Preface

A preface is the author's story of why and how this book came to be. It is not prerequisite for reading or using the book.

The gestation of this book. This book was thrice reconceived, gaining weight at each rebirth. It was first born as a text based on a capstone mathematics course for secondary teachers that I developed over several years. Some (still present) features of that text, that distinguished it from some other curricular materials, were the following:

- Emphasis from the beginning was given to the continuous real number line $\mathbb{R}$, with its multiple structures (algebraic, order, geometric). For example, division with remainder was treated, not as part of integer arithmetic, but as linear measurement on $\mathbb{R}$. This approach was resonant with the ideas of V. Davydov.
- Keeping inview the integers embedded in the reals, thus emphasizing a kind of duality, rather than contrast, between the discrete and continuous
- Powerful ideas from abstract algebra, notably group theory, were initially incorporated in concrete, not abstract ways. For example, I defined what an additive group of real numbers was without having defined what a group was. It was further defined what it meant for a real additive group A to be discrete, and it seemed natural to consider discrete additive groups, thus right away interweaving geometry with algebra. The following theorems about an additive group A were then proved as simple applications of (real) division with remainder. (i) A is either discrete or dense in $\mathbb{R}$. (ii) If A is discrete then $A = \mathbb{Z}a$ for a unique $a \geq 0$. (iii) $\mathbb{Z}a + \mathbb{Z}b$ is discrete if and only if a and b are commensurable, in which case we define $\gcd(a, b) = d \geq 0$ by, $\mathbb{Z}a + \mathbb{Z}b = \mathbb{Z}d$. More broadly, a major theme of the course was to study the additive and multiplicative groups of the rings closely related to school mathematics, including modular rings, $\mathbb{Z}/\mathbb{Z}m$.
- Emphasis was given to mathematical connections, for example seeing structural similarity between mathematical ideas and entities that may appear at first to be unrelated. This was enacted both in the curriculum, and in the design of some novel problem-solving activities. The latter are discussed here in Chapter 11, on Mathematical Connections.
- For example, the Chapter 9, on Discrete Calculus, is a striking confluence of several mathematical themes, thus nicely illustrating the broad connectivity of mathematics, something not often visible in school curricula.

As I tried to put these ideas into book form, and maintain some reasonable level of mathematical rigor and completeness, the manuscript grew a bit more ponderous, and lost some of the more lyrical intuitive approach I tried to maintain in the course. At that stage, my conception of the book shifted from "Mathematics for secondary teachers" to "A mathematical perspective on school mathematics," somewhat in the spirit of Felix Klein's *Elementary Mathematics from an Advanced Standpoint: Arithmetic, Algebra, Analysis.*[1] This conception has some affinity with the similarly inspired book of Usiskin and colleagues, *Mathematics for High School Teachers: An Advanced Perspective* (2003). In the course of writing this book there appeared the book, "Secondary Mathematics for Mathematicians and Educators" by my colleague, Michael Weiss.[2] Similarly inspired by Felix Klein, Weiss focuses more directly than I on the current secondary curriculum, and he offers many insightful analyses of the history and mathematical evolution of that curriculum.

Having made the investment in laying the foundations that enabled broader and more precise mathematical treatment, it was both easy and hard to resist adding rich topics proximal to, if not typically inside of, school mathematics. This seemed worthwhile since the topics were not only reasonably accessible to high school teachers, but because they also exposed the wide variety of questions that the mathematics "naturally" invites, and the corresponding diversity of ideas and techniques invented to address them. Whence the next emergent title, "The mathematical neighborhood of school mathematics." In this sense, it provides one aspect of what some educators have called *horizon content knowledge.*[3] There is no new mathematics here, but there are some new ways of representing, and talking about some basic mathematics. However, importantly, I have not shown here how this relates to teaching.

The final stage of production was my response to helpful feedback from reviewers of a preliminary draft. While they showed appreciation for some aspects of the text, it did not seem to fit any familiar genre, and, most importantly, it did not seem to have a clearly defined audience or obvious mode of use. In the course of addressing these issues, hopefully with some success, the material was reorganized, to fashion a more coherent story line, and some substantial new material was added. In particular, the title was changed by replacing "neighborhood" by "neighborhoods."

So, in a sense, the book wrote itself into being as much as being a preconception of its author.

[1] (Klein, 1932)
[2] (Routledge, 2021)
[3] (Ball, Thames, and Phelps, 2008)