CHAPTER 11

Mathematical Connections

Guide to the reader

Background resources. This chapter is written expressly for mathematics educators, and it requires no special technical background.

Overview. This book was designed to reveal the rich network of, often surprising, connections among diverse mathematical ideas and methods. This expresses what many mathematicians call the coherence and unity of mathematics. For the mathematics educator, this poses the question:

How can students themselves learn
to discern/discover/invent/apply such connections?

I call these practices *connection-oriented mathematical thinking.* The primary mode of mathematics learning is problem solving. This chapter assembles examples of several experimental problem-solving activity designs that I have devised and used with the intention of cultivating in my students a disposition and sensibility toward connection-oriented mathematical thinking.

For the selective reader. For mathematics educators, both K-12 and undergraduate, I hope that most of this chapter will be of interest.

Contents

11.1. Connection-oriented mathematical thinking

This closing chapter is contrasting and complementary to those preceding it, in that it attends explicitly to some resources for instruction. It shares with the earlier chapters an emphasis on mathematical connections. Indeed, it assembles here the various experiments I have tried, when teaching content courses to pre-service secondary math teachers, to cultivate connection-making.

The main body of this book presents and explains to the reader/learner a rich variety of connections. Notable examples include: The organizing power of the notion of group to bring coherence to a host of fundamental topics; The ubiquitous appearances of binomial coefficients, in polynomial algebra, probability, discrete calculus, etc.; The incarnations of Catalan numbers as the answer to a striking variety of different questions; the proof and numerous applications of the Inclusion-Exclusion Principle; and the impressive range of implications of the ABC Conjecture.

This chapter, in contrast, reports on some experimental ideas for cultivating, in the learner, the capacity to make/discover mathematical connections, and the disposition to seek/notice, and find ways to exploit, mathematical connections.[1] Research in cognitive science, especially around transfer, suggests that this kind of thinking is not commonplace, so that one cannot generally expect it if it is not explicitly taught and learned.

A primary mode of mathematics learning is through problem-solving. After some unpacking of the concept of mathematical connection in Section 11.1, I discuss connection-making in solving a single problem; for example, problems with multiple solutions, with multiple solution strategies, and in what I call a "cross-domain" problem (Section 11.2). Then, in Sections 11.3 and 11.4, I offer some examples of problem sets, and designs of problem-solving formats, that I have found to be helpful in prompting students toward various kinds of connection-making, the connections then being between different mathematical problems. In Section 11.5 I illustrate how a rich variety of different problems/questions can be spawned by focused study of a single mathematical object (concept, process), in this case the Euclidean Algorithm. The premise is that these examples could inform the design of instructional activities intended to nurture connection-oriented mathematical thinking.

DEFINITION 11.1.1. By a *mathematical connection*, I mean an articulated expression of structural relatedness between two or more distinct mathematical entities (mathematical domains, contexts, concepts, procedures, problems, solutions, proofs, etc.).

For example, the diverse kinds of connections between a pair mathematical entities A and B could include:

Sameness or deep similarity: A and B are structurally the same or analogous. Examples include
- Isomorphism of algebraic structures;
- congruence of geometric objects;
- different representations of the same thing, for example permutations of 3 objects and symmetries of an equilateral triangle;
- linear transformations and matrices;
- rate of change and slope of a tangent line; or
- the deep analogies between $\mathbb{Z}$ and $F[X]$, F a field.

Generalization or specialization: A is a generalization of B; or B is a special case of A. Denote this $B < A$. Examples include
- Squares < regular polygons < polygons;
- for numbers, natural < integers < rational < real < complex < quaternion;
- prime factorization of integers < unique factorization domains;
- polynomials < differentiable functions < continuous functions < functions

[1] This is related to the Common Core Mathematical Practice 7: *Look for and make use of structure.* I call this mathematical habit of mind *connection-oriented mathematical thinking.*

Linkage: A and B are mathematically linked, for example as components of some encompassing entity C. Examples include

- Addition (A) and multiplication (B) in a ring (C), linked by the distributive law;
- vertices and edges in a graph, linked by incidence relations;
- domain and range of a function;
- kernel and image of a homomorphism;
- a set and its closure in a metric space;

Opposites or inverses: B might be the set theoretic complement of A, or the logical negation of A, or the inverse process of A. Examples include

- The inverse of a function;
- the negation, or the converse of a proposition;
- for numbers, even and odd, positive and negative, rational and irrational;
- existence or non-existence of a problem solution;
- the orthogonal complement of a vector subspace;
- analysis (decomposing into parts) vs. synthesis (combining parts into a whole)

The emphasis here will be on connections that are subtle, and perhaps unexpected, for example when A and B appear, at first, to be unrelated. Many important, and now familiar, mathematical connections were first discovered in that state. And indeed, some of the greatest advances in mathematics have been of this kind. But they can occur at elementary levels as well. Connections are what confer the sense of coherence and unity of mathematics, and this is part of what makes connection-oriented mathematical thinking a worthy educational goal. Our focus here will be mainly on connections of the first kind – sameness, and deep similarity.

A primary mode of mathematics learning is problem solving. Accordingly, this chapter presents a variety of examples and designs of problem solving activities intended to cultivate connection-oriented mathematical thinking. The opportunities for connection-making will be housed in problems, or problem sets. We begin with a helpful survey of some problem types and formats.

11.1.2. Problem Types. Mathematical problems come in a variety of formats. Here are some common examples.

Solve: Find a mathematical object satisfying a set of conditions. For example, one might find a solution to an equation or the symmetries of a geometric figure, or find a formula for the number of ways to meaningfully insert parentheses into a non-associative product of n factors.

Solution space: Find *all* mathematical objects satisfying a set of conditions. This kind of problem tacitly subsumes a hierachy of sub-questions:

Existince: Does a solution exist? Existence can be proved by exhibiting a solution. Sometimes one can prove existence without being able to exhibit a solution. In general, non-existence is harder to prove. For example, Fermat's Last Theorem is a non-existence theorem: if $n \geq 3$, there is no solution in the non-zero integers (a, b, c) of $a^n + b^n = c^n$. The Four-Color Theorem is also a non-existence theorem: there does not exist a planar map that requires more than four colors to be colored so that adjacent countries have distinct colors.

Uniqueness: If a solution exists, is it unique?

Cardinal: If there are multiple solutions, how many are there? Finitely many? Infinitely many?

Properties: What special properties do the solutions have? If they are numbers, are they real? Positive? Integers? Even? If they are functions, are they bounded? Periodic? Continuous? Differentiable? Polynomial? If they are groups, are they finite, commutative, or cyclic?

Structure: How can you describe the solution space?[2] What is its structure? For example, the set of solutions (x, y) of $x^2+y^2 = 1$ is not just an infinite set of points; it is a circle. It can be rationally parametrized to find all Pythagorean triples. Even when finite, the solution space may have some significant structure, for example be a group. Within the solution space, one can further ask questions like, "Which solution(s) maximize(s) some measurable property?"

Classification: Describe (for example, list, or parametrize) the solution space when it is discrete.[3] Examples include the classification of types of quadrilaterals, of regular 3-D polyhedra ("Platonic solids"), of crystal structures in 3-D, or of finite simple groups.

Comprehensive Study: In-depth study of a particularly interesting mathematical object. For example, the icosahedron, or the symmetric groups, the Mandelbrot Set, or the "Monster Group," or the Galois group of the algebraic closure of $\mathbb{Q}$.

11.2. Connection-making in solving a problem

A mathematical connection is a relationship between two (or more) mathematical entities. Such connections are most striking and engaging when the entities involved seem, at first sight, to be unrelated. The following contexts can present opportunities for such connection-making.

11.2.1. Diverse representations of a problem solution. Allow me to illustrate this with the "pool border problem," commonly used in middle school:

`A square pool of side length` $s \in \mathbb{N}$ `is bordered with unit square tiles. Find` $B(s)$`, the number of tiles needed to make the border.`

This problem nicely admits a variety of solution strategies, and corresponding solution representations, both geometric and algebraic.

The geometric pictures in Figure 11.2 illustrate different ways to decompose and count the border. Then equating the different algebraic representations uses the basic "rules of arithmetic." While this diversity is attractive, it is interesting to consider what particular advantages each method or representation might afford. For example, all but the first of the geometric solutions are linear, whereas the first one, $(s + 2)^2 - s^2$, is quadratic.

[2]That is, the set of all solutions.

[3]That is, it has no continuous geometric structure.

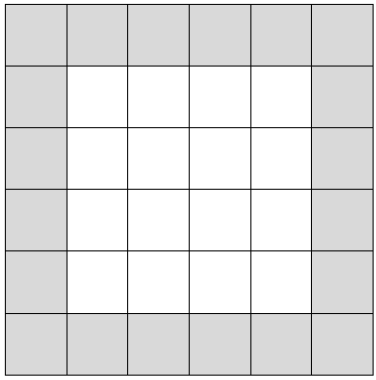

Figure 11.1. The pool border problem.

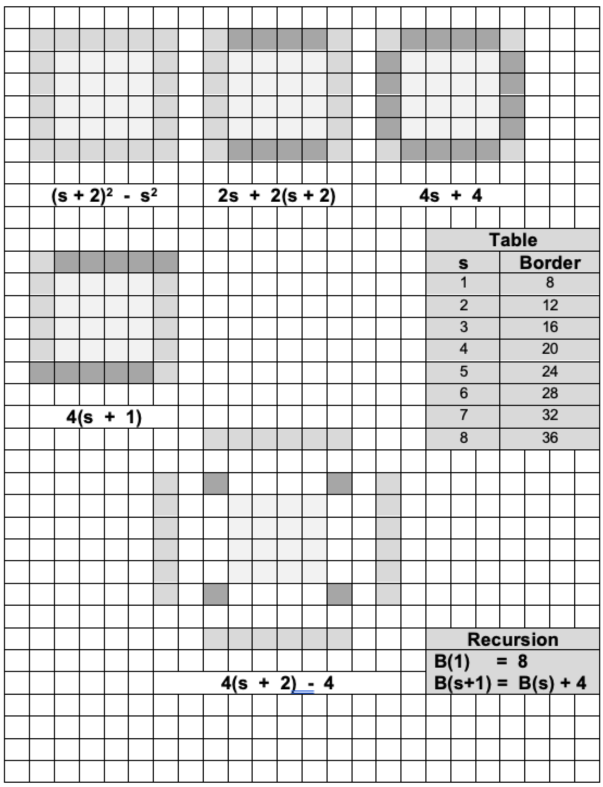

Figure 11.2. Solutions to the pool border problem.

Suppose now that we consider the 3-dimensionsal version of the problem, represented graphically in Figure 11.3:

`A cube of side length` $s \in \mathbb{N}$ `is bordered with unit cube tiles. Find` $B(s)$`, the number of tiles needed to make the border.`

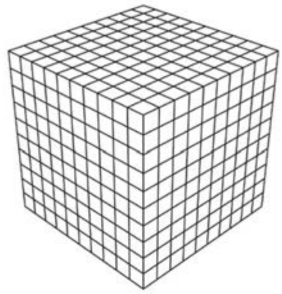

Figure 11.3. The 3-D pool border problem.

The analogue of the linear methods above become the various 2-dimensional methods of decomposing the surface of the cube. The 3-dimensional version of the quadratic method becomes cubic:

$$\begin{aligned} B(s) &= \text{Volume(cube of side length } s+2) \\ &\quad - \text{Volume(cube of side length } s) \\ &= (s+2)^3 - s^3 \\ &= 6s^2 + 12s + 8 \end{aligned}$$

Notice that the coefficients $6, 12, 8$ correspond to the numbers of faces, edges, and vertices of the (inner) cube. Moreover, the number of border tiles adjacent to each (face, edge, vertex) is $(s^2, s, 1)$. Thus, this algebraic derivation reflects a natural combinatorial decomposition of the boundary surface.

11.2.2. Problems with multiple solution strategies. Here we can relate different solution strategies. This too is illustrated in the above discussion of the pool border problem. Another example is to compare ways of proving a claim. For example, consider the claim

(11.2.2.1) The sum of the first n odd integers is n^2.

One can prove this by induction on n. Alternately, one can use the image in Figure 11.4 to establish the claim.

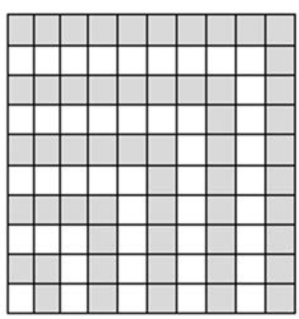

Figure 11.4. Proof of Claim (11.2.2.1).

Another example is to compare the many proofs of the Pythagorean Theorem. For example, one proof is based on the diagram in Figure 11.5.

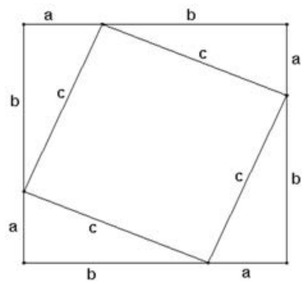

Figure 11.5. Proving the Pythagorean Theorem.

In fact, without mentioning Pythagorous, I like to present my students with the sequence of images seen in Figure 11.6 and ask (open-endedly), "What mathematical story might these images be telling?"

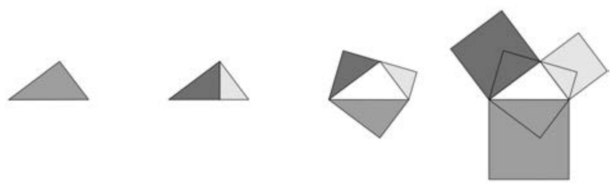

Figure 11.6. Telling a mathematical story

11.2.3. Problems with multiple solutions. In this case students can engage the problem at various levels: find a solution; find several solutions; find all solutions; prove that you have all solutions. A nice example that I have seen Deborah Ball use in teacher professional development is the "magic triangle problem" (see Figure 11.7). Its appeal here is the multiplicity of solutions, of methods for finding them, and the intricate relations between them.

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Place the numbers 1, 2, 3, 4, 5, 6 in the six circles,
one number in each circle, so that the sum of the
numbers along each side of the triangle is the same.

Find all such magic triangles.
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What can be the common side sum, s? The sum of all the numbers is $1 + 2 + 3 + 4 + 5 + 6 = 21$. However, note that there are three *vertex circles* and three *edge circles*, and, if we add together the three side sums, each vertex circle is included twice in the sum. Thus, if v is the sum of the numbers inside all vertex circles, then the sum of all three sides of the magic triangle is

$$3s = 21 + v$$

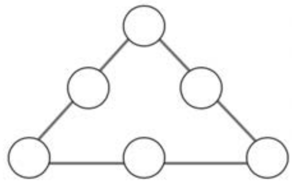

Figure 11.7. A magic triangle

Note that $21 - v$ is then the sum of the three edge numbers. Since v is a sum of three numbers, we have

$$1 + 2 + 3 = 6 \leq v \leq 4 + 5 + 6 = 15$$

Since $3s = 21 + v$, it follows that v is divisible by 3, and so we have that $v \in \{6, 9, 12, 15\}$, and so $s = \frac{1}{3}(21 + v) = 7 + \frac{v}{3} \in \{9, 10, 11, 12\}$. In fact, all of these are possible.

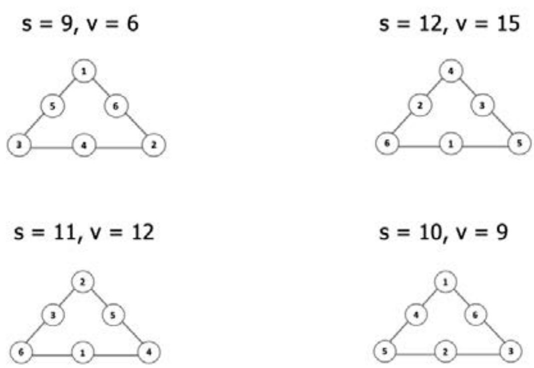

Figure 11.8. Solutions to the magic triangle problem

There are unique vertex sums to make $v = 6 = 1+2+3$ and $v = 15 = 4+5+6$. But $v = 9 = 1+2+6 = 1+3+5 = 2+3+4$ and $v = 12 = 1+5+6 = 2+4+6 = 3+4+5$. However, only $(1, 3, 5)$ for $v = 9$ and $(2, 4, 6)$ for $v = 12$ are realizable as vertex numbers for magic triangles. Given s and the vertex values, the edge values are determined. So, one can see by inspection that vertex values $(1, 2, 6)$ and $(2, 3, 4)$ for $v = 9, s = 10$ and $(1, 5, 6)$ and $(3, 4, 5)$ for $v = 12, s = 11$ are impossible. Thus, the four examples above are the only magic triangle, up to geometric symmetry.

Imagine that the circles are hinges on the triangle. Then if we pull the three edge circles outward we get a new triangle in which the edge circles now become the vertex circles. It is a nice exercise to show that, starting with a magic triangle with side sum s, this will produce a new magic triangle with side sum $21 - s$. This pairs the solutions with $s = 9$ and $s = 12$, and the solutions with $s = 10$ and $s = 11$.

11.2.4. Cross-domain problems. By a *cross-domain problem*, I mean a mathematical problem whose solution most naturally draws on resources from two or more mathematical domains (such as arithmetic, algebra, geometry, combinatorics, probability, calculus, etc.). As I stated in the Introduction, giving students such problems can help disrupt their tendency to type-cast a problem as belonging to some domain, and then unconsciously block the seeking of ideas or strategies from other domains. This may happen because the school curriculum often presents the domains in specialized, but disconnected courses. Here are some sample cross domain problems.

(1) *Show that any product of d consecutive intergers is divisible by $d!$.*

On its face, this is an arithmetic problem, difficult to solve directly in that context. But it is a corollary of perceptive observation of the combinatorial formula

$$\binom{n}{d} = \frac{n(n-1)(n-2)\cdots(n-d+1)}{d!}$$

(2) *Call a divisor d of a whole number N* separating *if* $\gcd(d, N/d) = 1$. *Find a formula for the number of separating divisors of N.*

This is clearly a number theory problem, but its solution draws on results from combinatorics.

(3) *Two ladders in a hallway are leaning against opposite walls, with their bases against opposite walls. They cross at 2 ft. above the floor. The (integer) heights of their tops above the floor are m ft. and n ft., respectively. What are the possibilities for (m, n)?*

This is clearly a geometry problem, but its solution requires finding all solutions of a Diophantine equation.

(4) In Figure 11.9, *Which shaded area is greater – the dark gray or the light gray?*

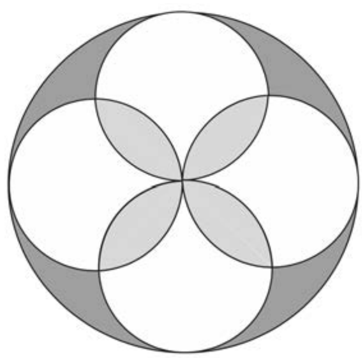

Figure 11.9. A problem in geometric measurement.

This is clearly a problem in geometric measurement, but it also entails some combinatorial reasoning.[4]

(5) *Circle Park has a network of circular trails for cyclists to use (see Figure* 11.10*). The trails have bridges so that they intersect only along the diagonal AB. Which is the shortest way to travel from A to B using this network of circular paths?*

This is a problem in (one-dimensional) geometric measure and proportionality.

[4]I thank Tetyana Berisovski for this problem. She has made many such problems based on beautiful geometric images.

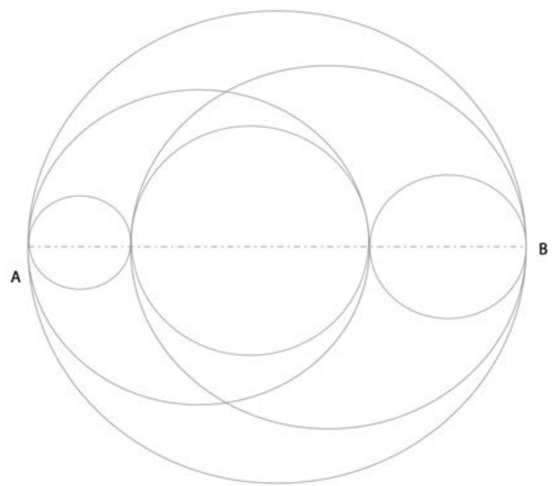

Figure 11.10. Circle Park

(6) *If a non-constant continuous periodic function $f(x)$ has two periods, p and q, how must p and q be related? What can be said about the set of all periods of f?*

While this is evidently a problem in analysis, its solution draws on results about additive groups of real numbers, and commensurability.

11.2.5. The problem of figuring out what the problems means. Mathematicians, as well as teachers and students are often unclear when discussing "order of operations," and what is the point of such conventions, often taught only as something to be memorized, perhaps with the support of some mnemonic. Here is a perhaps more productive approach to this issue.

Consider the expression

$$\frac{1}{2} - \frac{3}{4} \div \frac{5}{6} + \frac{7}{8} \times \frac{9}{10}$$

(a) Calculate the expression. Express the answer as a fraction, and as a finite decimal.
(b) By meaningfully inserting parentheses into the expression, find as many different values of the resulting expressions as you can.
(c) Prove, if you can, that you have found all possible values.
(d) Communicate the expression to another person by phone. Ask that person to calculate its value, and, on that basis, speculate which of your answers to (b) might have been used.

11.3. Connections between mathematical problems

We now consider ways that two (or more) mathematical problems may be mathematically related. Though the contexts of the problems may be mathematically significant, we do not allow that two problems, both involving stories about pizzas, to be, on that ground alone, considered to be mathematically related.

A very weak kind of relation is that the problems belong to a common mathematical domain (algebra, geometry, calculus, etc.). This relation can be sharpened by specifying a common subdomain, or a common problem type. For example, a common type of calculus problem is to ask students to evaluate a definite integral, or to maximize a differentiable function on an interval. These kinds of connections (such as same problem type) are still rather superficial. We want to detect deeper, structural, similarities between different problems. More importantly, there may be deep structural connections between two problems for which standard mathematical language affords no easy description. Thus, one might challenge students to not only discern such connections, but further, to find/invent the language needed to describe the connection(s). These are the kinds of issues discussed in this section.

11.3.1. Different problems that are mathematically the "same". Cognitive scientists call two problems A and B *isomorphic* if there is a bijection from the objects, relations, and allowable operations of A to those of B, so that a solution process of A is mapped to a solution process for B. We will then say that A and B are structurally the same. Here is a simple example:

(A) Arisha, Brianna, and Carmen run a race. Assuming no ties, what are all possible outcomes, 1st, 2nd, 3rd?

(B) In a 3×3 grid square, shade three of the nine (unit) squares in such a way that there is exactly one shaded square in each row and in each column. What are all ways of doing this?

One way to map (B) to (A) is to imagine a solution of (B) (see Figure 11.11) as a picture of the finish of the race, the finish line being at the top, with the columns, left-to-right, being the running lanes of Arisha, Brianna, Carmen, respectively, and the shaded squares the finishing positions of the runners. This gives a structure preserving bijection between the solution spaces of (B) and (A).

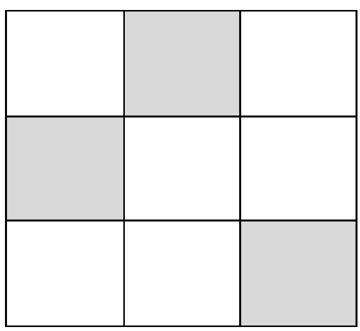

Figure 11.11. One solution to the 3×3 grid problem.

Problems (A) and (B) are mathematically the "same" not simply because there is a structure preserving bijection between their solution spaces, but because that bijection embodies a relation preserving bijection between the structural elements of the two solution spaces. For example, it does not suffice to say that the solution to both problems is the set of permutations of three objects. That is to say, problems with the same solution, or with the "same" solution space, are not necessarily the same structurally.

For example, consider the problem

(C) Find all symmetries of an equilateral triangle.

The solution space can be described as the set of permutations of the three vertices. Why is (C) not structurally the same as (A) and (B)? While there is a natural injection from the solution space of (C) to the set of permutations of the vertices, it is not intrinsic to the problem that this injection is bijective; that is an artifact of the case of equilateral triangles. For example, if, instead, we consider squares, or regular n-gons ($n \geq 3$) then the number of symmetries is $2n$, not $n!$ On the other hand, if we generalize (A) and (B) to $n \geq 3$, the analogous structural correspondence remains valid, and the solution spaces represent all $n!$ permutations.

11.3.2. Degrees of sameness; discernment tasks. Often problems may be significantly related without being structurally the same in the above sense. Of course, these connections can have a variety of forms, and with different levels of strength. This is something worth noticing, but may be hard to measure objectively or precisely. The important thing is to identify and give a mathematically explicit articulation of the nature of the connection. For example, with problems (A), (B), and (C) in 11.3.1, though (C) is not structurally the same as (A) and (B), all three problems can be described as determining some set of permutations of three objects, and that itself is a mathematical connection worth noting.

11.3.3. The Discernment Problem. To provide students with opportunities for such discernment I use the following task format:

Below are problems, labeled $A, B, C, \ldots$. The object is to place the letter of each problem in one of the boxes shown in Figure 11.12.

- Put letters in the same box if they are mathematically the "same" problem, apart from superficial differences of context.
- If problems in different boxes are closely related mathematically, connect their boxes by a line, or by a double line if the connection is very strong. (Note, you need not use all of the boxes, and you may reasonably answer this question even if you have not completely solved all of the individual problems.)
- Work first individually. Then compare and discuss answers in your group.
- With or without consensus, explain (to the whole group) your choices, in particular the nature of the connections.

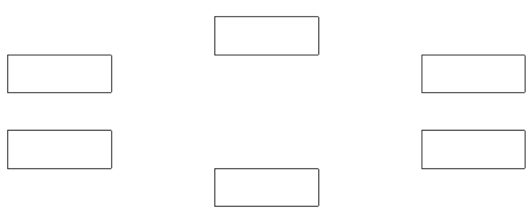

Figure 11.12. Boxes for the discernment problem

The student products in this format will be "connection networks," with explanation, and these can be quite variable. The processes of explaining them, and the efforts to reconcile differences across different groups can help to develop a probing discourse about structural relations among problems. The next two sections provide some problem sets used with the above discernment format.

11.3.4. The 3-permutation discernment problem set.

(A) What are all three-digit numbers that you can make using each of the digits 1, 2, 3, and using each digit exactly once?
(B) In a group of five students, how many ways are there to pick a team of three students?
(C) You are watching Arisha, Brianna, and Carmen on a merry-go-round. At each moment you see them in some order – left, middle, right. As the merry-go-round turns, what are all the different orders in which you see them?
(D) If Arisha, Brianna, and Carmen have a race, and there are no ties, what are all possible outcomes: first, second, third?
(E) From a bag full of many pennies, nickels, and quarters, I randomly choose three coins. What are all possible amounts of money that I might have?
(F) In a 3×3 grid square, color three of the nine unit squares blue, in such a way that there is exactly one blue square in each row and in each column. What are all the ways of doing this?
(G) What are all the symmetries of an equilateral triangle?

In this set, B and E are outliers, not deeply related to the other problems, or even to each other, despite the fact that they both have ten solutions. A structure involved in each of A, C, D, F, G, is the set of permutations of three objects (the vertices of the triangle in G), and the solution to each of these problems is in fact the full set of six permutations. In A, D, and F this outcome is demonstrably inherent in the problem, though this fact is least obvious for F. On the other hand, it can be argued that in C and G there is no a priori guarantee that all permutations will be achieved.

In piloting this example, with pre-service secondary teachers, all of them senior mathematics majors, they solved all of the individual problems without great difficulty, but they did not perceive the distinction of C and G from A, D, and F. To awaken their awareness of this, I asked them to formulate a parallel problem set with the number three replaced by four. For example, problem A became:

(A') What are all four-digit numbers that you can make using each of the digits 1, 2, 3, 4, and using each digit only once?

They easily constructed a four-based parallel problem set, A', B', . . . , G', for example with G' about symmetries of a square. I then asked them to repeat the connection network activity with this modified problem set. In this case, A', D', and F' still yielded the full set of $4!(= 24)$ permutations, whereas G' led to only $4 \cdot 2 = 8$ of them. Moreover C' leads to a surprisingly complex problem in combinatorial geometry that, suitably interpreted, leads to $4 \cdot 3 = 12$ solutions. Some of them were struck by the fact that the seemingly modest change from three to four made such a profound difference in the nature of the relations among the problems.

11.3.5. The 8-choose-3 discernment problem set.

(A) A taxi wants to drive from one corner to another that is 5 blocks north, and 3 blocks east. How many possible shortest routes are there to do this?
(B) On the number line, starting at 0, you are to take 8 steps, each of which is either distance 1 to the right, or distance 1 to the left, and in such a way that you end up at -2. How many different such walks are there?
(C) The home team won a soccer game 5 to 3. How many possible sequences of scoring were there as the game progressed?
(D) You have coins worth 3 cents and 5 cents. With 8 such coins, how many different values can you obtain?
(E) From a group of 8 students, you need to select a 5-person basketball team. How many different ways are there to do this?
(F) You are to cut a 9-inch ribbon into six pieces, each of length a whole number of inches. How many ways are there to do this?
(G) In the expansion of $(1+x)^8$, what is the coefficient of x^3?

Here there is one outlier, (D), with 9 solutions, $(8\cdot 3+2n)$ cents for $0 \leq n \leq 8$. The problem solution space of all of the other problems can be represented by the structure consisting of the set of all "binary sequences of length 8, with exactly 3 terms of one type." These are sequences $(x_1, x_2, \ldots x_8)$ in which each x_j takes one of two possible values, say 0 or 1, and exactly three of them take the value 1. In (A), the values would be N (north) and E (east). In (B) they would be -1 and $+1$. In (C) they would be H (home) and V (visitors). In (E), $x_j = 1$ if student j is not on the team, and 0 otherwise. In (F), $x_j = 0$ if you cut at inch j, $(1 \leq j \leq 8)$, and 1 otherwise. In (G), $x_j = 1$ or x, and the product of the x_js is x^3.

11.3.6. A more subtle example of common structure. Consider the following two problems:

(A) **The Wine and Tea Problem** Suppose I have a barrel of red wine, and you have a cup of green tea. Suppose also that I put a teaspoon of my wine into your cup of tea, then you take a teaspoon of the mixture in your teacup, and put it back into my wine barrel. Which is more now: the wine in the teacup or the tea in the wine barrel?
(E) **Trapezoid Diagonals Problem** The two diagonals a trapezoid divide the trapezoid into four triangles. What is the relation of the areas of the two triangles containing the legs (non-parallel sides) of the trapezoid?

At first sight, most observers see little, if any, mathematical connection between these two problems. I learned of (A) from Vladimir Arnold, who described it as a problem Russian parents give to very young, pre-mathematics-education children, who, according to Arnold, solve it more quickly and simply than mathematicians.

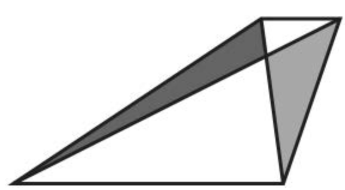

Figure 11.13. Trapezoid Diagonals Problem

I found problem (E) when trying to construct a geometric model of (A). Here is a solution to (E). Let **T** denote the trapezoid, and s and S its parallel sides, say at distance h apart. A diagonal d of **T** divides **T** into two triangles, t, with base s, and T, with base S, and both with height h. The other diagonal d' similarly divides **T** into triangles t' and T', and clearly Area(T) = Area(T'). It follows that Area$(T \setminus (T \cap T'))$ = Area$(T' \setminus (T \cap T'))$. Notice that $T \setminus (T \cap T')$ and $T' \setminus (T \cap T')$ are the two (shaded) triangles in Figure 11.13 containing the legs of **T**.

Now suppose that Area(**T**) represents the total volume of wine and tea in problem (A), and that t represents the tea in the teacup, and T represents the wine in the wine barrel. After the exchange, say t' represents the mixture in the teacup, and T' the mixture in the wine barrel. This makes sense since t and t', and T and T', have equal areas, respectively. Then $T \setminus T \cap T'$ represents the wine in the teacup, and $T' \setminus (T \cap T')$ represents the tea in the wine barrel. So, the amounts are the same.

This analysis reveals a (measurement) structure common to problems (A) and (E). Either problem can easily be solved independently, without this observation. But this structural connection has a mathematical significance beyond the separate solutions. I call these two *common structure problems.*

We'll next show that (A) and (E) are two of a set of five common structure problems, and how one can use this to frame a problem-solving activity to cultivate what I like to call "structure sense."

11.3.7. The Measure-Exchange common structure problem set.

(A) **The Wine and Tea Problem** Suppose I have a barrel of red wine, and you have a cup of green tea. Suppose also that I put a teaspoon of my wine into your cup of tea, then you take a teaspoon of the mixture in your teacup, and put it back into my wine barrel. Which is more now: the wine in the teacup or the tea in the wine barrel?

(B) **Heads Up** I place on the table a collection of pennies. I invite you to randomly select a set of these coins, as many as there were heads showing in the whole group. Next I ask you to turn over each coin in the set that you have chosen. Then I tell you, "The number of heads now showing in your group is the same as the number of heads in the complementary group." Explain how I know this is true.

(C) **Faces Up** I blindfold you and then place in front of you a standard deck of 52 playing cards in a single stack. I have placed exactly 13 of the cards face up, wherever I like in the deck. Your challenge, while still blindfolded, is to arrange the cards into two stacks so that each stack has the same number of face-up cards.

(D) **Tringale Medians** In a triangle, the medians from two vertices form two triangles that meet only at the intersection of the medians. How are the areas of these two triangles related? More precisely, let ABC be a triangle. Let A' be the mid-point of AC, B' the mid-point of BC, and D the intersection of AB' and BA'. How are the areas of $\triangle AA'D$ and $\triangle BB'D$ related?

(E) **Trapezoid Diagonals Problem** The two diagonals a trapezoid divide the trapezoid into four triangles. What is the relation of the areas of the two triangles containing the legs (non-parallel sides) of the trapezoid?

I have presented such common-structure problem sets with the following instructional format to students:

(1) Solve the individual problems.
(2) Identify and articulate a mathematical structure common to each of the problems, and demonstrate how it is involved in each of the problems.
(3) Notice ways in which the problems are significantly different, and analyze some of the consequences of these differences.

It is part 2 that is the most novel and most challenging aspect for the students, even when the individual problems have been solved. I do not provide a definition of "structure," thus leaving to the students what they deem to be a "structure" in the given context. Further, there may well be no ready-made language to describe what they have found, so they may have to improvise/invent descriptive language for this purpose. This is evidently cognitively challenging work, so I have found it best to assign it as collaborative small-group work, transacted over several lessons.

11.4. Different problems reducible to a common model

In a brief paper,[5] Zal Usiskin listed six nontrivial equivalent problems. Based in part on conversations with Usiskin, I have expanded his list to the thirteen problems displayed below. They are organized into four groups, belonging to different mathematics domains: arithmetic, rates, geometry, and algebra.

The connection-oriented mathematical thinking opportunity afforded by this problem set derives from the exposure it provides to a large set of quite diverse problems that are all modeled by essentially the same equation. Note, however, that these problems do not all share a common structure in the sense of 11.3. I describe below a design for using this problem set that I have used with some groups of teachers and teacher educators.

11.4.1. The expanded Usiskin problem set.

Arithmetic

(Ar_1) Find all ways to express $\frac{1}{2}$ as the sum of two unit fractions (that is, fractions of the form $1/n$, with n a positive integer).

(Ar_2) Find all rectangles with integer side lengths whose area and perimeter are numerically equal.

(Ar_3) The product of two integers is positive and twice their sum. What could those integers be?

(Ar_4) For which integers $n > 1$ does $n - 2$ divide $2n$?

Rates

(R_1) Which pairs of positive integers[6] have harmonic mean equal to 4?

(R_2) Nan can paint a house in n days, and her Mom can paint it in m days (n and m positive integers). Working together they can paint the house in 2 days. What are the possible values of (n, m)?

[5](Usiskin, Z. (1968). Six nontrivial equivalent problems. *The Mathematics Teacher*, 61(4), 388-390.)

[6]The harmonic mean h of n numbers $a_1, a_2, \ldots, a_n$ is such that

$$\frac{1}{h} = \frac{\sum_{i=1}^{n} \frac{1}{a_i}}{n}$$

(R_3) A turtle travels up a hill at n miles per hour, and returns down the hill at m miles per hour ($n \le m$ integers). Its average speed for the round trip is 4 miles per hour. What are the possible values of (n, m)?

Geometry

(G_1) Given a point P in the plane, find all integers n such that a small circular disk centered at P can by covered by non-overlapping congruent tiles shaped like regular n-gons that have P as a common vertex.

(G_2) Two vertical poles, N and M, have heights n meters and m meters, respectively, with $n, m \in \mathbb{Z}$. A wire is stretched from the top of pole N to the base of pole M, and another wire is stretched from the top of pole M to the base of pole N. These wires cross at a point 2 meters above the ground. What are the possible values of (n, m)?

(G_3) The base b and corresponding height h of a triangle are integers. A 2×2 square is inscribed in the triangle with one side on the given base, and other vertices on the other two sides. What are the possible values of the pair (b, h)?

Algebra

(A_1) For which positive numbers s does $p(x) = x^2 - sx + 2s$ have integer roots?

(A_2) Let u be a positive real number. Find all solutions (n, m, v) with n and m positive integers, and $v > 0$, of the equations:

$$(uv)^2 = u^n = v^m$$

(A_3) For which positive integers (m, n) is

$$2 \int_0^1 (x^{m-1} + x^{n-1})\, \mathrm{d}x = 1$$

Usiskin's original list of six problems is Ar_1, Ar_2, Ar_3, Ar_4, R_1, and G_1 above. I thank Zal for discussion of some of the other problems as well.

11.4.2. Presentation to the students. These problems appear to be quite diverse. I asked each student to choose one problem-theme, and so formed four student groups (arithmetic group, rates group, geometry group, and algebra group), each group to work collaboratively on its chosen thematic problem set, but the groups worked independently. Student choices were based mainly on things like the parts of mathematics they liked best or felt most confident with, or on which problems seemed, at first appearance, easiest for them to solve. Each group was assigned to solve its problem set, and to present its solutions to the class the following week. They were free to consult outside resources, including myself, but to prepare a presentation that would instruct the rest of the class about what they found, and what they found to be difficult.

I deliberately said little about the overall mathematical point of this assignment, other than to solve, and seek relations among, each group's interesting set of problems. In particular, there was no suggestion of why these particular problem sets were collected together, in particular that they might all be mathematically related in some way.[7]

[7]This design of the instruction was based on a model that Davida Fischman used for a professional development session when I gave her this problem set.

I had the groups make their presentations in the order of the list, the arithmetic group first. Problems Ar_1, Ar_2, Ar_3, Ar_4, in order, lead directly to the following Diophantine[8] equations:

$$\frac{1}{n} + \frac{1}{m} = \frac{1}{2} \tag{11.4.2.1}$$

$$2(n+m) = nm \tag{11.4.2.2}$$

$$nm = 2(n+m) > 0 \tag{11.4.2.3}$$

$$\text{For which } n > 1 \text{ is } 2n = (n-2)m \text{ for some integer } m? \tag{11.4.2.4}$$

Moreover, it is not difficult to see how equations (11.4.2.1)–(11.4.2.4), are algebraically equivalent. Hence, solving any one of them solves the others.

My students generally preferred to use (11.4.2.3) to express m in terms of n:

$$m = \frac{2n}{n-2} \tag{11.4.2.5}$$

They then did numerical experiments to find those n for which $2n/(n-2)$ is an integer.[9] The solutions they found were $(4,4), (3,6)$, or $(6,3)$. None of the students tried to work directly with (11.4.2.1), which is my preferred approach. Using the symmetric roles of m and n, we can assume that $n \le m$. Then $n \ge 3$; otherwise $1/n \ge 1/2$. Also $n \le 4$; otherwise $1/n + 1/m < 1/2$. Thus, either $n = 3$ (and so $m = 6$) or $n = 4$ (and $m = 4$).

Students working on the **Rate** problems gave an excellent survey of problems in which the harmonic mean arises. Problem R_1 corresponds to the equation

$$\frac{1}{4} = \frac{1}{2}\left(\frac{1}{n} + \frac{1}{m}\right) \tag{11.4.2.6}$$

which is (11.4.2.1) multiplied by 1/2. For R_3, if one travels distance d at speed v in time t, then $d = vt$ and $t = d/v$. Now suppose that one travels distance d at speed v_1 in time t_1, and then returns at speed v_2 in time t_2. Then the average speed for the whole trip is

$$\begin{aligned} v_{\text{avg}} &= \frac{\text{total distance}}{\text{total time}} \\ &= \frac{2d}{t_1 + t_2} \\ &= \frac{2d}{d/v_1 + d/v_2} \\ &= \frac{2}{1/v_1 + 1/v_2} \end{aligned}$$

Thus,

$$\frac{1}{v_{\text{avg}}} = \frac{1}{2}\left(\frac{1}{v_1} + \frac{1}{v_2}\right)$$

In other words, v_{avg} is the harmonic mean of v_1 and v_2. In problem R_2, d would be the work of painting the house, and n and m describe the rates at which Nan and

[8] "Diophantine" because one seeks (positive) integer solutions.

[9] Some students even graphed m in (11.4.2.5) as a function of $n > 0$, and highlighted the integer points on the graph.

her Mom do that job. The rate of doing it together (analogous to average speed) is the harmonic mean of the two rates.

Of course, the rate group sees that its work to solve (11.4.2.6), and hence each of the rate problems, is the same as the work already shown by the arithmetic group.

The **geometry** problems were less obviously related, but they too led to the same Diophantine equations. In problem G_1, let $\alpha(n)$ denote the (equal) interior angle(s) of a regular n-gon. Then it is known that $\alpha(n) = \frac{n-2}{n} \cdot 180°$. For some number, say m, of these regular n-gons to fit together to cover the area around a point P, we would need $m \cdot \frac{n-2}{n} \cdot 180° = 360°$. In other words,

$$m(n-2) = 2n$$

which is (11.4.2.4), the same equation treated by the arithmetic group.

For G_2 (also known as the "crossing ladders" problem), consider Figure 11.14. Using similar triangles, we have $(a+b)/n = b/2$ and $(a+b)/m) = a/2$. Adding these equations and dividing by $a+b$ gives

$$1/n + 1/m = 1/2$$

which is the same as (11.4.2.1).

For G_3, consider Figure 11.15. The big triangle and the one above the square are similar (since corresponding sides are parallel), and so $h/b = (h-2)/2$. Multiplying this by $2b$, we get

$$2h = b(h-2)$$

equivalent to (11.4.2.4), the same equation already treated by the arithmetic group.

The **algebra** group was the most challenged, because, it seems, the relevant algebraic methods were less familiar. In Al_1, they first tried using the quadratic formula, which did not conveniently make available the information that the roots are integers. If instead, we formally factor p

$$p(x) = x^2 - sx + 2s = (x-n)(x-m)$$

with $n, m \in \mathbb{Z}$, we find that $n+m = s$ and $nm = 2s$, whence n and m are positive, since s is, and so we have the equation

$$nm = 2(n+m)$$

which is equivalent to (11.4.2.2), already treated by the arithmetic group.

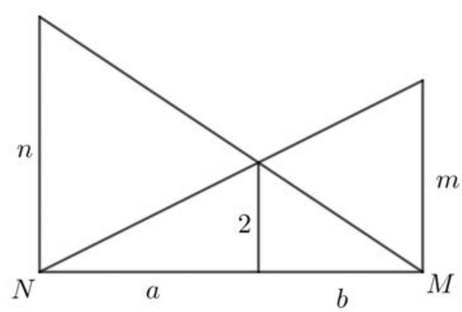

Figure 11.14. Crossing Ladders Problem

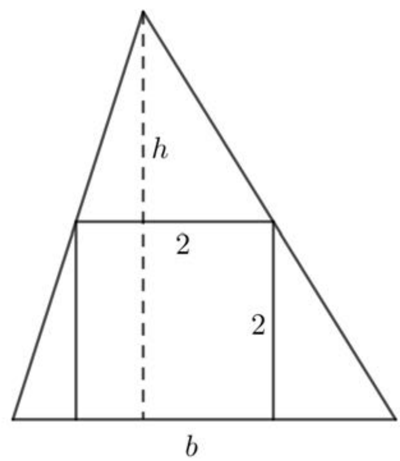

Figure 11.15. Diagram for Problem G_3

Students needed the most help with Al_2, where the mathematics is mainly happening in the exponents:

$$(uv)^2 = u^n = v^m$$

We first get, from $(uv)^2 = v^m$,

$$u^2 = v^{m-2} \Rightarrow v = u^{2/(m-2)}$$

Then, substituting v in $(uv)^2 = u^n$ gives $\left(uu^{2/(m-2)}\right)^2 = u^n$. Equating exponents then gives

$$n = 2\left(1 + \frac{2}{m-2}\right) = \frac{2m}{m-2}$$

whence, again, the same equation

$$2m = \frac{n}{m-2}$$

seen in (11.4.2.4).

In A_3, the evaluating the integral gives

$$2\left(\frac{x^m}{m} + \frac{x^n}{n}\right)\Bigg|_0^1 = 2\left(\frac{1}{m} + \frac{1}{n}\right)$$

and so

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{2}$$

11.4.3. Student reflections and further connections.

- The students were all surprised, and intrigued, to see that their diverse problems all led to essentially the same (Diophantine) equation:

  $$\frac{1}{n} + \frac{1}{m} = \frac{1}{2}$$

  and its variants.

- Many students wondered whether there was some way that they could have anticipated this commonality; but they saw no simple way they could have done this.
- Some students researched the web to see if they could find some standard discussion, or identification of this basic equation. The closest thing to this was connection with the harmonic mean, found by the rates group.
- This led many students to ask me how I found all these different problems with the "same" solutions, thinking that I was somehow being "sneaky."
- So, at this stage, the phenomenon seemed more like an intriguing coincidence, or uncanny craftiness – that so many different looking problems could be modeled by the same equation (and its variants).
- Apart from that, it did not seem to provide any more general kind of mathematical insight beyond the immediate observation that a single equation could mathematically model an extraordinary variety of mathematical problems.
- Nonetheless, everyone found the activity to be interesting and worthwhile, and looked forward to more of this kind of activity.

11.4.4. Relation to the classification of Platonic solids. Partly in response to this reaction, I went on to engage a quick discussion of Platonic Solids, and to illustrate how their classification could be carried out by solving a Diophantine equation that is a slight variant of equation (11.4.2.1) above. The point of this was to further emphasize that such Diophantine equations arise usefully in still more diverse contexts. This derivation is presented here in section 5.10. Using Euler's formula (5.10.2), the classification is based on the solutions of the Diophantine equation,

$$\frac{1}{2} + \frac{1}{E} = \frac{1}{n} + \frac{1}{m}$$

which is a variant of (11.4.2.1), and solved by similar methods.

11.5. Problems related to the Euclidean Algorithm

In this section, we recall the Euclidean Algorithm,[10] survey a surprising variety of topics related to it, and formulate a list of problems, labeled $(EA_1), (EA_2), \ldots,$ (EA_{11}), focused on these connections.

11.5.1. Division with Remainder (DwR). Given $a, b \in \mathbb{R}$, $b > 0$, we call

$$a = qb + r$$

a *DwR-equation* if $q \in \mathbb{Z}$ and $0 \leq r < b$. For given (a, b), there is a unique DwR-equation.[11] Moreover, if $a > b$, then $q > 0$.

11.5.2. The Euclidean Algorithm Process. The Euclidean Algorithm (EA) is the following process: Starting with real numbers $a \geq b > 0$, put $a_0 = a$

[10] See Section 1.2

[11] See Theorem 1.1.1.

and $a_1 = b$, and inductively construct,

$$\mathbf{EA_n}(a,b) = \begin{cases} a_0 &= q_1 a_1 + a_2 \\ a_1 &= q_2 a_2 + a_3 \\ a_2 &= q_3 a_3 + a_4 \\ &\vdots \\ a_{n-2} &= q_{n-1}a_{n-1} + a_n \\ a_{n-1} &= q_n a_n + a_{n+1} \end{cases}$$

so that each equation is a DwR-equation, and so, $a_0 \geq a_1 > a_2 > \cdots > a_n > a_{n+1} \geq 0$. If $a_{n+1} = 0$, we say that (EA) *terminates* at stage n, and we define

$$(*) \qquad a_n = \gcd(a, b)$$

If, on the other hand, $a_{n+1} > 0$, we can continue the process to construct $\mathbf{EA}_{n+1}(a,b)$. If $\mathbf{EA_n}(a,b)$ never terminates, then the (EA) produces an infinite, descending sequence of positive real numbers $a_0 > a_1 > a_2 \cdots > a_n > a_{n+1} > \ldots$.

Here are the first Euclidean Problems.

(EA_1) For $a, b \in \mathbb{Z}$, why does (*) agree with other definitions of $\gcd(a,b)$?[12]
(EA_2) Under what conditions on (a,b) does $\mathbf{EA_n}(a,b)$ terminate?[13]
(EA_3) If $\mathbf{EA_n}(a,b)$ is non-terminating, what is $\lim_n a_n$?[14]

11.5.3. The Euclidean square-tiling of an $a \times b$ rectangle $R(a,b)$. Let $a \geq b > 0$ be real numbers, as above. Suppose that we want to tile the $a \times b$ rectangle $R = R(a,b)$ with square tiles (of variable size). The Euclidean ("greedy") algorithm for square tiling is to first fill as much of R as possible with the largest possible $b \times b$ square tiles. Writing $a = qb + r$ by (DwR), we see that we can fit q ($b \times b$)-tiles, and what remains is a rectangle $R(b,r)$. Continuing in the same way with $R(b,r)$, etc., we end up producing what we call the Euclidean tiling T_E of $R(a,b)$. This provides a geometric picture of the Euclidean Algorithm $\mathbf{EA}(a,b)$. T_E is a finite tiling if and only if $\mathbf{EA}(a,b)$ terminates.[15]

(EA_4) With notation as in the systems of equations $\mathbf{EA_n}(a,b)$ above, assume that $a_{n+1} = 0$. For $e \in \{0,1,2\}$, define $S_e(a,b) = \sum_{1 \leq j} q_j a_j^e$.
  (a) Interpret $S_e(a,b)$ geometrically for each value of e.
  (b) Evaluate $S_e(a,b)$ for $e = 1$ and $e = 2$.

---

[12] It suffices to show, for each definition, that $\gcd(a - qb, b) = \gcd(a,b)$ for all $q \in \mathbb{Z}$.

[13] The answer is furnished by Theorem 1.2.3 and Subsection 1.2.4.

[14] The answer is furnished by Theorem 1.2.3.

[15] It is a classical theorem of Max Dehn that a rectangle admits a finite square tiling if and only if its side lengths are commensurable. See (Bass, Hyman (2011). "Vignette of Doing Mathematics: A Meta-cognitive Tour of the Production of Some Elementary Mathematics," *The Mathematics Enthusiast*: Vol. 8 : No. 1 , Article 2.)

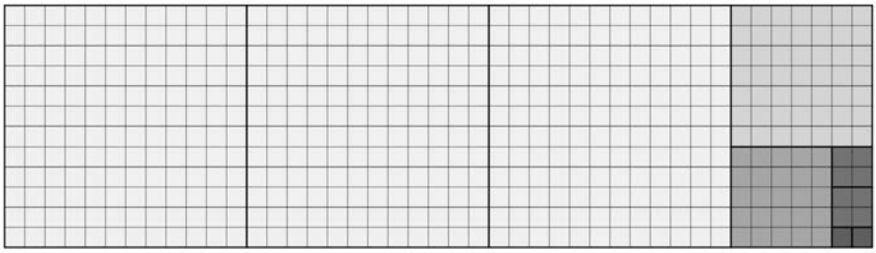

Figure 11.16. The Euclidean Tiling of $R(43, 12)$.

To answer (a), see that $S_0(a,b) = \sum_{1\le j} q_j$, which is the number of tiles of T_E, and $S_2(a,b) = \sum_{1\le j} q_j a_j^2 = ab$, which is the area of $R(a,b)$. What about $S_1(a,b)$? Observe that $S_1(a,b) = \sum_{1\le j} q_j a_j$ gives the sum of the side-lengths of the tiles in T_E, and we claim that $S_1(a,b) = a + b - \gcd(a,b)$, if $\mathbf{EA_n}(a,b)$ terminates. To see this, add the equations in $\mathbf{EA_n}(a,b)$ and simplify:

$$\sum_{0\le j\le n-1} a_j = \sum_{1\le j\le n} q_j a_j + \sum_{2\le j\le n} a_j$$

More generally, for any square tiling T of a rectangle R, we shall write $p(T)$ for the sum of the side lengths of the tiles in T, and call this the *perimeter of* T. Thus,

$$S_1(a,b) = \sum_{1\le j} q_j a_j = p(T_E)$$

11.5.4. Equi-distribution. Suppose that we want to distribute c cakes equally among s students ($c < s$). Then each student will receive c/s of a cake. We shall write $p(s,c)$ for the *minimum number of cake-pieces* needed to make this distribution. For any distribution D, we'll write $p(D) \ge p(s,c)$ for the number of cake-pieces in that distribution.

(EA_5) The *Euclidean distribution*, D_E, proceeds as follows: First cut a (c/s)-size piece from each of the c cakes, and distribute these pieces to c of the students. Then there remain c partial cakes, of size $(1-c/s)$ to be equally shared among $(s-c)$ students. Continue in the same manner with this (reduced) distribution, etc. Show that $p(D_E) = p(T_E)$, where T_E is the Euclidean tiling of $R(s,c)$.[16]

(EA_6) The *linear distribution*, D_L, proceeds as follows: Imagine that the cakes are thin rectangles in shape. Line them up end to end, treat this as one long "mega-cake", and cut it into s equal pieces, which are then the student shares. Show that $p(D_L) = p(D_E)$.[17]

[16]Suggestion: Imagine that the cakes are $1 \times s$ rectangles. Stack them in a pile to form a $c \times s$ rectangle R. Then relate the Euclidean distribution to the Euclidean square tiling of R.

[17]Suggestion: Choose linear units so that each rectangular cake has length s. Then, end-to-end they have total length $c \cdot s$. The cake separations occur at multiples of c, so there are $s-1$ of these. The student share separations (cuts) are at multiples of s; there are $c-1$ of these. The cuts common to these two sets occur at multiples of $m = \mathrm{lcm}(s,c)$. We have $c \cdot s = d \cdot m$, $d = \gcd(s,c)$ (from 3.6.10), so there are $d-1$ common cuts. Hence the total number of cuts is,

$$(s-1) + (c-1) - (d-1) = (s+c-d) - 1.$$

It is shown in Bass (loc. cit.) that $p(D_L) = p(D_E) = s + c - \gcd(s, c)$ is, in fact, the minimum number, $p(s, c)$, of cake pieces for any cake distribution. Further, for any square tiling T of a $c \times s$-rectangle, $p(T) \geq p(T_E) = p(s, c) = s + c - \gcd(s, c)$, a kind of Isoperimetric Theorem.

11.5.5. Multi-equi-distribution. Let M be some measurable quantity. A partition of M is a decomposition P of M into finitely many disjoint pieces. Let $p(P) = |P|$ be the number of pieces of P. A *refinement* of P is obtained by further partitioning each of the pieces of P. For a partition Q of M, we write $Q \rtimes P$ if Q is equivalent to a refinement of P.

For an integer $s \geq 1$, let $E(s)$ denote a partition of M into s pieces of equal measure. Let $(s_1, s_2, \ldots, s_n)$ be a sequenc of positive integers. Call a partition P of M an $(s_1, s_2, \ldots s_n)$-*equi-partition* if $P \rtimes E(s_j)$ for $1 \leq j \leq n$.

Let $p(s_1, s_2, \ldots s_n)$ be the least number of pieces of an $(s_1, s_2, \ldots, s_n)$-equi-partition of M. Clearly, $p(s_1, s_2, \ldots s_m) \leq m = \operatorname{lcm}(s_1, s_2, \ldots, s_n)$. In fact, $E(m)$ is an $(s_1, s_2, \ldots, s_n)$-equi-partition of M.

(EA_7) (This is an open problem.) Determine $p(s_1, s_2, \ldots, s_n)$.

Discussion of (EA_7). In the previous problem of sharing c cakes among s students, c and s play symmetric roles mathematically, so one can see that minimizing the number of cake pieces for an equi-distribution is equivalent to the case $n = 2$ of (EA_7). Hence, by the result cited in the footnote above, $p(s_1, s_2) = s_1 + s_2 - \gcd(s_1, s_2)$.

Imagine that M is measurably represented by the interval $[0, S]$, where $S = s_1 \cdot s_2 \cdots s_n$. Write $S = s_j s'_j$ for $1 \leq j \leq n$. Then we can realize $E(s_j)$ with the set C_j of $s_j - 1$ cuts of M at ts'_j with $1 \leq t \leq s_j - 1$.

Let $L(s_1, s_2, \ldots, s_n)$ be the partition of M obtained using all of the cuts $C = C_1 \cup C_2 \cup \cdots \cup C_n$. Clearly, $L(s_1, s_2, \ldots, s_n)$ is an $(s_1, s_2, \ldots, s_n)$-equi-partition of M. Hence,

$$p(s_1, s_2, \ldots, s_n) \leq p(L(s_1, s_2, \ldots, s_n)) = |C| + 1$$

(EA_8) Find an arithmetic formula for $p(L(s_1, s_2, \ldots, s_n))$.[18]

11.5.6. The diagonal of a $c \times s$ rectangle.

(EA_9) Consider the rectangle $R = R(c, s)$ tiled by $s \cdot c$ unit squares. Let Δ be a diagonal of R. Show that the number of unit tiles that Δ enters is $p(D_L)$.

Say R has vertices $(0, 0), (0, c), (s, 0)$ and (s, c), and Δ joins $(0, 0)$ to (c, s). Apart from the top edge and right edge of R, Δ meets c horizontal grid lines and s vertical grid lines. At each of these intersections, Δ enters a new unit tile, for at most $c + s$ unit tiles. If the crossing occurs at a grid vertex, the intersection of a horizontal and a vertical grid line, then the new unit tile entered is counted twice. Thus, the number of unit tiles that Δ enters is $c + s - d$, where d is the number of grid vertices other than (s, c) on Δ. Let $d = \gcd(s, c)$, and write $(s, c) = d \cdot (s', c')$, with $\gcd(s', c') = 1$. Then the set of grid vertices, other than (s, c), on Δ is $\{t(s', c') \mid 0 \leq t \leq d - 1\}$. Hence, the number of unit tiles through which Δ passes is $c + s - \gcd(c, s)$.

And so, the number of cake pieces is $s + c - \gcd(s, c) = S_1(s, c)$ as in (EA_4).

[18] Suggestion: use the inclusion-exclusion formula—Theorem 7.4.3—to find $|C|$.

This general line of investigation can be interestingly (and accessibly) pursued toward a discussion of the continued fraction representation of $a/b = a_0/a_1$ in terms of $(q_1, q_2, \ldots, q_n)$.

11.5.7. Continued fractions. Recall the Euclidean Algorithm from 11.5.2: Starting with real numbers $a \geq b > 0$, put $a_0 = a$ and $a_1 = b$, and inductively construct

$$\mathbf{EA_n}(a,b) = \begin{cases} a_0 & = q_1 a_1 + a_2 \\ a_1 & = q_2 a_2 + a_3 \\ a_2 & = q_3 a_3 + a_4 \\ & \vdots \\ a_{n-2} & = q_{n-1} a_{n-1} + a_n \\ a_{n-1} & = q_n a_n + a_{n+1} \end{cases}$$

so that each equation is a DwR equation, and so, $a_0 \geq a_1 > a_2 > \cdots > a_n > a_{n+1} \geq 0$.

For $0 \leq h < n$, put $R_h = a_h/a_{h+1} \geq 1$, so that $R_{h-1} = q_h + 1/R_h$. Hence,

$$\begin{aligned} R_0 &= q_1 + \frac{1}{R_1} \\ &= q_1 + \frac{1}{q_2 + \frac{1}{R_2}} \\ &= q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \frac{1}{R_3}}} \\ &= q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \frac{1}{q_4 + \frac{1}{R_4}}}} \\ &\vdots \end{aligned}$$

Given a finite sequnce $(q_1, q_2, \ldots, q_n)$ of positive integers, define the fraction $[q_1, q_2, \ldots, q_n]$ inductively by

$$[q_1] = q_1 \text{ and } [q_1, q_2, \ldots, q_n] = q_1 + \frac{1}{[q_2, q_3, \ldots, q_n]} \tag{11.5.7.1}$$

These expression are called *simple continued fractions*. One reason for their interest is that they give, computationally, the most efficient rational approximations to irrational numbers.

(EA_{10}) Let q be a positive integer, and put $\phi(q) = [q, q, q \ldots]$.

(a) Suppose that $x > 0$ and $x = q + 1/x$. Show that

$$x = \phi(q) = \frac{1}{2}\left(q + (q^2+4)^{1/2}\right)$$

$\phi(1) = (1+\sqrt{5})/2$ is called the *Golden Ratio*, and $\phi(2s) = s + (s^2 + 1)^{1/2}$, so $\phi(2) = 1 + \sqrt{2}$.

(b) Suppose that $x > 0$, and $x = 1 + 1/(q + x)$. Show that

$$x = [1, q+1, q+1, \ldots] = 1 + \frac{1}{\phi(q+1)}$$

and

$$\begin{aligned} x &= \frac{1}{2}\left(1 - q + ((q-1)^2 + 4(q+1))^{1/2}\right) \\ &= \sqrt{2} && \text{when } q = 1 \end{aligned}$$

11.5.8. Tony Gardiner's game of Euclid. This game is due to Tony Gardiner.[19] Each of the two players chooses a positive whole number and records it secretly. The two players then toss to decide who should start, before revealing their chosen numbers - say a and b. Player 1 then changes the pair a, b by subtracting any positive integer multiple of the smaller number from the larger to produce a new pair a', b'. Negative numbers are forbidden. Player 2 can then transform the new pair a', b' in the same way, and so on. The first player to produce a pair in which one of the two numbers is zero is the winner.

For example, suppose the players choose 43 and 8, respectively. The first player transforms $(43, 8) \to_1 (11, 8)$. The second player then transforms $(11, 8) \to_2 (3, 8)$. The game continues:

$$\begin{aligned} (3,8) &\to_1 (3,2) \\ (3,2) &\to_2 (3,1) \\ (3,1) &\to_1 (0,1) \end{aligned}$$

so Player 1 wins.

(EA_{11}) When can Player 1 force a win? How then should they play in order to win?

This is a challenging problem to formulate a precise answer, but the idea for constructing a suitable strategy is rather simple. You would like, if possible, to choose a move that forces your opponent's next move to be to your advantage. When, and how, is this possible? Think about the range of choices available at each move.

11.6. The magical marriage of two games

This section is based on something I learned in a conversation with Alan Schoenfeld.

11.6.1. Tic-Tac-Toe. The familiar game of tic-tac-toe involves two players, X and O, who take turns placing an X or an O, respectively, in the empty cells of a 3×3 grid. A winner is the first player to create a line of Xs or of Os, respectively. Except for the beginning, the play is mainly defensive, blocking the opponent from completing a line. Player 1 could be motivated to occupy the center cell, since it lies on the most (four) lines, thus blocking those lines to the opponent, and affording the most opportunities for a winning line. If Player 2 then chooses an edge cell, Player 1 can win, as follows: Occupy an adjacent vertex cell. Then Player 2 must block a diagonal. Then Player 1 chooses the vertex cell whose adjacent edge cells are unoccupied. This is a winning position. If, on the other hand, Player 2 first occupies a vertex cell, then Player 1 can at least force a tie.

[19](Gardiner, A. (2002) *Understanding Infinity.* Dover Publications, 2002.)

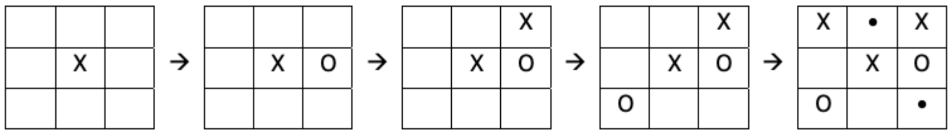

Figure 11.17. A winning position in tic-tac-toe

11.6.2. The 15-game. Two players, A and B, take turns choosing integers between 1 and 9, inclusive. Once a player has chosen a particular number, it can no longer be chosen by either player. The winner, if any, is the first player with three numbers that sum to 15.[20]

What are all the possible ways to get 15 as the sum of three of these numbers? There are various ways to answer this. For example, group the solutions by the largest of the three numbers involved:

Largest Number	
9	The other two add up to $6 : 1+5,\ 2+4$
8	The other two add up to $7 : 1+6,\ 2+5,\ 3+4$
7	The other two add up to $8 : 2+6,\ 3+5$
6	The other two add up to $9 : 4+5$

Thus, there are eight solutions:

$$(1,5,9),(2,4,9),(1,6,8),(2,5,8),(3,4,8),(2,6,7),(3,5,7),(4,5,6)$$

Grouping the numbers according to how many solutions contain them:

In two solutions:	$1,3,7,9$
In three solutions:	$2,4,6,8$
In four solutions:	5

11.6.3. Magic squares. We want to place the integers 1 through 9 into the nine cells of a 3×3 grid square so that the sum, s, of the numbers in each row, column, and diagonal is the same. Since $3s = 1+2+3+4+5+6+7+8+9 = 45$, then $s = 15$.

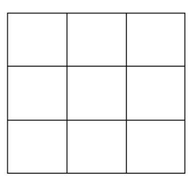

Figure 11.18. An empty grid.

In view of the discussion in 11.6.2, to make a magic square we must put 5 in the center cell, the even numbers in the vertex cells, and the remaining odd numbers in the edge cells. This determines the magic square up to geometric symmetry (rotations and reflections).

[20]This is most easily played with physical cards that have the integers 1 through 9 on them.

4	3	8
9	5	1
2	7	6

Figure 11.19. A solution of the magic square.

11.6.4. A strategy for the 15-game. If you are playing the 15-game, make a magic square. As each player chooses a number, place an X (if you are the player), or O, respectively in the cell occupied by that number. That converts the 15-game into a game of tic-tac-toe, without having to calculate sums.