Preface

This book is an introduction to Number Theory from a more geometric point of view than is usual for the subject, inspired by the idea that pictures are often a great aid to understanding. The title of the book, *Topology of Numbers*, is intended to express this visual slant, where we are using the term “Topology" with its general meaning of “the spatial arrangement and interlinking of the components of a system".

The other unusual aspect of the book is that, rather than giving a broad introduction to all the basic tools of Number Theory without going too deeply into any one, it focuses on a single topic, quadratic forms $Q(x,y) = ax^2 + bxy + cy^2$ with integer coefficients. Here there is a very rich theory that one can really immerse oneself into to get a deeper sense of the beauty and subtlety of Number Theory. Along the way we do in fact encounter many standard number-theoretic tools, with some context to show how useful they can be.

A central geometric theme of the book is a certain two-dimensional figure known as the *Farey diagram*, discovered by Adolf Hurwitz in 1894, which displays certain relationships between rational numbers beyond just their usual distribution along the one-dimensional real number line. Among the many things the diagram elucidates that will be explored in the book are Pythagorean triples, the Euclidean algorithm, Pell’s equation, continued fractions, Farey sequences, and two-by-two matrices with integer entries and determinant ± 1.

But most importantly for this book, the Farey diagram can be used to study quadratic forms $Q(x,y) = ax^2 + bxy + cy^2$ via John Conway’s marvelous idea of the *topograph* of such a form. The origins of the wonderfully subtle theory of quadratic forms can be traced back to ancient times. In the 1600s interest was reawakened by numerous discoveries of Fermat, but it was only in the period 1750-1800 that Euler, Lagrange, Legendre, and especially Gauss were able to uncover the main features of the theory.

The principal goal of the book is to present an accessible introduction to this theory from a geometric viewpoint that complements the usual purely algebraic approach. Prerequisites for reading the book are fairly minimal, hardly going beyond high school mathematics for the most part. One topic that often forms a significant part of elementary number theory courses is congruences modulo an integer n. It would be helpful if the reader has already seen and used these a little since we will not develop congruence theory as a separate topic and will instead just use congruences as the need arises, proving whatever nontrivial facts are required including several of the basic ones that form part of a standard introductory number theory course. Among these is quadratic reciprocity, where we give Eisenstein's classical proof since it involves some geometry.

The high point of the basic theory of quadratic forms $Q(x,y)$ is the *class group* first constructed by Gauss. This can be defined purely in terms of quadratic forms, which is how it was first presented, or by means of Kronecker's notion of ideals introduced some 75 years after Gauss's work. For subsequent developments and generalizations the viewpoint of ideals has proven to be central to all of modern algebra. In this book we present both approaches to the class group, first the older version just in terms of forms, then the later version using ideals.

Here is how the book is organized. A preliminary Chapter 0 gives a sample of some of the sorts of questions studied in Number Theory, in particular motivating the study of quadratic forms by seeing how they arise in understanding Pythagorean triples, the integer side-lengths of right triangles, such as 3,4,5 and 5,12,13.

After this introduction the next three chapters lay the groundwork for our approach to quadratic forms by introducing the Farey diagram and its first applications to visualizing the Euclidean algorithm and continued fractions, both finite and infinite.

The next four chapters are the heart of the book. Chapter 4 introduces the topograph of a quadratic form, which displays all its values visually in a convenient and effective picture. A variety of examples are given illustrating different kinds of qualitative behavior of the topograph. As applications, topographs give efficient ways to compute the values of periodic and eventually periodic continued fractions, and to find all the integer solutions of Pell's equation $x^2 - dy^2 = \pm 1$.

Chapter 5 develops the classification theory for quadratic forms $ax^2 + bxy + cy^2$ in terms of the discriminant $b^2 - 4ac$. There are only a finite number of essentially distinct forms of a given discriminant, and it is shown how to compute these. Forms with symmetry play a special role, and a fairly complete picture of these is developed.

Chapter 6 turns to the fundamental representation problem, which is to find all the values a given form takes on, or in other words, to determine when an equation $ax^2 + bxy + cy^2 = n$ has integer solutions. There are two central themes here: how the factorization of n into primes plays a key role, largely reducing the problem to the case that n itself is prime; and how congruences modulo the discriminant give useful criteria for solvability, particularly in the case of primes.

Chapter 7 completes the basic theory by presenting Gauss's discovery of a way to multiply forms of a given discriminant, refining the multiplication of the values of the forms. This leads to an explanation of the seemingly mysterious fact that while there is essentially only one form of a given discriminant that represents a given prime, there can be several different forms representing nonprimes.

Finally, the rather lengthy Chapter 8 goes in a different direction to give an exposition of the alternative viewpoint toward quadratic forms by expanding the set of rational numbers to sets of numbers $a + b\sqrt{n}$ with a and b rational. Here the deeper subtleties of quadratic forms are translated into subtleties with the factorization of such numbers into "primes" and the lack of uniqueness of such factorizations. In keeping with the viewpoint of the rest of the book, we strive to make this essentially algebraic theory as geometric as possible.

At the end of the book there are several tables giving the key data for quadratic forms of small discriminant.

This book will remain available online in electronic form for free downloading after it has been published in the traditional paper form. The web address where it can be found is

http://www.math.cornell.edu/˜hatcher

Also available here will be a list of corrections as well as possible revisions and additions to the book. Readers are encouraged to send comments and corrections to the email address posted on the web page.