Preface

> For more than two thousand years some familiarity with mathematics has been regarded as an indispensable part of the intellectual equipment of every cultured person. Today the traditional place of mathematics in education is in grave danger.

These opening sentences to the preface of the classical book *What Is Mathematics?* were written by Richard Courant in 1941. It is somewhat soothing to learn that the problems that we tend to associate with the current situation were equally acute sixty-five years ago (and, most probably, way earlier as well). This is not to say that there are no clouds on the horizon, and by this book we hope to make a modest contribution to the continuation of the mathematical culture.

The first mathematical book that one of our mathematical heroes, Vladimir Arnold, read at the age of twelve was *Von Zahlen und Figuren*[1] by Hans Rademacher and Otto Toeplitz. In his interview given to the "Kvant" magazine, published in 1990, Arnold recalls that he worked on the book slowly, a few pages a day. We cannot help hoping that our book will play a similar role in the mathematical development of some prominent mathematician of the future.

We hope that this book will be of interest to anyone who likes mathematics, from high school students to accomplished researchers. We do not promise an easy ride: the majority of results are proved, and it will take considerable effort from the reader to follow the details of the arguments. We hope that as a reward the reader, at least sometimes, will be filled with awe by the harmony of the subject (this feeling is what drives most mathematicians in their work!). To quote from *A Mathematician's Apology* by G. H. Hardy,

> The mathematician's patterns, like the painter's or the poet's, must be *beautiful*; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.

For us too, beauty is the first test in the choice of topics for our own research, as well as the subject for popular articles and lectures, and consequently, in the choice of material for this book. We did not restrict ourselves to any particular area (say, number theory or geometry); our emphasis is on the diversity and the unity of mathematics. If, after reading our book, the reader becomes interested in a more systematic exposition of any particular subject, (s)he can easily find good sources in the literature.

About the subtitle: the dictionary definition of the word *classic* is "judged over a period of time to be of the highest quality and outstanding of its kind".

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[1] *The Enjoyment of Mathematics*, in the English translation; the Russian title was a literal translation of the German original.

We tried to select mathematics satisfying this rigorous criterion. The reader will find here theorems of Isaac Newton and Leonhard Euler, Augustin Louis Cauchy and Carl Gustav Jacob Jacobi, Michel Chasles and Pafnuty Chebyshev, Max Dehn and James Alexander, and many other great mathematicians of the past. Quite often we include recent results of prominent contemporary mathematicians, such as Robert Connelly, John Conway and Vladimir Arnold.

There are about four hundred figures in this book. We fully agree with the dictum that a picture is worth a thousand words. The figures are mathematically precise—so a cubic curve is drawn by a computer as a locus of points satisfying an equation of degree three. In particular, the figures illustrate the importance of accurate drawing as an experimental tool in geometrical research. Two examples are given in Lecture 29: the Money-Coutts Theorem, discovered by accurate drawing as late as in the 1970s, and a very recent theorem by Richard Schwartz on the Poncelet grid which he discovered by computer experimentation. Another example of using the computer as an experimental tool is given in Lecture 3 (see the discussion of "privileged exponents").

We did not try to make the lectures similar in length and level of difficulty: some are quite long and involved whereas others are considerably shorter and lighter. One lecture, "Cusps", stands out: it contains no proofs but only numerous examples, richly illustrated by figures; many of these examples are rigorously treated in other lectures. The lectures are independent of each other, but the reader will notice some themes that reappear throughout the book. We do not assume much by way of preliminary knowledge: a standard calculus course will do in most cases, and quite often even calculus is not required (and this relatively low threshold does not leave out mathematically inclined high school students). We also believe that any reader, no matter how sophisticated, will find surprises in almost every lecture.

There are about two hundred exercises in the book, many provided with solutions or answers. They further develop the topics discussed in the lectures; in many cases, they involve more advanced mathematics (then, instead of a solution, we give references to the literature). More difficult exercises are marked by a single or a double asterisk.

This book stems from a good many articles we wrote for the Russian magazine "Kvant" over the years 1970–1990[2] and from numerous lectures that we gave over the years to various audiences in the Soviet Union and the United States (where we have lived since 1990). These include advanced high school students—the participants of the Canada/USA Binational Mathematical Camp in 2001 and 2002, undergraduate students attending the Mathematics Advanced Study Semesters (MASS) program at Penn State over the years 2000–2006, high school students—along with their teachers and parents—attending the Bay Area Mathematical Circle at Berkeley.

The book may be used for an undergraduate Honors Mathematics Seminar (there is more than enough material for a full academic year), various topics courses, Mathematical Clubs at high school or college, or simply as a "coffee table book" to browse through, at one's leisure.

To support the "coffee table book" claim, this volume is lavishly illustrated by an accomplished artist, Sergey Ivanov. Sergey was the artist-in-chief of the "Kvant" magazine in the 1980s and then continued in a similar position in the

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[2] Available, in Russian, online at http://kvant.mccme.ru/.

1990s at its English-language cousin "Quantum". Being a physicist by education, Ivanov's illustrations are not only aesthetically attractive, but they also reflect the mathematical content of the material.

We started this preface with a quotation; let us finish with another one. Max Dehn, whose theorems are mentioned here more than once, thus characterized mathematicians in his 1928 address [**22**]; we believe his words apply to the subject of this book:

> At times the mathematician has the passion of a poet or a conqueror, the rigor of his arguments is that of a responsible statesman or, more simply, of a concerned father, and his tolerance and resignation are those of an old sage; he is revolutionary and conservative, skeptical and yet faithfully optimistic.

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