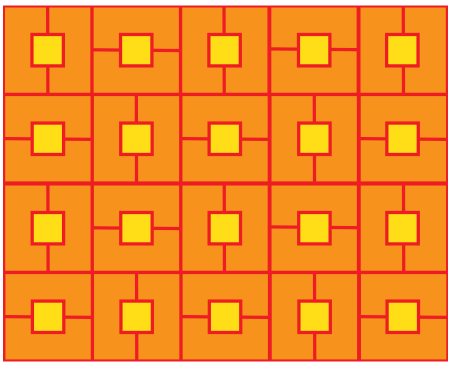


Figure 1.8: Nonconvex polygonal tiles can intersect each other in disconnected sets.

We conclude this section with several properties of tilings that follow from our definition.

Lemma 1.1

Any disk in the plane can intersect only finitely many tiles in a tiling.

Proof We define the **diameter of a tile** to be the greatest distance between any two points in the tile. Since there are only a finite number of prototiles in a protoset $\mathcal{P}$, there is an upper bound, call it $d_{\mathcal{P}}$, on the diameter of the prototiles and therefore on the diameter of every tile in the tiling.

There is also a lower bound, call it $\alpha_{\mathcal{P}}$, on the areas of the prototiles, since again, there are only finitely many of them. It is the immediate that $\alpha_{\mathcal{P}}$ is a lower bound on the area of every tile in the tiling.

Given a disk D in the plane of radius R, any tile that intersects D is completely contained in the disk D' with the same center and radius $R + d_{\mathcal{P}}$. Each tile contained entirely in D' has an area of at least $\alpha_{\mathcal{P}}$, and none of those areas overlap. But the total area of D' is $\pi(R + d_{\mathcal{P}})^2$, which is finite. So there is room inside D' for at most a finite number of tiles (in fact, fewer than $\pi(R + d_{\mathcal{P}})^2/\alpha_{\mathcal{P}}$). So D intersects only finitely many tiles. ■

Corollary 1.2

A tile T in a tiling can touch only a finite number of other tiles.

Proof We enclose T in a disk D of radius $d_{\mathcal{P}}$ centered at any point of T. Then any tile that touches T also touches D. But by Lemma 1.1, there are only finitely many tiles touching D, and hence only finitely many that can touch T. ■

To prove the next fact, we need a fundamental result from topology which we include here without proof. By a simple closed curve, we mean a non-self-intersecting loop.

Theorem 1.3. Jordan Curve Theorem

A simple closed curve in the plane separates the plane into an interior region and an exterior region.

In particular, this means that any path from a point in the interior region to a point in the exterior region must cross the curve. Although this result seems self-evident, the known proofs are highly nontrivial.

Lemma 1.4

Three tiles in a tiling cannot share three or more vertices.

Proof Suppose that T_1, T_2, and T_3 share vertices v_1, v_2, and v_3. Choose v_1, v_2, and v_3 as in Figure 1.9, so that v_2 is the intermediate shared vertex on the boundary of each tile. (We leave it as an exercise to prove there must be such a vertex.) Choose an arc through the interior of T_1 that ends at v_1 and v_2 and an arc through the interior of T_2 that also ends at ends at v_1 and v_2. Their union is a simple closed curve C_{12} that touches T_3 at most at the two points v_1 and v_2. Then $\mathring{T}_3$ is either to the interior or to the exterior of C_{12} by the Jordan Curve Theorem. If it is to the interior, then T_3 cannot touch v_3, a contradiction. So T_3 is to the exterior. Similarly, form a simple closed curve C_{23} out of an arc through the interior of T_1 that ends at v_2 and v_3 and an arc through the interior of T_2 that also ends at v_2 and v_3. Then in order to touch v_1, T_3 must be in the exterior of C_{23} as well. But then $\mathring{T}_3$ must be to the outside of $T_1 \cup T_2$ and the two regions bounded by C_{12} and C_{23}. But then T_3 cannot touch v_2, a contradiction. ■

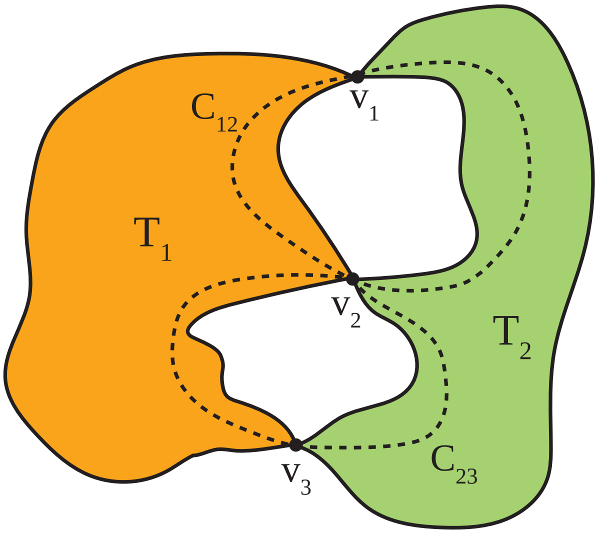


Figure 1.9: Two of three tiles that are sharing three vertices.

This next fact is something that we would certainly hope follows from our definition of tilings.

Lemma 1.5
A tile in a tiling has a finite number of vertices, and therefore a finite number of edges.

Proof Suppose not. That is, suppose there is a tile T_1 in a tiling $\mathscr{T}$ such that T_1 has infinitely many vertices. Then by Corollary 1.2, there is at least one tile T_2 that touches T_1 an infinite number of times in order to create the infinite number of vertices. (We are using the Pigeonhole Principle here).

In fact, another tile T_3 must also touch infinitely many of the vertices shared by T_1 and T_2 as every vertex in $T_1 \cap T_2$ must touch at least one other tile, and there are only finitely many tiles to go around. Hence we have three tiles that share three or more vertices, which in this case is infinitely many vertices, contradicting Lemma 1.4. ■

We also prove the following.

Lemma 1.6
The collection of tiles in a tiling $\mathscr{T}$ is a countable set.

Proof Consider the set $\mathbb{Q} \times \mathbb{Q}$ of all points with rational coordinates in the plane. As discussed in Section 0.2, this is a countable set. This is also a dense set in the plane, which is to say that every open disk in the plane contains such a point.

Then because each tile is topologically a disk, it has nonempty interior, which is an open set in the plane. So every tile contains at least one element from $\mathbb{Q} \times \mathbb{Q}$ in its interior.

For each tile in $\mathscr{T}$, choose a point from $\mathbb{Q} \times \mathbb{Q}$. This yields an injective map from the set of tiles in $\mathscr{T}$ into $\mathbb{Q} \times \mathbb{Q}$. Hence it maps the set of tiles bijectively to a subset of $\mathbb{Q} \times \mathbb{Q}$. But as in Exercise 3 of Chapter 0, a subset of a countable set is countable.

■

We need to consider several kinds of equivalency between tilings.

Definition 1.9

Tilings $\mathscr{T}_1$ and $\mathscr{T}_2$ are **congruent** *if there is an isometry of the plane that takes $\mathscr{T}_1$ to $\mathscr{T}_2$. The isometry could consist of a translation, a rotation and/or a reflection. The two tilings are* **equivalent** *if there is a scaling up or down of the plane (for example, think of the function $f(x, y) = (kx, ky)$) of $\mathscr{T}_1$ that makes it congruent to $\mathscr{T}_2$.*

For example, we can construct the regular hexagonal tiling out of hexagons of side-length 1, as on the right in Figure 1.4. If we rotate, translate, or reflect that tiling, the result is congruent to the original. If we construct the same tiling but out of hexagons of side-length 2, we obtain an equivalent tiling.

We need one more definition.

Definition 1.10

Given a tiling $\mathscr{T}$, we define a **patch** *to be a collection of tiles in $\mathscr{T}$, the union of which is a topological disk. Given a disk $D(p, R)$ centered at a point p with radius R, we define the patch corresponding to $D(p, R)$ to be the patch $\mathcal{A}(p, R)$ consisting of the union of all tiles that intersect the disk together with additional tiles that fill the holes that may be created when we take the union of those tiles, as in the orange tiles in Figure 1.10. For any subset S of $\mathscr{T}$, the patch $\mathcal{A}(S)$ corresponding to that subset consists of the union of all tiles that intersect S together with the tiles needed to fill any holes created by that union.*

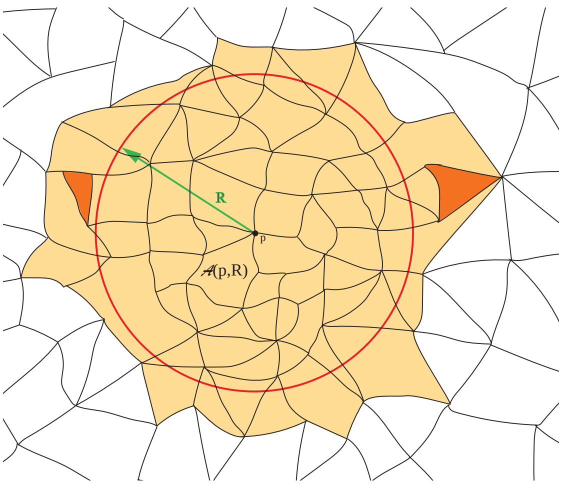


Figure 1.10: Defining the patch corresponding to a disk $D(p, R)$.

For example, as in Figure 1.11, fix a vertex p of the unit square tiling. For $R < 1$, $\mathcal{A}(p, R)$ is the union of the four tiles that contain p. For $1 \leq R < \sqrt{2}$, $\mathcal{A}(p, R)$ picks up two more tiles on each of the four edges of the previous patch. At $R = \sqrt{2}$, the corresponding patch picks up the four missing corner tiles.

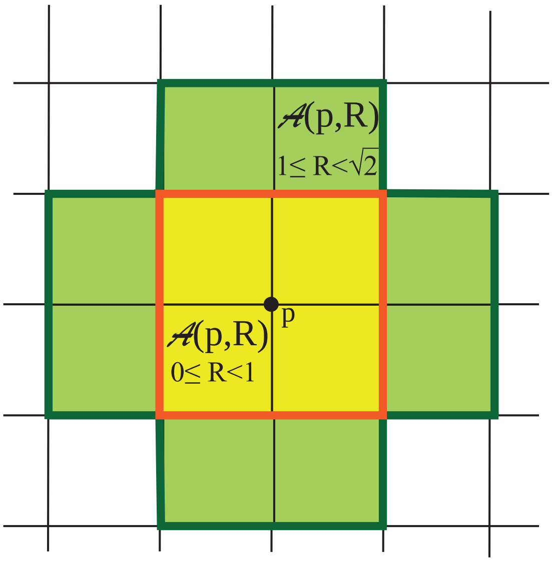


Figure 1.11: Patches in the unit square tiling.

A question that was open until recently asked whether for any tile T in a monohederal tiling, the neighborhood of T is equal to the patch of T. This was proved false by undergraduates working with Casey Mann by generating a counterexample from a variation on the Voderberg tile that appears in the spiral tiling in Figure K from the Introduction. (See Section 2.4 and Figure 2.41 for the details.) Patches will come up repeatedly in the subsequent sections.

Exercises for Section 1.1

1. Show that the set of tiles in each of the three regular tilings are in fact countable, by describing a method for identifying each consecutive positive integer with a specific tile in the tiling.
2. Prove that the only edge-to-edge regular tilings of the plane are the square, triangle, and hexagonal tilings. (Hint: Compute the angles on a regular polygon with n edges by first cutting it up into triangles with no new corners).
3. Show that there is an uncountable infinity of monohedral tilings, all of which are 3-valent.
4. Show that if the prototiles for a polygonal tiling are all convex, all corners occur at vertices.
5. Show that in a polygonal tiling, every vertex is a corner of at least two tiles.
6. Show that every tile in a tiling is incident to at least two vertices.
7. Find a monohedral polygonal tiling such that for every tile, two of its corners are not vertices and two of its vertices are not corners.
8. Find a tiling with protoset an equilateral unit triangle and a unit square such that all vertices have valency 5.
9. For all $n \geq 6$, find a polygonal tiling with all vertices of valency n. (Hint: Not all corners need to be vertices.)

10. Complete the proof of Lemma 1.4 by showing that if two tiles share three vertices, then one of the vertices is intermediate in the sense that there is no path from it out to ∞ that does not pass through at least one of the tiles after it leaves the vertex. (Hint: Use the Jordan Curve Theorem.)
11. Find the valencies of all of the vertices in the tilings in Figure 1.6. Find the number of edges in the boundary of a blue tile, a yellow tile, and a purple tile.
12. Find a tiling with two prototiles, one of which is an equilateral triangle, such that none of the corners of the triangle tiles are vertices of the tiling.
13. * Find an edge-to-edge monohedral tiling by triangles such that for any triangle T of the tiling, there is another triangle T' of the tiling such that $\mathcal{N}(T) = \mathcal{N}(T')$. That is to say, T and T' have exactly the same neighborhoods. (Hint: Think about what must be true at the vertices they do not share.)
14. For each of the regular tilings by equilateral triangles and regular hexagons with unit edge lengths, and p a vertex of each tiling, determine how many tiles are in $\mathcal{A}(p, R)$ for the various values of R such that $0 \leq R \leq 1$.
15. For the tiling in Figure 1.12, where the squares incident to p have edge length 1, draw the patch $\mathcal{A}(p, \sqrt{2})$.

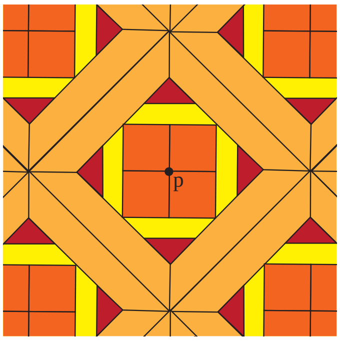


Figure 1.12: Find $\mathcal{A}(p, \sqrt{2})$.

16. Given a region R in the plane, define the patch $\mathcal{A}(R)$ to be the smallest patch that properly contains R. For a given tile T in a tiling $\mathscr{T}$, define $\mathcal{A}_0(T) = T$ and $\mathcal{A}_n(T) = \mathcal{A}(\mathcal{A}_{n-1}(T))$ for $n = 1, 2, \ldots$. For each of the three regular tilings that appear in Figure 1.4, find a polynomial formula in terms of n for a) the number of tiles b) the number of vertices and c) the number of edges in $\mathcal{A}_n(T)$. (Note that this problem has nine parts. You may find it helpful to use the fact $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.)

Chapter 2 Types of Tilings

In this chapter, we restrict ourselves to well-behaved tilings. By that we mean tilings with prototiles that are polygons, and with various other restrictions that limit the possibilities.

2.1 Uniform Tilings

We first consider the case of regular polygons, which is to say, polygons such that all edges have the same length and all angles are identical. We have already seen that if we restrict ourselves to edge-to-edge tilings with just a single regular polygon as prototile, we obtain only the three regular tilings by equilateral triangles, squares and regular hexagons as in Figure 1.4. But what happens if we allow more than one type of prototile? That is the subject of this section.

We denote a regular n-gon by {n}. Any regular n-gon $\{n\}$ can be cut into $n-2$ triangles by adding disjoint diagonals as in Figure 2.1.

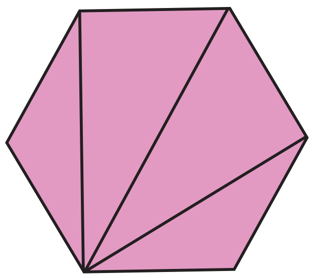


Figure 2.1: Cutting an n-gon into $n-2$ triangles.

Therefore, the sum of the angles at its corners must be equal to the sum of the angles of the $n-2$ triangles, which is $(n-2)\pi$. Since the angle at each of the n corners is the same, there must be an angle of $(n-2)\pi/n$ at each corner.

If we have an n_1-gon, n_2-gon, ..., n_r-gon with corners meeting at a vertex v, the sum of the angles at their corners must add to 2π. Therefore, after dividing by π, this yields the **Vertex Restriction Equation**:

$$\frac{(n_1-2)}{n_1} + \frac{(n_2-2)}{n_2} + \dots \frac{(n_r-2)}{n_r} = 2 \tag{2.1}$$

One can check that there are only 17 possible choices for positive integers $n_1, n_2, \dots, n_r$ that satisfy this equation. (We will do some of that checking in the exercises.) For four of those choices, there are two distinct ways to place the corresponding polygons around the vertex, not counting a reflection at the vertex, yielding a total of 21 local patterns for polygons at a vertex.

Species number	1	2	3	4	5	6
Type	3.3.3.3.3.3	3.3.3.3.6	3.3.3.4.4 3.3.4.3.4	3.3.4.12* 3.4.3.12*	3.3.6.6* 3.6.3.6	3.4.4.6* 3.4.6.4

7	8	9	10	11	12	13
3.7.42**	3.8.24**	3.9.18**	3.10.15**	3.12.12	4.4.4.4	4.5.20**

14	15	16	17
4.6.12	4.8.8	5.5.10**	6.6.6

Table 2.1: Types of vertices.

Definition 2.1

A **species** *of a vertex is a set of positive integers that satisfy Equation 2.1. We denote a species by $n_1, n_2, \ldots, n_r$. A* **type** *of a vertex is a choice of the order of the polygons surrounding a vertex for a given species. For polygons encountered in the cyclic order around the vertex $\{n_1\}, \{n_2\}, \ldots, \{n_r\}$, we denote the type by $n_1.n_2.\ldots.n_r$*

For instance, the types of the vertices for the regular tilings are 3.3.3.3.3.3, 4.4.4.4, and 6.6.6 and these are the only tilings such that for all vertices there is a single vertex type with just one integer appearing in that type. For convenience, an integer c repeated consecutively n times in a vertex type is denoted c^n. So we denote the vertex types of the regular tilings by 3^6, 4^4, and 6^3. And when denoting a type of a vertex, we choose the representative that is lexicographically first. For instance, we use 3.4.6.4 rather than options such as 4.6.4.3, 6.4.3.4, or 4.3.4.6. See Table 2.1 for the complete list, where types are grouped together when they correspond to the same species. As we will see, the vertex types with a * cannot exist as the only type of vertex in a tiling. The vertex types with a ** cannot exist as vertex types in any tiling.

In Figure 2.2 we see pictures of all the possibilities for types of vertices, keeping in mind that we consider the reflection of a picture the same type of vertex.

We might imagine that restricting to a single species would dramatically limit the number of possible tilings. But as in Figure 2.3, if we restrict to the fifth species 3,3,6,6 corresponding to two triangles and two hexagons, then we can slide rows of tiles so that in adjacent rows, we either match triangle edges to triangle edges or we match triangle edges to hexagon edges. If we represent the first possibility with a 0 and the second possibility with a 1, we get a bi-infinite sequence of 0's and 1's that represent each such tiling. For example, we might have $\ldots 100111010101110 \ldots$. Two such sequences represent the same tiling if one is a shift or a reflection of another. We will see in the exercises that this yields an uncountable infinity of distinct tilings.

Another example is the species 3, 4, 4, 6 corresponding to a triangle, two squares, and a hexagon. On the left in Figure 2.4, we see a tiling with all vertices of type 3.4.6.4. However, for each hexagon, we can take the patch consisting of that hexagon and its neighboring tiles, outlined in black. Then we can rotate it by $\pi/6$, and reinsert it to obtain a new tiling with vertex types

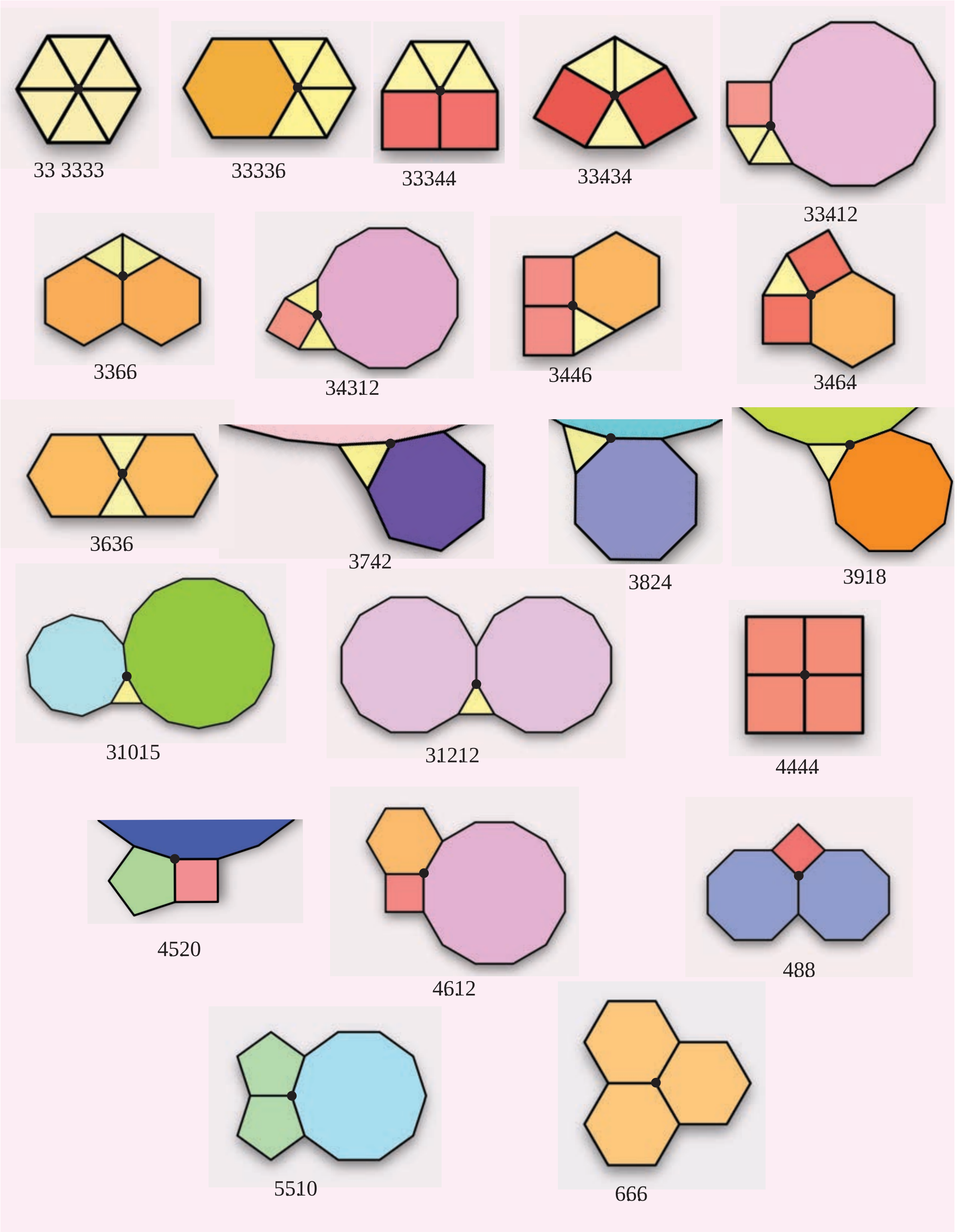


Figure 2.2: The types of vertices.

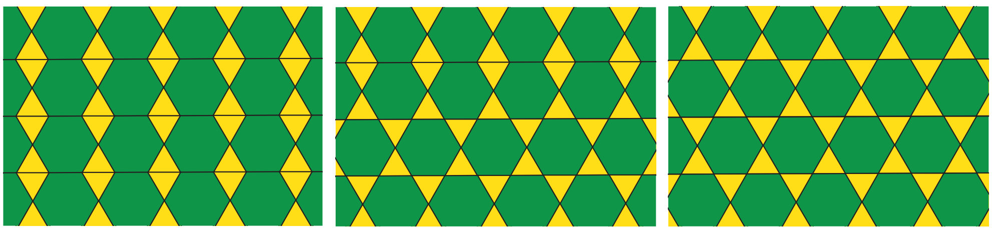


Figure 2.3: Infinitely many tilings with all vertices of a given species.

3.4.6.4 and 3.4.4.6, but still all vertices of species 3, 4, 4, 6, as appears on the right. Since we can choose an infinite collection of such patches, all disjoint, for similar reasons as in the previous example, we have an uncountable infinity of tilings with all vertices of this species.

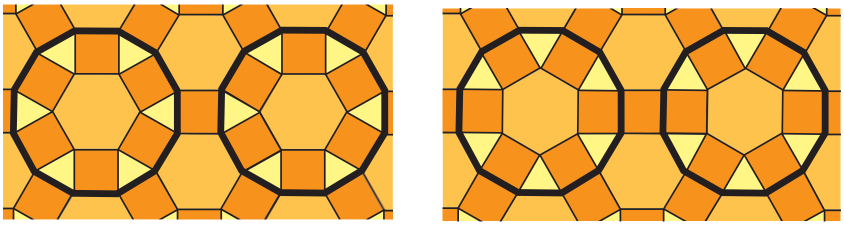


Figure 2.4: Another example of infinitely many tilings with all vertices of a given species.

So to limit the numbers of tilings to consider, it makes sense to restrict to tilings, all the vertices of which have the same type.

Definition 2.2

An edge-to-edge tiling by regular polygons is called a **uniform tiling** *if all vertices have the same type. These are also often called* **Archimedean tilings** *or* **semiregular tilings***. We denote such a tiling with vertices of type $a.b.c\dots$ by $(a.b.c\dots)$.*

The requirement that all vertices have the same type is equivalent to insisting the tilings are monogonal. The cyclically ordered set of regular polygons at each vertex determines the vertex and incident edges up to congruence and vice versa.

Simple examples of uniform tilings are given by the regular tilings. More generally, since our uniform tilings are edge-to-edge, the prototiles must be a finite collection of regular polygons, all with the same edge length. With these restrictions, the possibilities are limited.

Theorem 2.1

There are 11 uniform tilings, which are listed here:
$(3^6), (3^4.6), (3^3.4^2), (3^2.4.3.4), (3.4.6.4), (3.6.3.6), (3.12^2), (4^4), (4.6.12), (4.8^2),$ *and* (6^3).

The uniform tilings appear in Figure 2.5. With the exception of $(3^4.6)$, each of them has a unique form. In the case of $(3^4.6)$, there are two corresponding tilings related through reflection, yielding an enantiomorphic pair of tilings. This enumeration of the uniform tilings was first published by the German astronomer and mathematician Johannes Kepler (1571–1630) in 1619 [52].

Proof We first eliminate the 10 types of vertices that do not appear in the statement of the theorem (those that either have a ** or a * in Table 2.1) by showing that if we start with a vertex of that type, we cannot extend to a uniform tiling of the plane. It is enough to consider how tiles would fit around a polygon with an odd number of sides.

For example, consider the vertex type 3.3.4.12. Let v_1 be a vertex as in Figure 2.6. Consider the triangle T_1 sharing an edge with the 12-gon. The second vertex v_2 that they share must be of the same type, so it is shared by another triangle T_2 and square. But the new triangle T_2 must share an edge with the first triangle. So the third vertex v_3 of T_1 is shared by three distinct triangles, which makes it impossible for that vertex to be of type 3.3.4.12. We leave it to the homework to argue in a similar manner that the other nine do not correspond to tilings.

So now we are left with only 11 possible vertex types for uniform tilings. But we still need to show that tilings exist for each of them. Although Figure 2.3 shows a tiling for each (and two for 3^4.6), a picture is not a proof. One can draw what appears to be tilings by regular polygons that in fact are not. The angles and edges are just slightly off. So we need to know that these actually tile the plane. In the cases of (3^6), (4^4) and (6^3), these are the regular tilings which we do know tile the plane. The first two come from sets of parallel lines and the third comes from amalgamating triangles in the first.

The two tilings for $(3^4.6)$ and the tiling $(3.6.3.6)$ are obtained from (3^6) by gluing sets of six triangles together to obtain hexagons. So these tilings are clearly valid.

The tiling $(3^3.4^2)$ is obtained by strips of squares and strips of equilateral triangles glued together alternately, the strips of which are subsets of (3^6) and (4^4), so this tiling is also valid. The tiling (3.12^2) can be obtained by starting with the hexagonal tiling (6^3) and at each vertex v, creating a small equilateral triangle with its three vertices lying on the three edges leaving v. Then slowly expanding all such triangles at the vertices at the same time until their edge lengths exactly equal the edge length on the original tiling outside the equilateral triangles and between their vertices, we obtain the tiling (3.12^2). Similar arguments work to show the rest of the uniform tilings are valid. ■

We have defined the uniform tilings to be those tilings that consist of regular polygons such that there is only one type of vertex, which is equivalent to saying they are monogonal. However, it is straightforward to check that in fact, each one of them satisfies the stronger condition that it is isogonal, which is to say vertex-transitive. So we could have defined a uniform tiling to be an edge-to-edge tiling by regular polygons that is isogonal, and we would have obtained the same set of tilings. This motivates the following definition.

Definition 2.3

*A tiling is k-**uniform** if it is an edge-to edge tiling made up of regular polygons that has k transitivity classes of vertices.*

Note that then a k-uniform tiling can have at most k different vertex types, but it can also have fewer. To denote a k-uniform tiling, we simply list the k different vertex types in order inside parentheses. So for instance, we could have the 2-uniform tiling $(3^6; 3^2.6^2)$. The following result is due to Krötenheert in 1969 [54].

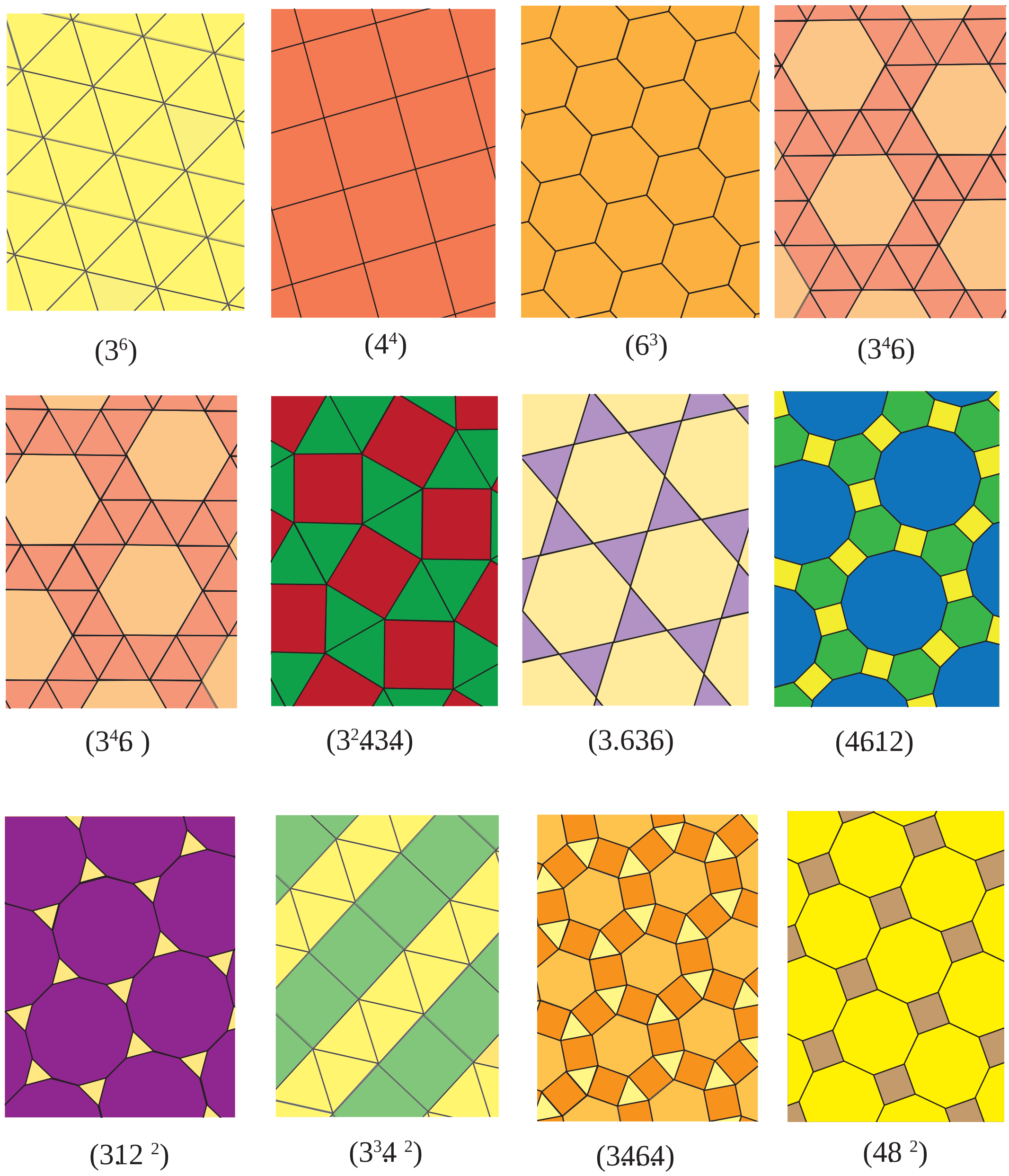


Figure 2.5: The uniform tilings.

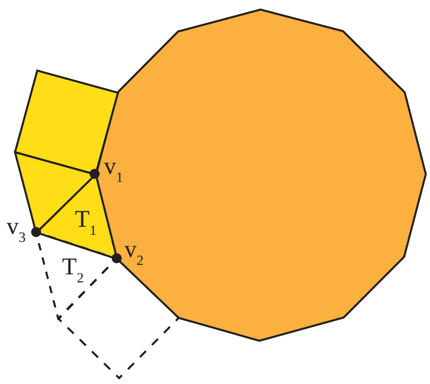


Figure 2.6: Arguing that there is no vertex of type 3.3.4.12.

Theorem 2.2

There are exactly 20 2-uniform edge-to-edge tilings.

The proof of this is similar in spirit to the proof that there are 11 uniform tilings, but the details become substantially more complicated. We show several of the 2-uniform tilings in Figure 2.7. The solid dots correspond to one vertex from each vertex transitivity class.

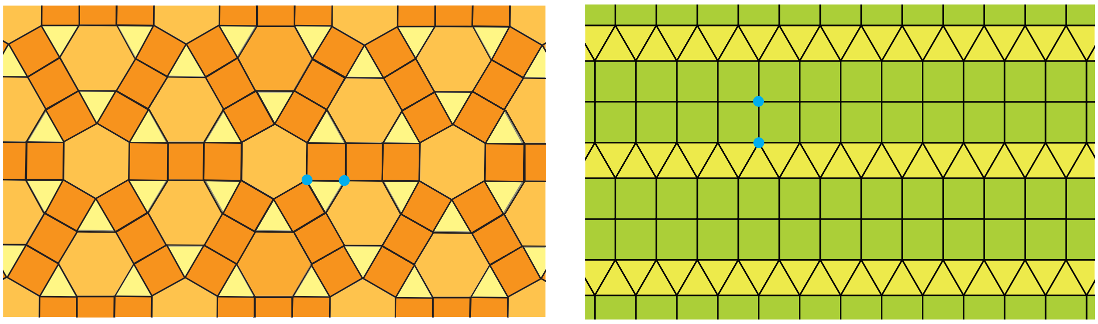


Figure 2.7: Several 2-uniform tilings.

In Figure 2.8, we see a 3-uniform tiling.

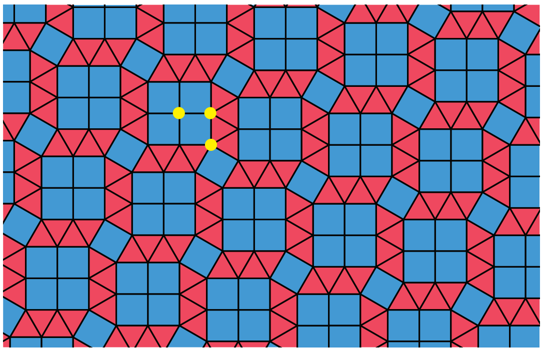


Figure 2.8: A 3-uniform tiling.

We can also look for tilings that use as many different types of vertices as possible. In Figure 2.9, we see a tiling with 14 vertex types, which we will see in the exercises is the most possible. This tiling was found by Jihoon Kim, who was a student in my tiling class at the time.

Exercises for Section 2.1

1. Use the following steps to show that the set of equivalence classes of bi-infinite sequences of 0's and 1's, up to shifting and reflection, is uncountably infinite.
   (a) Show that the set of bi-infinite sequences $(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)$ is uncountable, where each x_i is either 0 or 1. (Hint: See the diagonalization argument in Section 0.2.)
   (b) Show that if we identify the sequence $(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)$ to $(\dots, x_2, x_1, x_0, x_{-1}, x_{-2}, \dots)$, we still have an uncountable set.

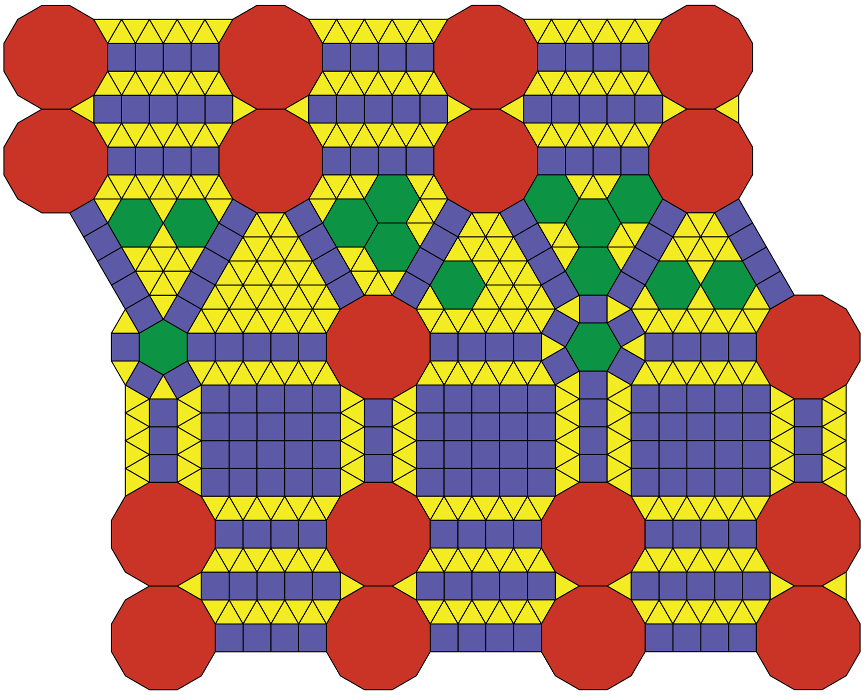


Figure 2.9: A tiling with 14 different vertex types.

(c) Show that if we further identify all sequences that are shifts of one another (so x_i in the first sequence becomes x_{i+n} in the second sequence for all i and a fixed n), there are still an uncountable infinity of possibilities.

(d) Explain why this implies that any non-rectangular parallelogram admits an uncountable infinity of monohedral tilings. (It also proves it for tilings with species 3,3,6,6 as in the discussion in this section.)

2. Find a tiling with D_3 symmetry group such that all vertices are of type 3.3.3.4.4 and 3.3.4.3.4.
3. Show that vertex types 3.7.42, 3.8.24, 3.9.18, 3.10.15, 4.5.20 and 5.5.10 cannot appear as vertices in any edge-to-edge tiling by regular polygons.
4. From the previous exercise, we know that only 15 vertex types are possible in any tiling. In Figure 2.9, we see a tiling with 14 of the vertex types. Show that this is the most possible by determining the missing vertex type and then showing that this particular vertex type can only occur in one tiling, the vertices of which all have that one vertex type.
5. Find a periodic tiling with all 14 vertex types such that a period parallelogram has the least number of tiles possible.
6. Use the Vertex Restriction Equation to show that the only vertex types of length three (so they appear as a.b.c) are those listed.
7. Determine the number of vertex types and the number of transitivity classes of vertices for the tiling in Figure 2.10.
8. Prove that for each of the following vertex types, there is no tiling of the plane by regular

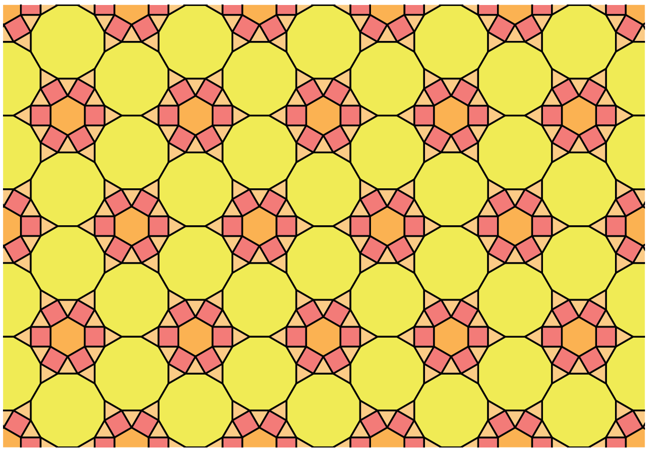


Figure 2.10: Determine the numbers of vertex types and transitivity classes of vertices.

polygons that has only that vertex type:

(a) 3.3.4.12

(b) 3.4.3.12

(c) 3.3.6.6

(d) 3.4.4.6

9. Prove that the tilings $(3.4.6.4)$ and (4.8^2) appearing in Figure 2.3 are valid tilings. That is to say, explain how to obtain them from one of the tilings (3^6), (4^2), or (6^3).
10. Find a 2-uniform tiling that uses only hexagons and triangles.
11. Generalize the tiling that appears on the right in Figure 2.7 to obtain tilings from squares and equilateral triangles that are k-uniform for all $k \geq 3$.
12. Determine the symmetry groups for each of the tilings appearing in Figures 2.7 and 2.8.
13. Determine the symmetry group of each of the 11 uniform tilings in Figure 2.5.

2.3 Tilings that are not Edge-to-Edge

Up to now, we have primarily considered polygonal tilings that are edge-to-edge. In other words, all vertices are also corners of the polygonal tiles on which they lay. But many beautiful tilings result if we do not restrict ourselves in this way. For instance, in Figure 2.16, we see a tiling that is not edge-to-edge. Other examples we have already seen include Figures 1.36, 1.37, 1.39, 1.40, and 1.43.

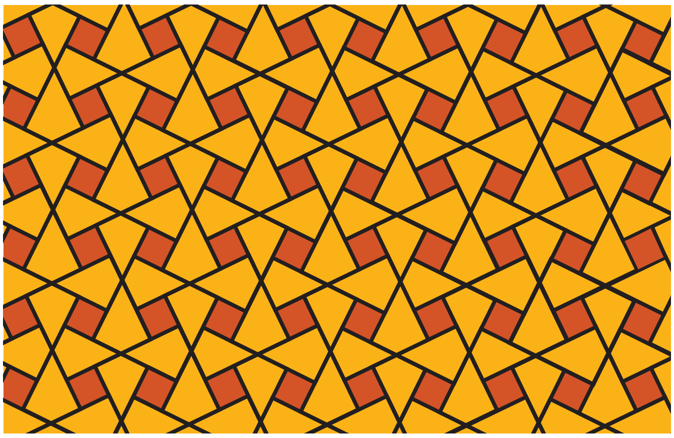


Figure 2.16: A dihedral tiling that is not edge-to-edge.

Some of the most functional tilings are not edge-to-edge. For instance, as in Figure 2.17, brickwork is usually not edge-to-edge. In Figure 2.18, we see a tiling by 1×2 dominoes that is obtained by first tiling a square and then using copies of that square to tile the plane.

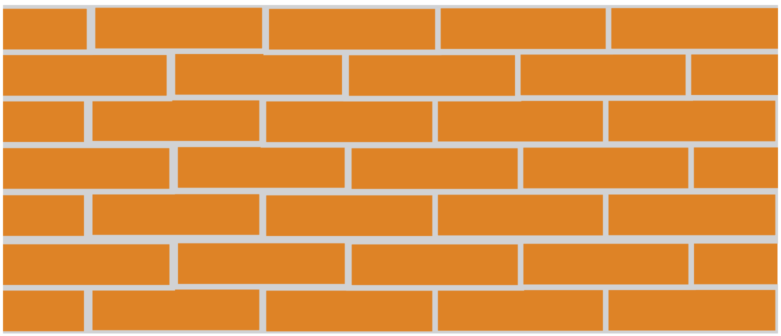


Figure 2.17: Brickwork is typically not edge-to-edge.

In this section, we see what can happen when we consider tilings by regular polygons that are not edge-to-edge. This freedom from gluing tiles edge-to-edge dramatically increases the number of possibilities. For instance, in the case of either of the regular tilings by equilateral triangles or squares, we can slide rows relative to one another to obtain uncountably many tilings. But in fact, this is all we can do. Any monohedral tiling by either equilateral triangles or squares is obtained in this way. On the other hand, there is just the one tiling by regular hexagons. There is no sliding possible.

Once we no longer restrict ourselves to a single prototile, the number of possible tilings explodes, so much so that we need to add some restrictions. For instance we can restrict to

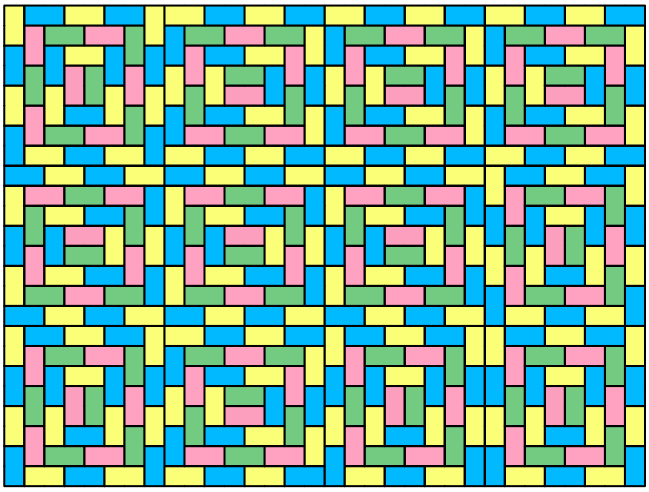


Figure 2.18: A tiling by dominoes.

uniform tilings, where by the term uniform here we mean a tiling made up of regular polygons that is isogonal. So the symmetries of the tiling act transitively on the vertices.

Theorem 2.4

There are eight families of uniform tilings by regular polygons that are not edge-to-edge, each depending on a real parameter.

We see representatives of each of the families in Figure 2.19. The first four come from the equilateral triangle and square regular tilings by sliding rows by an amount α. The next three are all dihedral, and the parameter α is the ratio of the edge lengths of the smaller prototile to the larger prototile. In the last case, there are three sizes of triangles, and α is the ratio of the edge length of the smallest to the edge length of the largest, which determines the edge length of the intermediate sized triangle. In the exercises, you will have a chance to think about how to prove this theorem.

Note that the tiling in Figure 2.16 is a superposition of the tiling by two squares in Figure 2.19(f) drawn at an angle in the plane with a reflection of that tiling through a vertical line, as seen in Figure 2.20.

The same method can be used to obtain other intricate latticework tilings. For example, in Figure 2.21, we see a tiling obtained in a similar manner applied to the tiling in Figure 2.19(g).

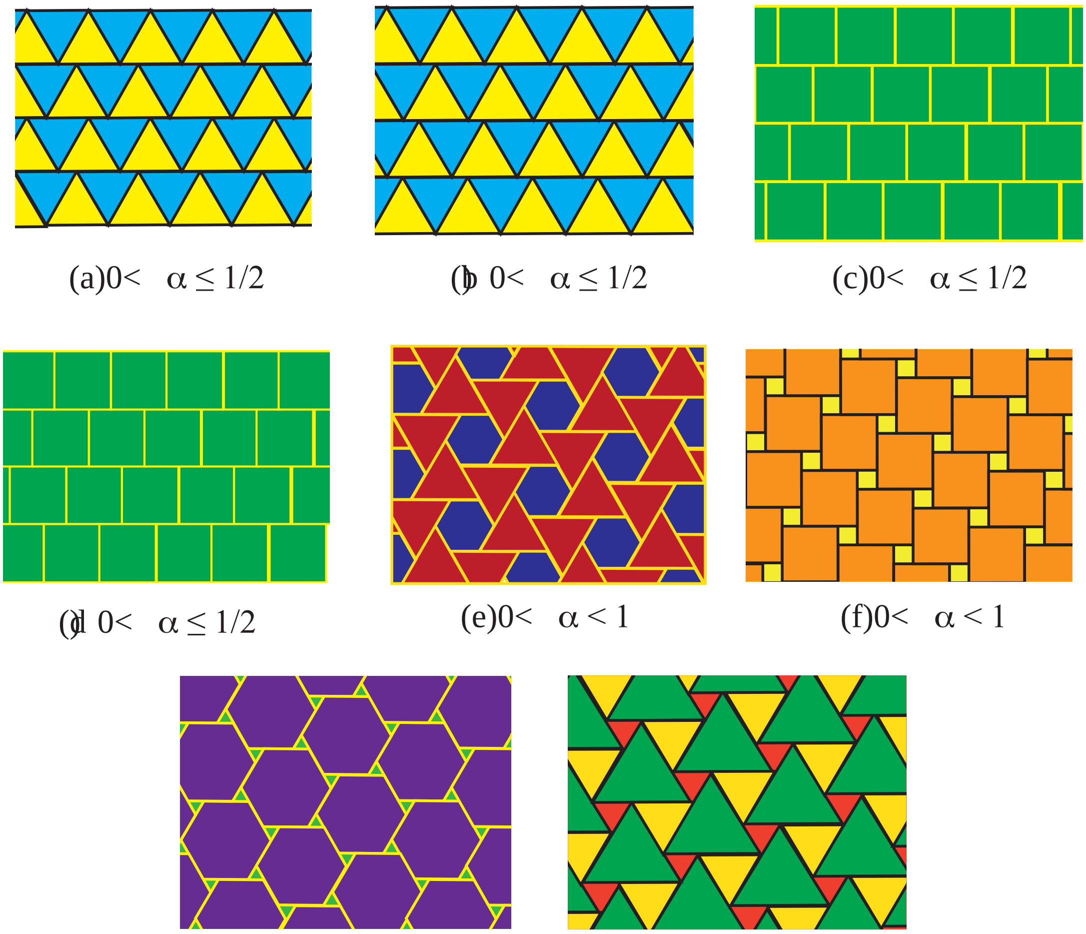
We can restrict our tilings to a single transitivity class for each prototile.

Definition 2.6

A polygonal tiling is **equitransitive** *if all congruent tiles are in the same transitivity class.*

Thus, in the case of a monohedral tiling, equitransitivity is equivalent to isohedrality. But if there are multiple prototiles in the protoset, each will correspond to a single unique transitivity class of tiles.

Equitransitivity limits tilings in useful ways. For instance, we mention the following theorem



(a) $0 < \alpha \le 1/2$ (b) $0 < \alpha \le 1/2$ (c) $0 < \alpha \le 1/2$

(d) $0 < \alpha \le 1/2$ (e) $0 < \alpha < 1$ (f) $0 < \alpha < 1$

(g) $0 < \alpha < 1$ (h) $0 < \alpha \le 1/2$

Figure 2.19: The uniform tiling families that are not edge-to-edge.

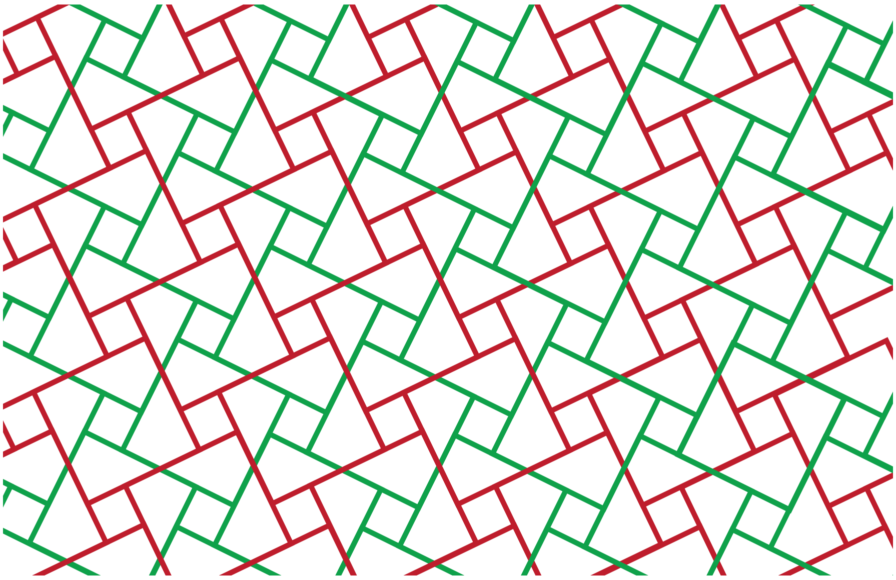


Figure 2.20: The superposition of two copies of the sixth uniform tiling yields the Islamic design in Figure 2.16.

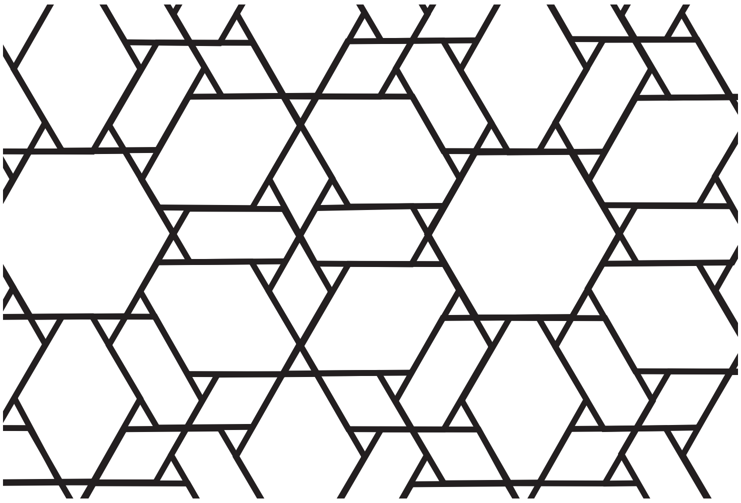


Figure 2.21: The superposition of two copies of the seventh uniform tiling.

with what seems like a somewhat arbitrary number. A periodic equitransitive tiling by convex polygons can contain 66-gons but cannot contain n-gons for $n > 66$ (see [16]).

There are still a vast array of equitransitive tilings even when we restrict to regular polygons. So we restrict further to those in which all the prototiles are n-gons for the same fixed n. For instance, we might only allow equilateral triangles in the protoset. Or we might only allow squares. In Figure 2.22, we see equitransitive tilings by two or five sizes of equilateral triangles.

But again, there are too many possibilities, so we restrict further.

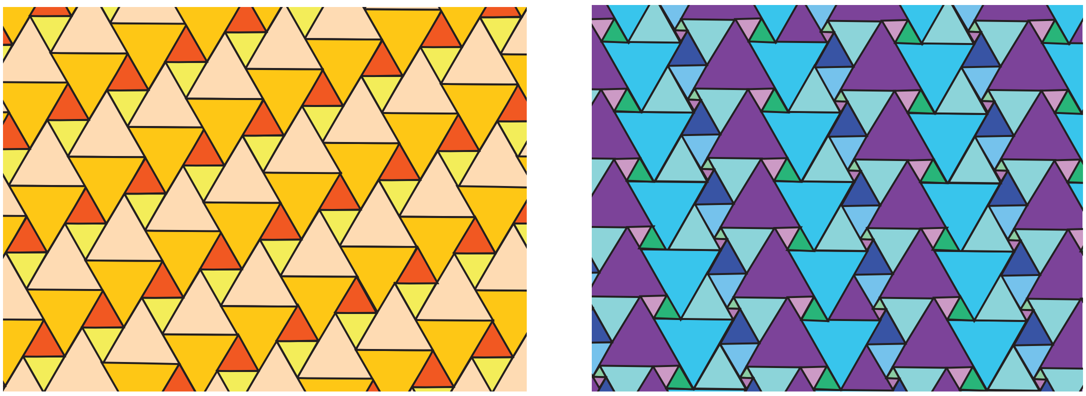


Figure 2.22: Equitransitive non-edge-to-edge tilings such that the tiles are equilateral triangles of two or five different sizes. Tiles are the same color if they are equivalent under translation.

Definition 2.7

A tiling by regular polygons that is not edge-to-edge is called **unilateral** *if every edge of the tiling is a side of at most one polygon. That is to say, the edge must be a proper subset of the side of at least one of the two tiles that share the edge.*

Note that neither of the tilings by triangles in Figure 2.22 are unilateral since the smallest triangle shares full edges with copies of itself. But the tiling by triangles in Figure 2.19(h) is a unilateral equitransitive tiling. In fact, in 2017 in [4], it was proved that Figure 2.19(h) represents all of the possible unilateral equitransitive tilings by equilateral triangles, keeping in mind that the two smaller triangles can have the same size. So there can be either two or three sizes of triangles.

We see three examples of unilateral equitransitive tilings by squares in Figure 2.23. In the first tiling, which has three sizes of squares, it is always the case that the length of the side of the largest square is the sum of the length of the sides of the two smaller squares. In fact, we can prove that when the protoset consists of three sizes of squares, this is always the case. In [74], Doris Schattschneider proved that there are exactly eight families of unilateral equitransitive tilings with three sizes of squares. In work by Casey Mann with undergraduates in 2015 [61], it was determined that there are 39 families of unilateral equitransitive tilings with four sizes of squares.

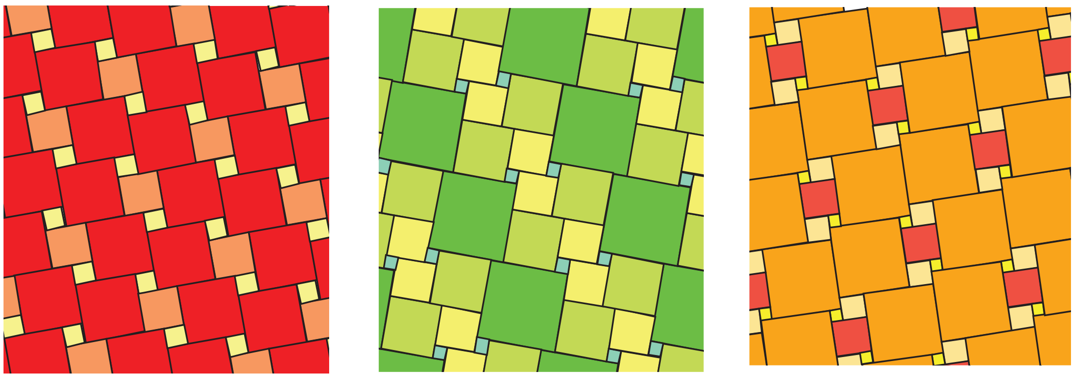


Figure 2.23: Unilateral equitransitive tilings by squares of three and four sizes.

Open Questions

1. *Which sets of side lengths $s_1, s_2, \ldots, s_r$, correspond to a unilateral equitransitive tiling by squares?*
2. *Show that there does not exist a tiling of the plane by infinitely many equilateral triangles, all of different sizes. (Note that this would not be a valid tiling in the way we have defined it, since it would require infinitely many prototiles.)*

One method for finding sets of valid side lengths is to find decompositions of either a large square or a large rectangle into square tiles with these sizes. Then we can tile the entire plane with copies of this larger tile and produce a tiling with these square tiling sizes. Decomposing a square into smaller squares is called the **squared square problem**. In Figure 2.24, we see a square tiled with 21 smaller squares, due to A. J. W. Duijvestijn [20]. This is an example of a **perfect squared square** in that all of the smaller squares have different side-lengths, which are the numbers in each tile. It is known that this is the least possible number of squares in a perfect squared square.

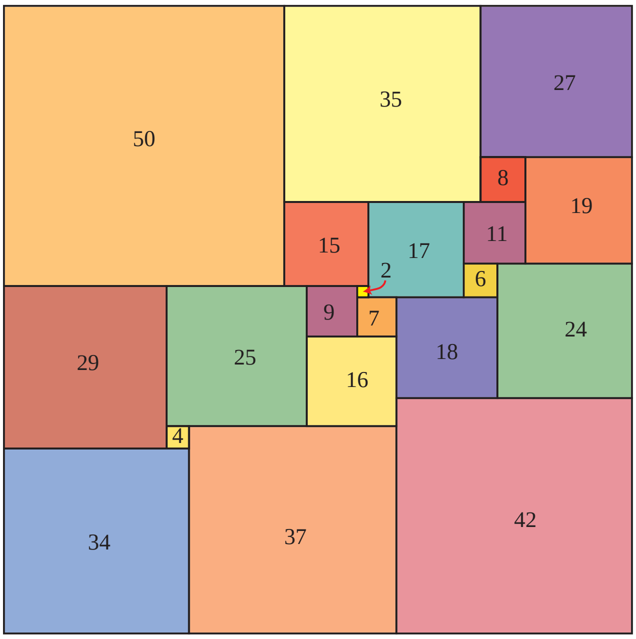


Figure 2.24: Subdividing a square into smaller squares.

Exercises for Section 2.3

1. Try superimposing a tiling from Figure 2.19 with a copy of itself that has been rotated, reflected, and/or translated to obtain an interesting tiling.
2. Show that for each positive integer n, there exist a tiling of the plane with protoset consisting of n squares of different sizes.
3. Show that for each positive integer n, there exist a tiling of the plane with protoset consisting of n equilateral triangles of different sizes.
4. Show that for any positive integers t and h with $t \geq h$, there is a protoset that consists of t equilateral triangles and h regular hexagons that admits a tiling.
5. Show that for any positive integers t and h with $t < h$, there is a protoset that consists of t equilateral triangles and h regular hexagons that admits a tiling.
6. Show that for any positive integers t and s, there is a protoset that consists of t equilateral triangles and s squares that admits a tiling.
7. One of the edge-to-edge uniform tilings in Figure 2.5 is not equitransitive. Determine which one it is.
8. Prove that any equitransitive tiling is periodic.
9. Find an equitransitive tiling that has protoset two sizes of regular hexagons and one size of equilateral triangle.
10. (a) Prove that there are only three types of vertices that can occur for uniform tilings that are not edge-to-edge. (By uniform here, we mean that the tiling consists of regular polygons and the tiling is isogonal.)
    (b) Use this to show that the only possibilities for such tilings are those with all prototiles

triangles, all prototiles squares, or one prototile a triangle and one a hexagon.

(c) Given a uniform tiling that is not edge-to-edge, show that if all prototiles are triangles, there are at most three sizes of triangles.

(d) Given a uniform tiling that is not edge-to-edge, such that there are three prototiles which are three sizes of triangles, then the edge length of the largest triangle must be the sum of the edge lengths of the other two.

11. Prove that if a uniform tiling that is not edge-to-edge has protoset consisting of two squares of different sizes, it must appear as in the sixth family of Figure 2.19. (Hint: First prove every vertex must be incident to two large squares and one small square.)

12. Find an example of a tiling that consists of all equilateral triangles of edge length one and one copy of an additional tile that is

(a) an equilateral triangle of edge length any positive number less than one.

(b) a parallelogram of angles $\pi/3$ and $2\pi/3$ and all edge lengths less than one.

(c) a polygon that is neither of the two options above.

13. If a tiling consists of all squares of edge length 1 and only one other tile that has diameter less than 1, determine the possible shapes of that tile.

14. Define a polygonal tiling to be monolateral if each edge is a side of exactly one of the two tiles that share it. Find an example of a monolateral equitransitive tiling by n-gons for some fixed n. (Hint: See past figures.)

15. Show that for a unilateral equitransitive tiling with three sizes of squares, the sum of the edge lengths of the two smaller squares must equal the edge length of the larger square.

16. Show that there is a perfect squared square with $20n + 1$ differently sized squares for every positive integer n.