Instructor Preface

This text is intended for a one-semester course on geometry. We have tried to write a book that honors the Greek tradition of synthetic geometry and at the same time takes Felix Klein's **Erlanger Programm** seriously. The primary focus is on transformations of the plane, specifically isometries (rigid motions) and similarities, but every effort is made to integrate transformations with the traditional geometry of lines, triangles, and circles. On one hand, we discuss in detail the concrete properties of transformations *as geometric objects*; on the other hand, we try to show by example how transformations can be used *as tools* to prove interesting theorems, sometimes with greater insight than traditional methods provide.

We have been surprised and pleased at how far this idea can be taken. We hope we have made concrete the usually abstract dictum of the *Erlanger Programm*:

a geometry is determined by its symmetry group.

For example, we have tried to show the intimate relationship between *Fagnano's Problem* (inscribe in a given triangle a triangle of minimal perimeter) and the problem of computing the product of three reflections. (This latter problem is natural since by the *First Structure Theorem* of §II.4 every isometry is the product of at most three reflections.) From this, one can prove the concurrency of the altitudes of a triangle using only reflections, not similarities as does the traditional proof. As a consequence, one can later conclude that the concurrency of altitudes holds equally well in elliptic geometry and (to the extent possible) hyperbolic geometry. In the other direction, traditional geometric reasoning is used in showing that every strict similarity transformation has a fixed point and the computation of the product of two rotations is interpreted in terms of traditional geometry.

The *Erlanger Programm* is often treated as a component of the larger topic of group theory. We first introduce the group concept in Chapter III and use it constantly throughout the subsequent material; in particular, it plays a central role in Chapters VII and VIII. But groups are never investigated for themselves; they are always subservient to the geometry. Nevertheless, students who have taken a course from this book will have a store of examples to make coping with group theory in a subsequent abstract algebra course easier and more meaningful.

We have expended considerable effort to give a self-contained development and to make explanations as clear as possible. Our students have found the text to be quite readable; we hope the same will be true for your students.

Our text is, in fact, the first of a projected two-volume work. The second volume will go beyond traditional Euclidean geometry by introducing coordinates, discussing different geometries — affine and non-Euclidean (hyperbolic and spherical/elliptical) — in a projective setting, and ending with an interpretation of Einstein's *Special Theory of Relativity* as an analog in higher dimensions of hyperbolic plane geometry. Until the completion of the second volume we hope this text will stand on its own as a treatment of plane Euclidean geometry that deepens the reader's understanding of symmetry and its natural place in geometry.

This book can support several different types of courses. Chapter I gives a fairly complete axiomatic development of plane Euclidean geometry, largely inspired by Moise, *Elementary Geometry from an Advanced Standpoint*. This can be made a substantial part of the course or be used entirely as a reference or something in between. The heart of the book is composed of Chapters II to V. Here we develop the basic facts about isometries (Chapters II and III) and similarities (Chapter IV) and attempt to integrate the Kleinian transformational viewpoint with geometry as formulated traditionally (Chapter V).

Chapters VI through IX present applications of the material from Chapters II to V. Different choices of material from these chapters will result in courses of quite different flavors. In particular Chapter VI, Chapters VII and VIII, and Chapter IX are essentially independent of each other.

Chapter VI uses the transformational approach to establish very traditional theorems of plane geometry. The chapter begins with the classical coincidences — the circumcenter, incenter, centroid, and orthocenter. In the treatment of the centroid, we emphasize the role of the medial triangle (the triangle formed by the midpoints of sides) and in particular the fact that it is similar to the original triangle by a similarity that stretches by 2 and rotates by 180°. This immediately leads to the Euler line and provides relationships that carry us on to the nine-point circle. We attempt to reveal the depth of this remarkable configuration. Several other topics which can be approached naturally via transformations, including *Fagnano's Theorem* and its relationship to the orthic triangle, *Napoleon's Theorem*, and the *Fermat Problem*, are also discussed.

Chapters VII and VIII are devoted to understanding symmetric figures. They describe the classification of discrete groups of plane isometries — rosette groups, frieze groups, and wallpaper groups. We have attempted to give a careful treatment. Especially in the case of the wallpaper groups, we hope the ingredients that go into the classification are brought out clearly and that the argument, as well as the final result, will be memorable (in the sense that your students will actually be able to remember it!). The group theoretic notion of a split extension is the basic tool, but we treat it in a concrete fashion. In particular, this concept helps to organize both the classification and its justification.

Chapter IX studies scaling and dimension and makes a brief foray into fractals. It discusses the area of plane figures, from familiar area formulas to Jordan content for more exotic shapes. The treatment is relatively abbreviated and provides many opportunities for students to fill in the details (with generous hints).

Throughout the book the approach is concrete, and we try to give complete explanations. The need for justification and proof is taken for granted, and there are many opportunities for students to construct their own arguments. Because of this and because of the concrete and accessible nature of the material, this text might form the basis for a bridge course to introduce students to mathematical reasoning.

Student Preface

This book is about Euclidean geometry, the same subject you studied in high school. However, the viewpoint is probably very different. The goal of this text is to present geometry in a way that honors the ideas of transformation and symmetry that have so profoundly shaped the modern scientific view of the world. To set the stage for the book, we offer you a thumbnail sketch of its historical roots.

Geometry as a deductive science was invented by the Greeks. Their work on the subject, the work of many thinkers over several centuries, was collected and woven together brilliantly by Euclid of Alexandria around 300 B.C. For nearly two thousand years, Euclid's *Elements*, expounding Greek geometry, was the high point of human intellectual achievement.

Though universally revered and admired, there were aspects of Euclid's *Elements* that worried some readers. His system of geometry was built on several "common notions" — easily accepted principles of reasoning which applied to many areas — together with five "postulates," accepted facts that were specifically about geometry. Four of these postulates were short, simple, and easy to accept. The fifth, however, was troublesome to those who thought seriously about the subject. Here is an English translation of what it said.

> If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

The statement is rather long, about as long as the other four postulates put together. But it was not its length that unnerved thoughtful students of Euclid. It was the "if produced indefinitely". It asserts that two lines which cross a third line in a certain way must eventually intersect. But where? If the interior angles of intersection are much smaller than 90°, the lines will intersect nearby. However, if the angles are close to 90°, the lines may intersect far away — possibly far, far away: in the next state or in Europe or beyond the Moon or outside our solar system or even our galaxy (of whose existence Euclid and everyone else before the twentieth century was blissfully unaware).

This is the difficulty with the *Fifth Postulate*: it makes an assertion about the structure of space *in the large*; in fact, infinitely large. (Perhaps such a statement was easier to make in ancient times, when people had little inkling of how large things could be.) There is no way we can physically check the truth of Euclid's *Fifth Postulate* because we can *never* physically confirm that two lines do not intersect; no matter how far we go without finding an intersection point, we can never rule out that such a point exists "just a little further" along the lines.

So people who were inclined to worry about such logical and aesthetic issues were unhappy with the *Fifth Postulate*. Many attempted to eliminate the need for it by turning it into a theorem, i.e., they tried to show it followed as a logical necessity from the first four postulates. However, no one succeeded in doing so although many purported "proofs" were constructed over the centuries. Though always flawed in some way, several of these "proofs" carried their reasoning quite far and established important results that do, in fact, follow from denying the *Fifth Postulate*.

After so many failed attempts over two millennia, the suspicion began to grow in the early nineteenth century that there might indeed be "other geometries" in which the *Fifth Postulate* was false. The first to publish a description of a non-Euclidean geometry (1829) was Nikolai Lobachevsky, professor at the University of Kazan in Russia. Lobachevsky's work appeared in the Bulletin of Kazan University. In that pre-internet (indeed, pre-theory of electricity and magnetism!) era, communications could be rather slow, and thus mathematicians further west remained unaware of Lobachevsky's work. In 1832, an independent account of a non-Euclidean geometry was published by Janos Bolyai, a young Hungarian. These two papers inaugurated the post-Euclidean era in geometry.

These were the first leaks in a dam that had been holding back human thought. The next decades saw a flood of research in geometry and the creation of a great profusion of geometric systems. The traditional meaning of geometry could not encompass the new wealth of phenomena. A revised understanding of the nature of geometry was urgently needed.

The basis for this new understanding came from a subject invented at nearly the same time as non-Euclidean geometry — the theory of *groups*. Unlike the questions surrounding the *Fifth Postulate* which were in the minds of many mathematicians at the time, group theory was the invention of one extraordinary individual, Evariste Galois.[1]

[1] Galois was inspired by a problem which had also attracted much attention — the problem of giving formulas for the solutions of polynomial equations. However, he applied his own unique and revolutionary approach to the problem.

Galois was too smart for his own good. A student revolutionary who once toasted the king of France by burying a knife in a table, he died in a duel over a woman at the age of twenty-two. That was in 1832, the same year as Bolyai's publication on non-Euclidean geometry. Fortunately for mathematics, Galois spent the night before the duel furiously writing down his ideas about groups and their applications to solving polynomial equations.

Galois' ideas were hard for his contemporary mathematicians to understand. However, they did get studied and appreciation for their power gradually percolated through the mathematical community during the same period when geometric research was roaring full throttle. Indeed, a realization of the relevance of group theory to geometry began to grow, with groups arising from *symmetries* — transformations of a system which preserve the essential characteristics of the system.

The seminal connection between geometry and group theory was discovered by Felix Klein. It was the custom in German universities of that era for new professors to give an inaugural lecture on their research to the full faculty.[2] In 1872, at the University of Erlangen, Felix Klein, then only twenty-three years old, presented to his colleagues striking ideas about how to unify geometry by means of symmetry via group theory. This proposal has become known as Klein's **Erlanger Programm**.[3]

Klein's first observation was that geometry is not about individual figures but about *classes of equivalent figures*. For example, in Euclidean geometry there is a notion of *congruence*. Any two congruent figures are "the same" from the point of view of Euclidean geometry — they have the same geometric properties. Furthermore, you can tell if two figures are congruent by *transforming* or moving one so that it becomes (more correctly, coincides with) the other. The transformation you use should be a *rigid motion*, i.e., it should preserve distances and angles.

Thus, at the core of Euclidean geometry, as well as at the core of most other geometries constructed after 1830, there was a notion of *geometric equivalence*, defined by a specified collection of transformations known as the *symmetries* of the geometry. (In Euclid's *Elements*, this idea was somewhat hidden and never explicitly acknowledged, but the attentive reader can see it used at certain critical places.) Klein further observed that

[2]It was a much bigger deal to be a professor in those days — in most universities there was often just one professor in a subject. They were addressed as "Herr Professor Doktor," the title a bit long but admirably distinguished.

[3]The "r" on the end of "Erlanger" is not a misprint. It's how German grammar works.

(i) the set of symmetries formed a group in the sense of Galois and

(ii) you can reconstruct the geometry from its symmetry group.[4]

In short, the fundamental idea in geometry is that of *symmetry*, and a given geometry is governed by the nature of its symmetries.

Klein's ideas and related work sparked a second wave of remarkable discovery that produced, among other things, a classification of the building blocks of all possible symmetry groups of geometries that are *continuous* in that they allow continuous movement. While this classification listed familiar objects such as Euclidean and non-Euclidean geometries and their higher-dimensional cousins, it also included a few exotic systems of symmetries whose associated geometries are still only partially understood.

Through the end of the nineteenth century this was all *pure* mathematics, inspired by nagging questions in the field and divorced from practical goals. In 1905, however, Albert Einstein introduced his *Special Theory of Relativity* that explained the troubling results of some experiments (e.g., the Michelson-Morley experiment) made to probe Maxwell's theory of electromagnetism. A year later, Hermann Minkowski, who had been Einstein's mathematics teacher at the University of Koenigsberg and was embarrassed by the unsophisticated level of the mathematics in Einstein's paper, reinterpreted Einstein's results in terms of a non-Euclidean geometry of four-dimensional space-time. Transformations had played a key role in the interpretation of the Michelson-Morley experiment and in Einstein's theory — Minkowski found the four-dimensional geometry for which they comprised the group of symmetries.

Since the appearance of Einstein's paper, symmetries and transformations have played an ever-greater role in theoretical physics. It is not too much of an exaggeration to say that symmetry and groups have been a dominant theme in modern physics. In particular, the structure of atoms, which leads to the chemistry that shapes all of biology, has at its base a structure of exquisite symmetry. Thus group theory and symmetry, which were first introduced in response to questions about solutions of equations, proved later to be fundamental for understanding geometry, and still later, for understanding the deeper truths of our real physical world. This history is a beautiful example of how mathematical ideas, pursued for their own sake, can have a dramatic impact and practical consequences in domains far beyond their original birthplace.[5]

[4]Actually the reconstruction of the geometry requires a little additional information along with the symmetry group. But the symmetry group remains the central object.

[5]The same point can be made in an even stronger way for the investigations that led to the discovery of non-Euclidean geometry. At face value, these seemed to be archetypically useless academic pursuits — they were not even going to produce new theorems, only tidy up the system

Our book takes Klein's *Erlanger Programm* seriously, while still retaining the flavor of a classical study of geometry in which triangles, circles, quadrilaterals, and other simple shapes are the primary objects of investigation. The study of transformations is integrated with serious attention to the beautiful results of synthetic geometry. We do this in two ways: transformations are studied *as geometric objects*, emphasizing their concrete geometric properties, and transformations are also used *as tools* to understand interesting concepts in geometry such as the circumcircle, incircle, centroid, orthocenter, Euler line, and nine-point circle. The authors have been surprised at the extent to which this unified viewpoint can succeed.

We hope this book presents the philosophy of the *Erlanger Programm* — that symmetry is the basis of geometry — not merely as an abstract, organizational viewpoint, but as a practical approach that enhances your understanding and highlights the beauty of this timeless subject.

Above all, we hope you enjoy the journey you are about to begin.

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Cross-Reference Conventions. Text cross-references to theorems, figures, equations, and other labeled items are handled as follows. Within Chapter V, for example, cross-references such as Theorem 1.2, Figure 5.3, and (3.3) refer to items with those labels *contained in* Chapter V. However, within Chapter V, cross-references such as Theorem II.1.2, Figure II.5.3, and (II.3.3) refer to items with those labels in Chapter II, hence *outside of* Chapter V.

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of postulates. As it turned out, they brought us to a great watershed in thought, producing massive reverberations in mathematics, science, and philosophy that have shaped and continue to shape the nature of our thinking.